# THE DYNAMICAL STRESSES PRODUCED IN A THICK PLATE BY THE ACTION OF SURFACE FORCES

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**1. Introduction.** The first discussion of the propagation of elastic waves in a thick plate was given by Lamb [1] for the two-dimensional problem of a harmonic wave travelling in a direction parallel to the medial plane of the plate. Lamb derived equations relating the thickness of the plate to the phase velocities of two types of wave, one symmetric with respect to the medial plane and the other antisymmetric. The symmetric modes of propagation introduced by Lamb have been studied by Holden [2] and the antisymmetric modes have been studied by Osborne and Hart [3]. More recently Pursey [4] has shown how the amplitude of the disturbance is related to a given distribution of stress, varying harmonically with time, applied to the free surfaces of the plate ; two types of source are considered by Pursey, one producing a two-dimensional field of the Lamb type, and the other having circular symmetry about an axis normal to the surface of the plate.

The purpose of the present paper is to derive formulae for the components of the displacement vector and of the stress tensor at an interior point of the plate when time-dependent pressures are applied to the free surfaces of the plate. As in the other papers cited, it is assumed that the displacements and strains are small and that the substance is homogeneous, isotropic and satisfies Hooke's law, so that the equations of the classical theory of elasticity apply. The method of solution follows that outlined in [5] and developed for the infinite solid in [6]. By a systematic use of the theory of integral transforms, expressions are established for the stress and displacement in the general three-dimensional case in the form of triple integrals. The simpler solutions corresponding to the case of axial symmetry and the two-dimensional problem are also derived. In general the evaluation of the integrals occurring in these formulae presents formidable difficulties. In one case—that in which the stress is due to pulses of pressure moving uniformly along the boundaries—the integrals in the two-dimensional theory reduce to single integrals which can be calculated numerically. The method is illustrated by the calculation of the normal component of stress in the medial plane of the plate. Finally the solution of the statical problem is derived from the " steady-state " solution by letting the velocity of the applied pulses tend to zero.

**2. The General Solution of the Equations of Motion.** We shall consider the distribution of stress in the interior of an infinite plate of homogeneous isotropic elastic material bounded by the parallel planes $z = \pm d$. We shall assume that there are no body forces operative in the interior of the plate. If we take a pair of orthogonal $x$- and $y$-axes in the central plane of the plate, $z = 0$, then in terms of the rectangular Cartesian co-ordinates $x$, $y$ and $z$ the displacement vector may be denoted by the components $(u, v, w)$ and the stress tensor by the components $\sigma_x$, $\sigma_y$, $\sigma_z$, $\tau_{xy}$, $\tau_{yz}$ and $\tau_{zx}$. In this system of co-ordinates the equations of motion of the solid may, in the absence of body forces, be written in the form

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = (\lambda + 2\mu)\frac{\partial^2 u}{\partial \tau^2}, \quad \text{.................................(2.1)}$$



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$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = (\lambda + 2\mu)\frac{\partial^2 v}{\partial \tau^2}, \qquad (2.2)$$

$$\frac{\partial \tau_{xz}}{\partial z} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = (\lambda + 2\mu)\frac{\partial^2 w}{\partial \tau^2}, \qquad (2.3)$$

where $\lambda$ and $\mu$ denote Lamé's elastic constants and $\tau$ is a space-like variable related to the time $t$ through the equation

$$\tau = c_1 t, \qquad (2.4)$$

where $c_1$ is the velocity of $P$-waves in the solid, so that

$$c_1^2 = \frac{\lambda + 2\mu}{\rho}. \qquad (2.5)$$

The relation between the stress tensor and the displacement vector may similarly be expressed by the set of six equations

$$(\sigma_x, \sigma_y, \sigma_z) = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu\left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}\right), \qquad (2.6)$$

$$(\tau_{xy}, \tau_{yz}, \tau_{zx}) = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right). \qquad (2.7)$$

We shall discuss the case in which the normal component of stress, $\sigma_z$, is prescribed uniquely at every point of the bounding planes $z = \pm d$, and each of the shearing stresses $\tau_{xz}$ and $\tau_{yz}$ vanish identically at each point of these boundaries. No new principle is involved in the calculation of the components of stress in the general case in which the shearing stresses instead of being identically zero on the boundaries, assume prescribed values, and the analysis extends along lines precisely similar to those outlined below. We therefore assume that our boundary conditions are :—

$$\tau_{xz} = \tau_{yz} = 0, \quad \text{on the planes } z = \pm d; \qquad (2.8)$$

$$\sigma_z = -p_1(x, y, \tau), \text{ on the plane } z = +d; \qquad (2.9)$$

$$\sigma_z = -p_2(x, y, \tau), \text{ on the plane } z = -d. \qquad (2.10)$$

To solve the set of nine partial differential equations symbolised by (2.1), (2.2), (2.3), (2.6) and (2.7), subject to the boundary conditions (2.8), (2.9) and (2.10), we introduce the three-dimensional Fourier transform defined by the equation

$$\bar{f}(\xi, \eta, z, \omega) = (2\pi)^{-3/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y, z, \tau)\, e^{i(\xi x + \eta y + \omega\tau)}\, dx\, dy\, d\tau \qquad (2.11)$$

of all the quantities occurring in these equations. If we multiply both sides of each of our nine partial differential equations by $\exp\{i(\xi x + \eta y + \omega\tau)\}$, integrate from $-\infty$ to $+\infty$ with respect to each of the variables $x$, $y$ and $\tau$, and make use of well-known properties of Fourier transforms, [7, p. 27 and p. 43], we obtain the set of simultaneous ordinary linear differential equations

$$i\xi\bar{\sigma}_x + i\eta\bar{\tau}_{xy} - D\bar{\tau}_{xz} = (\lambda + 2\mu)\omega^2\bar{u}, \qquad (2.12)$$

$$i\xi\bar{\tau}_{xy} + i\eta\bar{\sigma}_y - D\bar{\tau}_{yz} = (\lambda + 2\mu)\omega^2\bar{v}, \qquad (2.13)$$

$$i\xi\bar{\tau}_{xz} + i\eta\bar{\tau}_{yz} - D\bar{\sigma}_z = (\lambda + 2\mu)\omega^2\bar{w}, \qquad (2.14)$$

$$i(\bar{\sigma}_x, \bar{\sigma}_y, \bar{\sigma}_z) = \lambda(\xi\bar{u} + \eta\bar{v} + iD\bar{w}) + 2\mu(\xi\bar{u}, \eta\bar{v}, iD\bar{w}), \qquad (2.15)$$

$$i(\bar{\tau}_{xy}, \bar{\tau}_{yz}, \bar{\tau}_{zx}) = \mu(\eta\bar{u} + \xi\bar{v}, \eta\bar{w} + iD\bar{v}, iD\bar{u}, +\xi\bar{w}), \qquad (2.16)$$

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where we have written $D$ to denote the operator $\partial/\partial z$. If we now substitute from equations (2.15) and (2.16) into equations (2.12)–(2.14) we obtain the set of three simultaneous ordinary differential equations

$$(\beta^2\xi^2+\eta^2-D^2-\beta^2\omega^2)\bar{u}+(\beta^2-1)\xi\eta\bar{v}+i\xi D(\beta^2-1)\bar{w} = 0, \qquad \text{(2.17)}$$

$$(\beta^2-1)\xi\eta\bar{u}+(\xi^2+\beta^2\eta^2-D^2-\beta^2\omega^2)\bar{v}+i\eta(\beta^2-1)D\bar{w} = 0, \qquad \text{(2.18)}$$

$$i\xi(\beta^2-1)D\bar{u}+i\eta(\beta^2-1)D\bar{v}+(\xi^2+\eta^2-\beta^2D^2-\beta^2\omega^2)\bar{w} = 0, \qquad \text{(2.19)}$$

for the determination of the Fourier transforms of the components of the displacement vector. In these equations we have written

$$\beta^2 = \frac{\lambda+2\mu}{\mu}. \qquad \text{(2.20)}$$

By the ordinary methods for the solution of such differential equations we can readily show that

$$\bar{u} = \xi\Theta_1\cosh(n_1z)+\xi\Theta_2\sinh(n_1z)+n_2\Theta_3\cosh(n_2z)+n_2\Theta_5\sinh(n_2z), \qquad \text{(2.21)}$$

$$\bar{v} = \eta\Theta_1\cosh(n_1z)+\eta\Theta_2\sinh(n_1z)+n_2\Theta_4\cosh(n_2z)+n_2\Theta_6\sinh(n_2z), \qquad \text{(2.22)}$$

$$\bar{w} = in_1\Theta_2\cosh(n_1z)+in_1\Theta_1\sinh(n_1z)+i(\xi\Theta_5+\eta\Theta_6)\cosh(n_2z)+i(\xi\Theta_3+\eta\Theta_4)\sinh(n_2z), \qquad \text{(2.23)}$$

where $\Theta_1$, $\Theta_2$, $\Theta_3$, $\Theta_4$, $\Theta_5$, and $\Theta_6$ are arbitrary constants, and

$$n_1^2 = \xi^2+\eta^2-\omega^2, \quad n_2^2 = \xi^2+\eta^2-\beta^2\omega^2. \qquad \text{(2.24)}$$

It should be observed that though the arbitrary constants $\Theta_i$ ($i = 1, 2, \ldots, 6$) are independent of $z$ they may involve the parameters $\xi$, $\eta$ and $\omega$. If we substitute from equations (2.21)–(2.23) into the second and third equations of the set (2.16) we find that

$$\bar{\tau}_{xz} = \mu[2n_1\xi\Theta_1\sinh(n_1z)+2n_1\xi\Theta_2\cosh(n_1z) + \{(n_2^2+\xi^2)\Theta_3+\xi\eta\Theta_4\}\sinh(n_2z)+\{(n_2^2+\xi^2)\Theta_5+\xi\eta\Theta_6\}\cosh(n_2z)]$$

and that

$$\bar{\tau}_{yz} = \mu[2n_1\eta\Theta_1\sinh(n_1z)+2n_1\eta\Theta_2\cosh(n_1z) + \{\xi\eta\Theta_3+(n_2^2+\eta^2)\Theta_4\}\sinh(n_2z)+\{\xi\eta\Theta_5+(n_2^2+\eta^2)\Theta_6\}\cosh(n_2z)].$$

Now it follows from the boundary conditions (2.8) that both $\bar{\tau}_{xz}$ and $\bar{\tau}_{yz}$ will vanish on each of the planes $z = \pm d$. We must therefore have

$$2n_1\xi\Theta_1\sinh(n_1d)+\{(n_2^2+\xi^2)\Theta_3+\xi\eta\Theta_4\}\sinh(n_2d) = 0,$$

$$2n_1\xi\Theta_2\cosh(n_1d)+\{(n_2^2+\xi^2)\Theta_5+\xi\eta\Theta_6\}\cosh(n_2d) = 0,$$

$$2n_1\eta\Theta_1\sinh(n_1d)+\{(n_2^2+\eta^2)\Theta_4+\xi\eta\Theta_3\}\sinh(n_2d) = 0,$$

$$2n_1\eta\Theta_2\cosh(n_1d)+\{(n_2^2+\eta^2)\Theta_6+\xi\eta\Theta_5\}\cosh(n_2d) = 0.$$

It is readily shown that the solution of these algebraic equations is

$$\left.\begin{aligned}\Theta_1 &= (\xi^2+\eta^2-\tfrac{1}{2}\beta^2\omega^2)\sinh(n_2d)\Phi_1,\\ \Theta_2 &= (\xi^2+\eta^2-\tfrac{1}{2}\beta^2\omega^2)\cosh(n_2d)\Phi_2,\\ \Theta_3 &= -n_1\xi\sinh(n_1d)\Phi_1,\\ \Theta_4 &= -n_1\eta\sinh(n_1d)\Phi_1,\\ \Theta_5 &= -n_1\xi\cosh(n_1d)\Phi_2,\\ \Theta_6 &= -n_1\eta\cosh(n_1d)\Phi_2,\end{aligned}\right\} \qquad \text{(2.25)}$$

where $\Phi_1$ and $\Phi_2$ are arbitrary.

If we substitute these values for the $\Theta$'s into equations (2.21)–(2.23) and then substitute these expressions into the third equation of the set (2.15) we find that

$$\bar{\sigma}_z = 2i\mu[(\xi^2+\eta^2-\tfrac{1}{2}\beta^2\omega^2)^2\{\sinh(n_2d)\cosh(n_1z)\Phi_1+\cosh(n_2d)\sinh(n_1z)\Phi_2\} - n_1n_2(\xi^2+\eta^2)\{\sinh(n_1d)\cosh(n_2z)\Phi_1+\cosh(n_1d)\sinh(n_2z)\Phi_2\}].$$

Now it follows from equation (2.9) that

$\bar{\sigma}_z = -\bar{p}_1(\xi, \eta, \omega)$ when $z = +d$, and that $\bar{\sigma}_z = -\bar{p}_2(\xi, \eta, \omega)$ when $z = -d$,

so that the equations for the determination of the constants $\Phi_1$ and $\Phi_2$ are

$$\frac{i\bar{p}_1}{2\mu} = (\xi^2+\eta^2-\tfrac{1}{2}\beta^2\omega^2)^2\{\sinh(n_2d)\cosh(n_1d)\Phi_1+\cosh(n_2d)\sinh(n_1d)\Phi_2\} - n_1n_2(\xi^2+\eta^2)\{\sinh(n_1d)\cosh(n_2d)\Phi_1+\cosh(n_1d)\sinh(n_2d)\Phi_2\}.$$

$$\frac{i\bar{p}_2}{2\mu} = (\xi^2+\eta^2-\tfrac{1}{2}\beta^2\omega^2)^2\{\sinh(n_2d)\cosh(n_1d)\Phi_1-\cosh(n_2d)\sinh(n_1d)\Phi_2 - n_1n_2(\xi^2+\eta^2)\{\sinh(n_1d)\cosh(n_2d)\Phi_1-\cosh(n_1d)\sinh(n_2d)\Phi_2\}.$$

Solving these equations for $\Phi_1$ and $\Phi_2$ we find that

$$\Phi_1 = \frac{\bar{P}}{2i\mu}, \quad \Phi_2 = \frac{\bar{Q}}{2i\mu}, \qquad \text{......(2.26)}$$

where

$$\bar{P} = -\tfrac{1}{2}(\bar{p}_1+\bar{p}_2)\{(\xi^2+\eta^2-\tfrac{1}{2}\beta^2\omega^2)^2\cosh(n_1d)\sinh(n_2d)-n_1n_2(\xi^2+\eta^2)\sinh(n_1d)\cosh(n_2d)\}^{-1}, \qquad \text{......(2.27)}$$

and

$$\bar{Q} = -\tfrac{1}{2}(\bar{p}_1-\bar{p}_2)\{(\xi^2+\eta^2-\tfrac{1}{2}\beta^2\omega^2)^2\sinh(n_1d)\cosh(n_2d)-n_1n_2(\xi^2+\eta^2)\cosh(n_1d)\sinh(n_2d)\}^{-1}. \qquad \text{......(2.28)}$$

Substituting from equations (2.25)–(2.28) into equations (2.21)–(2.23) we obtain finally for the Fourier transforms of the components of the displacement vector

$$\bar{u} = \frac{\xi(\xi^2+\eta^2-\tfrac{1}{2}\beta^2\omega^2)}{2i\mu}\{\bar{P}\sinh(n_2d)\cosh(n_1z)+\bar{Q}\cosh(n_2d)\sinh(n_1z)\} - \frac{n_1n_2\xi}{2i\mu}\{\bar{P}\sinh(n_1d)\cosh(n_2z)+\bar{Q}\cosh(n_1d)\sinh(n_2z)\}, \qquad \text{......(2.29)}$$

$$\bar{v} = \frac{\eta(\xi^2+\eta^2-\tfrac{1}{2}\beta^2\omega^2)}{2i\mu}\{\bar{P}\sinh(n_2d)\cosh(n_1d)+\bar{Q}\cosh(n_2d)\sinh(n_1z) - \frac{n_1n_2\eta}{2i\mu}\{\bar{P}\sinh(n_1d)\cosh(n_2z)+\bar{Q}\cosh(n_1d)\sinh(n_2z)\}, \qquad \text{......(2.30)}$$

$$\bar{w} = \frac{n_1(\xi^2+\eta^2-\tfrac{1}{2}\beta^2\omega^2)}{2\mu}\{\bar{P}\sinh(n_2d)\sinh(n_1z)+\bar{Q}\cosh(n_2d)\cosh(n_1z)\} - \frac{n_1(\xi^2+\eta^2)}{2\mu}\{\bar{P}\sinh\{n_1d)\sinh(n_2z)+\bar{Q}\cosh(n_1d)\cosh(n_2z)\}. \qquad \text{......(2.31)}$$

The expressions for the components themselves are then found by inverting these formulae by means of the Fourier inversion theorem for multiple transforms [**7**, p. 45] :—

$$(u, v, w) = (2\pi)^{-3/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\bar{u}, \bar{v}, \bar{w})e^{-i(\xi x+\eta y+\omega\tau)}\,d\xi\,d\eta\,d\omega. \qquad \text{......(2.32)}$$

Similar expressions can be obtained for the components of the stress tensor. For example, it follows from the third equation of the set (2.15) and the equations (2.29)–(2.31) that

$$\bar{\sigma}_z = \bar{P}\{(\xi^2+\eta^2-\tfrac{1}{2}\beta^2\omega^2)^2 \cosh(n_1 z)\sinh(n_2 d) - n_1 n_2(\xi^2+\eta^2)\sinh(n_1 d)\cosh(n_2 z)\}$$
$$+\bar{Q}\{(\xi^2+\eta^2-\tfrac{1}{2}\beta^2\omega^2)^2 \sinh(n_1 z)\cosh(n_2 d) - n_1 n_2(\xi^2+\eta^2)\cosh(n_1 d)\sinh(n_2 z)\}. \quad \text{...(2.33).}$$

In the symmetrical problem in which $p_1 = p_2$ the expression for $\bar{\sigma}_z$ on the central plane $z = 0$ assumes a much simpler form. We find that

$$[\bar{\sigma}_z]_{z=0} = \bar{P}\{(\xi^2+\eta^2-\tfrac{1}{2}\beta^2\omega^2)^2\sinh(n_2 d) - n_1 n_2(\xi^2+\eta^2)\sinh(n_1 d)\}, \quad \text{...........(2.34)}$$

where $\bar{P}$ is given by equation (2.27) with $\bar{p}_2 = \bar{p}_1$. Inverting equation (2.34) by means of the appropriate Fourier theorem we then obtain an expression for the normal component of stress at the base of an elastic strip $0 \leqslant z \leqslant d$ when the base $z = 0$ rests on a rigid foundation and a pressure $p_1$ is applied to the surface $z = d$.

**3. Solution of the Equations of Motion in the Case of Axial Symmetry.** We shall now derive a formal solution of the problem for the case in which the pressures $p_1$ and $p_2$ which are applied to the boundaries of the thick plate are both distributed symmetrically about the $z$-axis. In these circumstances we can describe the displacement vector by a pair of components $u_r$ and $u_z$, where $(r, \theta, z)$ are the cylindrical polar co-ordinates of a point. The non-vanishing components of the stress tensor may then be denoted by $\sigma_r$, $\sigma_\theta$, $\sigma_z$ and $\tau_{rz}$. If we substitute from the stress-strain relations

$$(\sigma_r, \sigma_\theta, \sigma_z) = \lambda\left(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{\partial u_z}{\partial z}\right)+2\mu\left(\frac{\partial u_r}{\partial r}, \frac{u_r}{r}, \frac{\partial u_z}{\partial z}\right) \quad \text{..................(3.1)}$$

and

$$\tau_{rz} = \mu\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right) \quad \text{..........................................(3.2)}$$

into the equations of motion

$$\frac{\partial \sigma_r}{\partial r}+\frac{\partial \tau_{rz}}{\partial z}+\frac{\sigma_r-\sigma_\theta}{r} = (\lambda+2\mu)\frac{\partial^2 u_r}{\partial \tau^2}, \quad \text{.......................................(3.3)}$$

$$\frac{\partial \tau_{rz}}{\partial r}+\frac{\partial \sigma_z}{\partial z}+\frac{\tau_{rz}}{r} = (\lambda+2\mu)\frac{\partial^2 u_z}{\partial \tau^2}, \quad \text{.......................................(3.4)}$$

we find that the components of the displacement vector satisfy the pair of simultaneous partial differential equations

$$\beta^2\left(\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{u_r}{r^2}\right)+(\beta^2-1)\frac{\partial^2 u_z}{\partial r\,\partial z}+\frac{\partial^2 u_r}{\partial z^2} = \beta^2\frac{\partial^2 u_r}{\partial \tau^2}, \quad \text{..................(3.5)}$$

$$\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}+(\beta^2-1)\frac{\partial}{\partial z}\left(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}\right)+\beta^2\frac{\partial^2 u_z}{\partial z^2} = \beta^2\frac{\partial^2 u_z}{\partial \tau^2}. \quad \text{..................(3.6)}$$

If we now multiply both sides of equation (3.5) by $e^{i\omega\tau}rJ_1(\xi r)$ and both sides of equation (3.6) by $e^{i\omega\tau}rJ_0(\xi r)$ and integrate with respect to $r$ from 0 to $\infty$ and with respect to $\tau$ from $-\infty$ to $+\infty$ we find that these equations are equivalent to the pair of simultaneous ordinary differential equations

$$(\beta^2\xi^2-D^2-\beta^2\omega^2)\bar{u}_r+\xi(\beta^2-1)D\bar{u}_z = 0, \quad \text{..........................(3.7)}$$

$$-\xi(\beta^2-1)D\bar{u}_r+(\xi^2-\beta^2 D^2-\beta^2\omega^2)\bar{u}_z = 0, \quad \text{..........................(3.8)}$$

for the transforms $\bar{u}_r$ and $\bar{u}_z$ defined by the equations

$$\bar{u}_r = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} e^{i\omega\tau}\,d\tau\int_0^\infty r u_r J_1(\xi r)\,dr, \quad \text{..........................(3.9)}$$

$$\bar{u}_z = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} e^{i\omega\tau}\,d\tau \int_0^{\infty} r u_z J_0(\xi r)\,dr. \quad (3.10)$$

In equations (3.7) and (3.8), $D$ denotes the operator $\partial/\partial z$ as before, and $\beta$ is given by equation (2.20). It is readily shown that the solution of the pair of equations (3.7) and (3.8) is contained in the relations

$$\bar{u}_r = \xi\{A \cosh(n_1 z) + B \sinh(n_1 z)\} + n_2\{C \cosh(n_2 z) + D \sinh(n_2 z)\}, \quad (3.11)$$

$$\bar{u}_z = -n_1\{A \sinh(n_1 z) + B \cosh(n_1 z)\} - \xi\{C \sinh(n_2 z) + D \cosh(n_2 z)\}, \quad (3.12)$$

where $A$, $B$, C and $D$ are arbitrary constants and

$$n_1^2 = \xi^2 - \omega^2, \quad n_2^2 = \xi^2 - \beta^2\omega^2. \quad (3.13)$$

If the conditions on the boundary surfaces are

$$\tau_{rz} = 0, \text{ on } z = \pm d, \quad (3.14)$$

$$\sigma_z = -p_1(r, \tau) \text{ on } z = +d, \quad (3.15)$$

$$\sigma_z = -p_2(r, \tau) \text{ on } z = -d, \quad (3.16)$$

then it follows from equations (3.14) and (3.2) that

$$D\bar{u}_r - \xi\bar{u}_z = 0, \text{ on the planes } z = \pm d. \quad (3.17)$$

From this we find that

$$\left.\begin{aligned} A &= (\xi^2 - \tfrac{1}{2}\beta^2\omega^2)\sinh(n_2 d)\Theta_1, & B &= (\xi^2 - \tfrac{1}{2}\beta^2\omega^2)\cosh(n_2 d)\Theta_2, \\ C &= -\xi n_1 \sinh(n_1 d)\Theta_1, & D &= -\xi n_1 \cosh(n_1 d)\Theta_2, \end{aligned}\right\} \quad (3.18)$$

where $\Theta_1$ and $\Theta_2$ are arbitrary constants.

Making use of the fact that the boundary conditions (3.15) and (3.16) are equivalent to the relations

$$\bar{\sigma}_z = -\bar{p}_1(\xi, \omega) \text{ on } z = d\,; \quad \bar{\sigma}_z = -\bar{p}_2(\xi, \omega) \text{ on } z = -d,$$

where

$$\bar{p}_{1,2}(\xi, \omega) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} e^{i\omega\tau}\,d\tau \int_0^{\infty} r p_{1,2}(r, \tau) J_0(\xi r)\,dr, \quad (3.19)$$

we find that

$$\Theta_1 = -\bar{P}/(2\mu), \quad \Theta_2 = -\bar{Q}/(2\mu), \quad (3.20)$$

where

$$\bar{P} = -\tfrac{1}{2}(\bar{p}_1 + \bar{p}_2)\{(\xi^2 - \tfrac{1}{2}\beta^2\omega^2)^2 \cosh(n_1 d)\cosh(n_2 d) - \xi^2 n_1 n_2 \sinh(n_1 d)\cosh(n_2 d)\}^{-1}, \quad (3.21)$$

$$\bar{Q} = -\tfrac{1}{2}(\bar{p}_1 - \bar{p}_2)\{(\xi^2 - \tfrac{1}{2}\beta^2\omega^2)^2 \sinh(n_1 d)\cosh(n_2 d) - \xi^2 n_1 n_2 \cosh(n_1 d)\sinh(n_2 d)\}^{-1}. \quad (3.22)$$

The components of the displacement vector therefore possess the transforms

$$\begin{aligned} \bar{u}_r = {} & \xi(\xi^2 - \tfrac{1}{2}\beta^2\omega^2)^2 [\cosh(n_1 z)\sinh(n_2 d)\Theta_1 + \sinh(n_1 z)\cosh(n_2 z)\cosh(n_2 d)\Theta_2] \\ & - \xi n_1 n_2 [\sinh(n_1 d)\cosh(n_2 z)\Theta_1 + \cosh(n_1 d)\sinh(n_2 z)\Theta_2], \end{aligned} \quad (3.23)$$

$$\begin{aligned} \bar{u}_z = {} & -n_1(\xi^2 - \tfrac{1}{2}\beta^2\omega^2)[\sinh(n_1 z)\sinh(n_2 d)\Theta_1 + \cosh(n_1 z)\cosh(n_2 d)\Theta_2] \\ & + \xi^2 n_1 [\sinh(n_1 d)\sinh(n_2 z)\Theta_1 + \cosh(n_1 d)\cosh(n_2 z)\Theta_2], \end{aligned} \quad (3.24)$$

where $\Theta_1$ and $\Theta_2$ are given uniquely by the equations (3.19)–(3.22). The final solution of the problem is therefore given by the integral expressions

$$u_r = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} e^{i\omega\tau}\,d\omega \int_0^{\infty} \xi\bar{u}_r J_1(\xi r)\,d\xi, \quad (3.25)$$

$$u_z = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} e^{i\omega\tau}\,d\omega \int_0^{\infty} \xi\bar{u}_z J_0(\xi r)\,d\xi, \quad (3.26)$$

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where $\bar{u}_r$ and $\bar{u}_z$ are given by the expressions (3.23) and (3.24).

**4. The General Solution of the Two-Dimensional Problem.** The solution of the two-dimensional problem in which the physical conditions are identical in all planes parallel to the $xz$-plane (plane strain) can be derived by writing down the equations of plane strain and the corresponding stress-strain relations and solving them by a process similar to that employed in § 2 for the three-dimensional case. We may however derive the two-dimensional solution from the three-dimensional one by considering the solution in the latter case when the applied pressures $p_1$ and $p_2$ are functions of $x$ and $\tau$ only, that is

$$p_1 = p_1(x, \tau), \quad p_2 = p_2(x, \tau), \qquad (4.1)$$

so that, in the notation of § 2,

$$\left.\begin{aligned} \bar{p}_1(\xi, \eta, \omega) &= (2\pi)^{\frac{1}{2}}\bar{p}_1(\xi, \omega)\delta(\eta), \\ \bar{p}_2(\xi, \eta, \omega) &= (2\pi)^{\frac{1}{2}}\bar{p}_2(\xi, \omega)\delta(\eta), \end{aligned}\right\} \qquad (4.2)$$

where $\delta(\eta)$ denotes the Dirac delta function of argument $\eta$ and

$$\bar{p}_{1,2}(\xi, \omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{1,2}(x, \tau)e^{i(\xi x+\omega\tau)}\,dx\,d\tau \qquad (4.3)$$

are the two-dimensional Fourier transforms of the functions $p_1(x, \tau)$, $p_2(x, \tau)$. On substituting from equations (4.2) into equations (2.29)–(2.31) we obtain the expressions

$$\begin{aligned} \bar{u} = {} & \left(\frac{\pi}{2}\right)^{1/2}\frac{\xi(\xi^2-\frac{1}{2}\beta^2\omega^2)\,\delta(\eta)}{i\mu}\{\bar{\mathfrak{P}}\sinh(m_2d)\cosh(m_1z)+\bar{\mathfrak{Q}}\cosh(m_2d)\sinh(m_1z)\} \\ & -\left(\frac{\pi}{2}\right)^{1/2}\frac{m_1m_2\xi\,\delta(\eta)}{i\mu}\{\bar{\mathfrak{P}}\sinh(m_1d)\cosh(m_2z)+\bar{\mathfrak{Q}}\cosh(m_1d)\sinh(m_2z)\}, \end{aligned} \qquad (4.4)$$

$$\bar{v} = 0, \qquad (4.5)$$

$$\begin{aligned} \bar{w} = {} & \left(\frac{\pi}{2}\right)^{1/2}\frac{m_1(\xi^2-\frac{1}{2}\beta^2\omega^2)\,\delta(\eta)}{\mu}\{\bar{\mathfrak{P}}\sinh(m_2d)\sinh(m_1z)+\bar{\mathfrak{Q}}\cosh(m_2d)\cosh(m_1z)\} \\ & -\left(\frac{\pi}{2}\right)^{1/2}\frac{m_1\xi^2}{\mu}\{\bar{\mathfrak{P}}\sinh(m_1d)\sinh(m_2z)+\bar{\mathfrak{Q}}\cosh(m_1d)\cosh(m_2z)\}, \end{aligned} \qquad (4.6)$$

for the Fourier transforms of the components of the displacement vector. In these equations

$$m_1^2 = \xi^2-\omega^2, \quad m_2^2 = \xi^2-\beta^2\omega^2, \qquad (4.7)$$

$$\begin{aligned} \bar{\mathfrak{P}} = {} & -\tfrac{1}{2}[\bar{p}_1(\xi, \omega)+\bar{p}_2(\xi, \omega)]\,\{(\xi^2-\tfrac{1}{2}\beta^2\omega^2)^2\cosh(m_1d)\sinh(m_2d) \\ & -m_1m_2\xi^2\sinh(m_1d)\cosh(m_2d)\}^{-1}, \end{aligned} \qquad (4.8)$$

$$\begin{aligned} \bar{\mathfrak{Q}} = {} & -\tfrac{1}{2}[\bar{p}_1(\xi, \omega)-\bar{p}_2(\xi, \omega)]\,\{(\xi^2-\tfrac{1}{2}\beta^2\omega^2)^2\sinh(m_1d)\cosh(m_2d) \\ & -m_1m_2\xi^2\cosh(m_1d)\sinh(m_2d)\}^{-1}, \end{aligned} \qquad (4.9)$$

and the transforms $\bar{p}_1(\xi, \omega)$, $\bar{p}_2(\xi, \omega)$ are defined by equations (4.3). Inserting these expressions into Fourier's inversion theorem (2.32) we obtain the expressions

$$\begin{aligned} u = {} & \frac{1}{4i\pi\mu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\xi(\xi^2-\tfrac{1}{2}\beta^2\omega^2)\{\bar{\mathfrak{P}}\sinh(m_2d)\cosh(m_1z) \\ & \qquad +\bar{\mathfrak{Q}}\cosh(m_2d)\sinh(m_1z)\}\,e^{-i(\xi x+\omega\tau)}\,d\xi\,d\omega \\ & -\frac{1}{4i\pi\mu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}m_1m_2\xi\{\bar{\mathfrak{P}}\sinh(m_1d)\cosh(m_2z) \\ & \qquad +\bar{\mathfrak{Q}}\cosh(m_1d)\sinh(m_2z)\}\,e^{-i(\xi x+\omega\tau)}\,d\xi\,d\omega, \end{aligned} \qquad (4.10)$$

$$v = 0, \qquad (4.11)$$

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$$w = \frac{1}{4\pi\mu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} m_1(\xi^2 - \tfrac{1}{2}\beta^2\omega^2)\{\overline{\mathfrak{P}} \sinh(m_2 d)\sinh(m_1 z) + \overline{\mathfrak{Q}}\cosh(m_2 d)\cosh(m_1 z)\} e^{-i(\xi x+\omega\tau)}\, d\xi\, d\omega$$
$$- \frac{1}{4\pi\mu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} m_1\xi^2\{\overline{\mathfrak{P}}\sinh(m_1 d)\sinh(m_2 z) + \overline{\mathfrak{Q}}\cosh(m_2 d)\cosh(m_2 z)\} e^{-i(\xi x+\omega\tau)}\, d\xi\, d\omega, \quad \text{......(4.12)}$$

for the components of the displacement vector in this problem of plane strain.

In problems in which the applied normal forces are symmetrical with respect to the medial plane of the strip so that the boundary conditions are

$$\sigma_z = -p(x, \tau), \quad \tau_{xz} = 0, \quad z = \pm d, \quad \text{..........................(4.13)}$$

we find that $v = 0$ as before and that

$$u = -\frac{1}{4\pi i\mu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\overline{p}(\xi,\omega)}{f(\xi,\omega)}[\xi(\xi^2 - \tfrac{1}{2}\beta^2\omega^2)\sinh(m_2 d)\cosh(m_1 z) - m_1 m_2 \xi \sinh(m_1 d)\cosh(m_2 z)] e^{-i(\xi x+\omega\tau)}\, d\xi\, d\omega, \quad \text{......(4.14)}$$

$$w = -\frac{1}{4\pi\mu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\overline{p}(\xi,\omega)}{f(\xi,\omega)}[m_1(\xi^2 - \tfrac{1}{2}\beta^2\omega^2)\sinh(m_2 d)\sinh(m_1 z) - m_1\xi^2 \sinh(m_1 d)\sinh(m_2 z)] e^{-i(\xi x+\omega\tau)}\, d\xi\, d\omega, \quad \text{......(4.15)}$$

where

$$f(\xi,\omega) = (\xi^2 - \tfrac{1}{2}\beta^2\omega^2)^2\cosh(m_1 d)\sinh(m_2 d) - m_1 m_2\xi^2\sinh(m_1 d)\cosh(m_2 d), \quad \text{...(4.16)}$$

and

$$\overline{p}(\xi,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,\tau)\, e^{i(\xi x+\omega\tau)}\, dx\, d\tau. \quad \text{.......................(4.17)}$$

Substituting from equations (4.14) and (4.15) into the stress-strain relations we obtain the expressions

$$\sigma_x + \sigma_z = \frac{\beta^2 - 1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\overline{p}(\xi,\omega)}{f(\xi,\omega)}\,\omega^2(\xi^2 - \tfrac{1}{2}\beta^2\omega^2)\sinh(m_2 d)\cosh(m_1 z)\, e^{-i(\xi x+\omega\tau)}\, d\xi\, d\omega, \quad \text{...(4.18)}$$

$$\sigma_z - \sigma_x = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\overline{p}(\xi,\omega)}{f(\xi,\omega)}\{m_1 m_2\xi^2\sinh(m_1 d)\cosh m_2 z - (\xi^2 - \tfrac{1}{2}\omega^2)(\xi^2 - \tfrac{1}{2}\beta^2\omega^2)\sinh(m_2 d)\cosh(m_1 z)\} e^{-i(\xi x+\omega\tau)}\, d\xi\, d\omega, \quad \text{......(4.19)}$$

$$\tau_{xz} = \frac{i}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\overline{p}(\xi,\omega)}{f(\xi,\omega)}\, m_1\xi(\xi^2 - \tfrac{1}{2}\beta^2\omega^2)\{\sinh(m_2 d)\sinh(m_1 z) - \sinh(m_1 d)\sinh(m_2 z)\} e^{-i(\xi x+\omega\tau)}\, d\xi\, d\omega, \quad \text{...........(4.20)}$$

for the determination of the stress components.

**5. Pulses of Pressure Moving Uniformly along the Boundaries.** We shall now consider the particular case of pulses of pressure applied to the bounding surfaces and moving along them with uniform velocity $V$. If the pulses are symmetrical in the sense of the last section and if they have " shape " $p(x)$, then we have

$$p(x,\tau) = p(x - Vt) = p(x - \beta_1\tau),$$

where $\beta_1 = V/c_1$. Inserting this expression in formula (4.17), performing the integration with respect to $\tau$ and making use of the formal properties of the Dirac delta function we find that

$$\overline{p}(\xi,\omega) = \overline{p}(\xi)\delta(\omega + \beta_1\xi), \quad \text{...................................(5.1)}$$

where we have written

$$\bar{p}(\xi) = \int_{-\infty}^{\infty} p(\theta)\, e^{i\xi\theta}\, d\theta. \quad \text{.......(5.2)}$$

If we then substitute from equation (5.1) into equations (4.14) and (4.15) we find that, after we perform the integration with respect to $\omega$, the expressions for the components of the displacement vector reduce to

$$u = -\frac{1}{4\pi i\mu}\int_{-\infty}^{\infty} \frac{\bar{p}(\xi)\xi^3}{f(\xi, -\beta_1\xi)}[(1-\tfrac{1}{2}\beta_2^2)\sinh(\kappa_2\xi d)\cosh(\kappa_1\xi z) - \kappa_1\kappa_2\sinh(\kappa_1\xi d)\cosh(\kappa_2\xi z)]\, d\xi,$$

$$w = -\frac{1}{4\pi\mu}\int_{-\infty}^{\infty} \frac{\bar{p}(\xi)\xi^3}{f(\xi, -\beta_1\xi)}[\kappa_1(1-\tfrac{1}{2}\beta_2^2)\sinh(\kappa_2\xi d)\sinh(\kappa_1\xi z) - \kappa_1\sinh(\kappa_1\xi d)\sinh(\kappa_2\xi z)]\, d\xi,$$

where

$$\beta_2^2 = \beta^2\beta_1^2 = V^2/c_2^2, \quad \text{.......(5.3)}$$

and

$$\kappa_1^2 = 1-\beta_1^2, \quad \kappa_2^2 = 1-\beta_2^2. \quad \text{.......(5.4)}$$

If we assume further that $p(\theta)$ is an even function of $\theta$, and make a change of variable in these integrals from $\eta$ to $u$ where $u = \eta d$, the expressions for the components of the displacement vector become

$$u = \frac{1}{2\pi\mu}\int_0^{\infty} \frac{\bar{p}(u/d)}{f(u)}[(1-\tfrac{1}{2}\beta_2^2)\sinh u_2\cosh(zu_1/d) - \kappa_1\kappa_2\sinh u_1\cosh(zu_2/d)] \times \sin\left\{\frac{(x-Vt)u}{d}\right\} du \quad \text{......(5.5)}$$

and

$$w = -\frac{\kappa_1}{2\pi\mu}\int_0^{\infty} \frac{\bar{p}(u/d)}{f(u)}[(1-\tfrac{1}{2}\beta_2^2)\sinh u_2\sinh(zu_1/d) - \sinh u_1\sinh(zu_2/d)] \times \cos\left\{\frac{(x-Vt)u}{d}\right\} du, \quad \text{......(5.6)}$$

where we have written $u_1 = \kappa_1 u$, $u_2 = \kappa_2 u$, and

$$f(u) = (1-\tfrac{1}{2}\beta_2^2)^2\cosh u_1\sinh u_2 - \kappa_1\kappa_2\sinh u_1\cosh u_2. \quad \text{.......(5.7)}$$

In a similar way we may derive the values of the components of the stress tensor from the equations

$$\sigma_x + \sigma_z = \frac{(\beta^2-1)\beta_1^2(1-\tfrac{1}{2}\beta_2^2)}{\pi d}\int_0^{\infty} \frac{\bar{p}(u/d)}{f(u)}\sinh u_2\cosh(zu_1/d)\cos\left\{\frac{(x-Vt)u}{d}\right\} du, \quad \text{.......(5.8)}$$

$$\sigma_x - \sigma_z = -\frac{2}{\pi d}\int_0^{\infty} \frac{\bar{p}(u/d)}{f(u)}[\kappa_1\kappa_2\sinh u_1\cosh(zu_2/d) - (1-\tfrac{1}{2}\beta_1^2)(1-\tfrac{1}{2}\beta_2^2)\sinh u_2\cosh(zu_1/d)] \times \cos\left\{\frac{(x-Vt)u}{d}\right\} du, \quad \text{.......(5.9)}$$

$$\tau_{xz} = \frac{(1-\tfrac{1}{2}\beta_2^2)\kappa_1}{\pi d}\int_0^{\infty} \frac{\bar{p}(u/d)}{f(u)}[\sinh u_2\sinh(zu_1/d) - \sinh u_1\sinh(zu_2/d)] \times \sin\left\{\frac{(x-Vt)u}{d}\right\} du. \quad \text{.......(5.10)}$$

It will be observed that the $z$-component of the displacement vector is zero everywhere on the medial plane $z = 0$, so that this solution also applies to the case of an elastic strip of thickness $d$ resting upon a perfectly rigid foundation whose boundary forms the co-ordinate plane $z = 0$, with a pulse of pressure moving with uniform velocity $V$ along the upper surface $z = d$ of the elastic strip. The pressure transmitted by this elastic strip to the rigid foundation is of some interest; it is determined by the formula

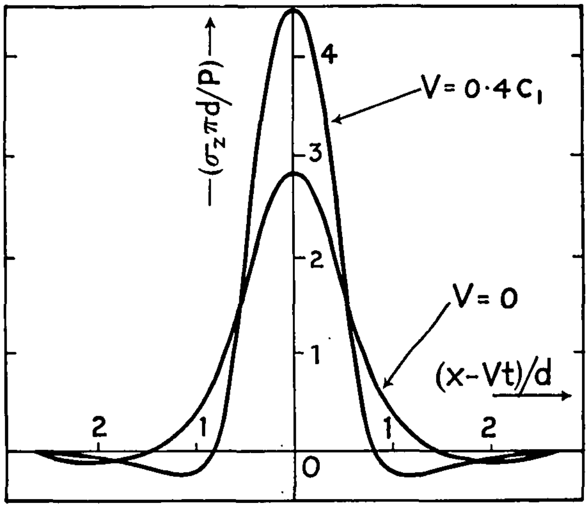


FIG. 1

$$(\sigma_z)_{z=0} = -\frac{1}{\pi d}\int_0^\infty \frac{\bar{p}(u/d)}{f(u)}\left[(1-\tfrac{1}{2}\beta_2^2)^2 \sinh u_2 - (1-\beta_1^2)^{1/2}(1-\beta_2^2)^{1/2}\sinh u_1\right] \times \cos\left\{\frac{(x-Vt)u}{d}\right\} du. \qquad \text{.............(5.11)}$$

In particular, if the loading is a point force of magnitude $P$, then $p(x) = P\delta(x)$, and $\bar{p}(\xi) = P$, so that equation (4.31) reduces to

$$(\sigma_z)_{z=0} = -\frac{P}{\pi d}\int_0^\infty \frac{[(1-\frac{1}{2}\beta_1^2)^2 \sinh u_2 - (1-\beta_1^2)^{1/2}(1-\beta_2^2)^{1/2}\sinh u_1]}{f(u)} \cos\left\{\frac{(x-Vt)u}{d}\right\} du.$$

This integral can be evaluated easily by Filon's method [8] and the variation of the pressure $(\sigma_z)_{z=0}$ with $x - Vt$ is shown graphically in Fig. 1, for the case in which $\lambda = \mu$ (i.e. Poisson's ratio is $\frac{1}{4}$) and $V = 0{\cdot}4c_1$. For comparison the corresponding pressure in the statical case $V = 0$ is also plotted. Comparing the two curves we see that when the force is moving the pressure is greater immediately below the point of application of the force than it is in the statical case but that it falls off more rapidly on either side. These calculations would appear to indicate that in considering the effect of dynamical loads on elastic structures it is not sufficient to take the values of the stress components given by the statical theory.

**6. Solution of the Statical Problem.** It is a simple matter to deduce from the last section the solution of the statical problem in which the infinite strip $-\infty < x < +\infty$, $-d \leqslant z \leqslant +d$ is deformed by the application of normal stresses $p(x)$ to each of the surfaces $z = \pm d$ in the case where $p(x)$ is an even function of $x$. From the definitions of $u_1$ and $u_2$

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it follows from equation (5.7) that, for small values of $V$,

$$f(u) = \tfrac{1}{4}(\beta_1^2 - \beta_2^2)(\sinh 2u + 2u).$$

Similarly we can show that

$$(1 - \tfrac{1}{2}\beta_2^2) \sinh u_2 \sinh (zu_1/d) - \sinh u_1 \sinh (zu_2/d)$$
$$= \tfrac{1}{2}(\beta_1^2 - \beta_2^2)[u \cosh u \sinh (zu/d) - (zu/d) \sinh u \cosh (zu/d)] - \tfrac{1}{2}\beta_2^2 \sinh u \sinh (zu/d),$$

and that

$$(1 - \tfrac{1}{2}\beta_2^2) \sinh u_2 \cosh (zu_1/d) - \kappa_1\kappa_2 \sinh u_1 \cosh (zu_2/d)$$
$$= \tfrac{1}{2}\beta_1^2 \sinh u \cosh (zu/d) - \tfrac{1}{2}(\beta_1^2 - \beta_2^2)[(zu/d) \sinh u \sinh (zu/d) + u \cosh u \cosh (zu/d)].$$

Inserting these expressions into equations (5.5) and (5.6), letting $V$ tend to zero and making use of the results

$$\lim_{V\to 0} \frac{\beta_1^2}{\beta_1^2 - \beta_2^2} = -(1-2\nu), \quad \lim_{V\to 0} \frac{\beta_2^2}{\beta_1^2 - \beta_2^2} = -2(1-\nu),$$

in which $\nu$ denotes Poisson's ratio, we find that the components of the displacement vector in the statical case are

$$u = -\frac{1}{\pi\mu}\int_0^\infty \bar{p}\left(\frac{u}{d}\right)\left\{\frac{u \cosh u \cosh (zu/d) + (zu/d) \sinh u \sinh (zu/d) + (1-2\nu) \sinh u \cosh (zu/d)}{\sinh 2u + 2u}\right\} \times \sin\left(\frac{zu}{d}\right) du,$$

$$w = -\frac{1}{\pi\mu}\int_0^\infty \bar{p}\left(\frac{u}{d}\right)\left\{\frac{u \cosh u \sinh (zu/d) - (zu/d) \sinh u \cosh (zu/d) + 2(1-\nu) \sinh u \sinh (zu/d)}{\sinh 2u + 2u}\right\} \times \cos\left(\frac{xu}{d}\right) du,$$

in agreement with a known result [**7**, p. 412].

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