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CHARACTERISTIC POLYNOMIALS OF GRAPH COVERINGS

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In this note, a formula for the characteristic polynomial of any (regular or irregular) graph covering is described.

Let G be a finite simple graph with vertex set $V(G) = \{v_1, v_2 \dots, v_m\}$. The *adjacency matrix* $A(G) = (a_{ij})$ is the $m \times m$ matrix with $a_{ij} = 1$ if v_i and v_j are adjacent and $a_{ij} = 0$ otherwise. The *characteristic polynomial* of G, denoted by $\Phi(G; \lambda)$, is the characteristic polynomial $\det\big(\lambda I - A(G)\big)$ of $A(G)$.

A *covering projection* (or simply *covering*) from a graph $\widetilde{G}$ to another G is a surjection $p : V(\widetilde{G}) \to V(G)$ such that $p\,|_{N(\widetilde{v})} : N(\widetilde{v}) \to N(v)$ is a bijection for all vertices $v \in V(G)$ and $\widetilde{v} \in p^{-1}(v)$, where $N(v)$, the neighbourhood of v, is the set of vertices adjacent to v. Sometimes, a graph $\widetilde{G}$ is also called a covering of G with the projection $p : \widetilde{G} \to G$, and it is n-fold if p is n-to-one.

Every edge of a graph G gives rise to a pair of oppositely directed edges. By $e^{-1} = vu$, we mean the reverse directed edge to a directed edge $e = uv$. A directed edge is also called an *arc* and the set of arcs of the graph G is denoted by $D(G)$. Let S_n be the symmetric group on $\Omega = \{1, 2, \dots, n\}$. A *voltage assignment* ϕ of G is a function $\phi : D(G) \to S_n$ with the property that $\phi(e^{-1}) = \phi(e)^{-1}$ for each $e \in D(G)$. The *derived graph* G^ϕ from a voltage assignment ϕ is defined as $V(G^\phi) = V(G) \times \Omega$, and (u, i) and (v, j) are adjacent if $uv \in D(G)$ and $j = i^{\phi(uv)}$. The first coordinate projection $p_\phi : G^\phi \to G$ is an n-fold covering. Let $C^1(G; n)$ denote the set of all voltage assignments $\phi : D(G) \to S_n$ of G. Gross and Tucker [**2**] showed that every n-fold covering $\widetilde{G}$ of a graph G can be derived from a voltage assignment in $C^1(G; n)$.

Characteristic polynomials of some graph coverings have already been computed. Chae, Kwak and Lee [**1**] have done it for double coverings of a graph. The characteristic polynomial of a graph covering when its voltages lie in an Abelian group or in a dihedral group was computed by Kwak and others [**3, 4**]. Mizuno and Sato [**5**] gave a formula for the characteristic polynomial of a regular covering. In this note, a formula for the characteristic polynomial of any (regular or irregular) graph covering is described, as an extension of all of the previous works.

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Let $\vec{G}$ denote the digraph obtained from G by replacing each edge of G with a pair of oppositely directed edges and let $\phi \in C^1(G,n)$. For each $\gamma \in S_n$, let $\vec{G}_{(\phi,\gamma)}$ denote the spanning subgraph of the digraph $\vec{G}$ whose directed edge set is $\phi^{-1}(\gamma)$. Let $V(G) = \{v_1, v_2, \ldots, v_m\}$ again. We define an order relation $\leqslant$ on $V(G^\phi)$ as follows: for $(v_i, s), (v_j, t) \in V(G^\phi)$, $(v_i, s) \leqslant (v_j, t)$ if and only if either $s < t$ or $s = t$ and $i \leqslant j$. Let $P(\gamma)$ denote the $n \times n$ permutation matrix associated with $\gamma \in S_n$, *that is*, its (s,t)-entry $P(\gamma)_{st} = 1$ if $s^\gamma = t$ and $P(\gamma)_{st} = 0$ otherwise. The tensor product $A \otimes B$ of the matrices A and B is considered as the matrix B having the element b_{st} replaced by the matrix Ab_{st}. Kwak and Lee ([**3**]) expressed the adjacency matrix $A(G^\phi)$ of a graph covering G^ϕ as

$$A(G^\phi) = \sum_{\gamma \in S_n} A(\vec{G}_{(\phi,\gamma)}) \otimes P(\gamma). \tag{1}$$

Let Γ be a finite group. A *representation* ρ of a group Γ over the complex field $\mathbb{C}$ is a group homomorphism from Γ to the general linear group $\mathrm{GL}(r, \mathbb{C})$ of invertible $r \times r$ matrices over $\mathbb{C}$. The number r is called the *degree* of the representation ρ (see [**6**]). Suppose that $\Gamma \leqslant S_n$ is a permutation group on Ω. It is clear that $P : \Gamma \to \mathrm{GL}(r, \mathbb{C})$ defined by $\gamma \to P(\gamma)$, where $P(\gamma)$ is the permutation matrix associated with $\gamma \in \Gamma$ corresponding to the action of Γ on Ω, is a representation of Γ. It is called the *permutation representation*. Let $\rho_1 = 1, \rho_2, \ldots, \rho_\ell$ be the irreducible representations of Γ and let f_i be the degree of ρ_i for each $1 \leqslant i \leqslant \ell$, where $f_1 = 1$ and $\sum_{i=1}^{\ell} f_i^2 = |\Gamma|$. It is well-known [**6**] that the permutation representation P can be decomposed as the direct sum of irreducible representations. In other words, there exists an invertible matrix M such that

$$M^{-1} P(\gamma) M = \bigoplus_{i=1}^{\ell} (\rho_i(\gamma) \otimes I_{m_i}) \tag{2}$$

for any $\gamma \in \Gamma$, where $m_i \geqslant 0$ is the multiplicity of the irreducible representation ρ_i in the permutation representation P and $\sum_{i=1}^{\ell} m_i f_i = n$. Notice that m_1 is the number of orbits under the action of the group Γ on Ω. So $m_1 \geqslant 1$.

Now let $\phi \in C^1(G,n)$ and $\Gamma = \langle \phi(e) \mid e \in D(G) \rangle$, the subgroup generated by the voltages $\phi(e)$. Noting that $\sum_{i=1}^{\ell} m_i f_i = n$, from equations (1) and (2) we have

$$(I_m \otimes M)^{-1} (\lambda I_{mn} - A(G^\phi))(I_m \otimes M) = \bigoplus_{i=1}^{\ell} \left[\left(\lambda I_{mf_i} - \sum_{\gamma \in \Gamma} A(\vec{G}_{(\phi,\gamma)}) \otimes \rho_i(\gamma) \right) \otimes I_{m_i} \right].$$

Since $\rho_1(\gamma) = 1$ for any $\gamma \in \Gamma$ and $A(G) = \sum_{\gamma \in \Gamma} A(\vec{G}_{(\phi,\gamma)})$, we get

$$\Phi(G^\phi; \lambda) = \Phi(G; \lambda)^{m_1} \prod_{i=2}^{\ell} \left[\det\left(\lambda I_{mf_i} - \sum_{\gamma \in \Gamma} A(\vec{G}_{(\phi,\gamma)}) \otimes \rho_i(\gamma) \right) \right]^{m_i}.$$

Summarising our discussions, we have the following theorem.

MAIN THEOREM. *Let G be a graph with m vertices, $\phi \in C^1(G,n)$ a voltage assignment on G and $\Gamma = \langle \phi(e) \mid e \in D(G) \rangle$. Let $\rho_1 = 1, \rho_2, \dots, \rho_\ell$ be the irreducible representations of Γ and let f_i be the degree of ρ_i for each $1 \leqslant i \leqslant \ell$ with $f_1 = 1$. Then the characteristic polynomial of the n-fold covering G^ϕ of G derived from the voltage assignment ϕ is*

$$\Phi(G^\phi; \lambda) = \Phi(G; \lambda)^{m_1} \prod_{i=2}^{\ell} \left[\det\Big(\lambda I_{mf_i} - \sum_{\gamma \in \Gamma} A(\vec{G}_{(\phi,\gamma)}) \otimes \rho_i(\gamma) \Big) \right]^{m_i},$$

where m_i is the multiplicity of ρ_i in the permutation representation P of Γ.

Since $m_1 \geqslant 1$, it gives that for every covering graph G^ϕ of the graph G, the characteristic polynomial $\Phi(G; \lambda)$ is a divisor of the characteristic polynomial $\Phi(G^\phi; \lambda)$, in [**1**, Corollary 1]. When Γ is a regular subgroup of S_n, the permutation representation P of Γ is equivalent to the (right) regular representation and the covering G^ϕ is a regular covering of G. In this case, each multiplicity m_i is equal to f_i, the degree of the irreducible representation ρ_i. Therefore, Mizuno and Sato's [**5**, Theorem 2] can be derived from the main theorem. Furthermore, When Γ is Abelian or Γ is the dihedral group of order $2n$, the same results as in [**3**] and in [**4**] can also be deduced.

We close this note by giving a computational example which could not be done by using any formula that was known before. Let G be any graph, $\phi \in C^1(G,4)$ a voltage assignment on G and $\Gamma = \langle \phi(e) \mid e \in D(G) \rangle = S_4$. Note that the symmetric group S_4 can be generated by (12) and (1234). Then, the permutation representation P of S_4 can be decomposed by $P = \rho_1 \oplus \rho_2$, where $\rho_1 = 1$, the trivial representation, and ρ_2 is defined on the generators of Γ by

$$\rho_2\big((12)\big) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \rho_2\big((1234)\big) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Therefore, the characteristic polynomial of the 4-fold covering G^ϕ of G derived from the voltage assignment ϕ is

$$\Phi(G^\phi; \lambda) = \Phi(G; \lambda) \det\Big(\lambda I_{3|V(G)|} - \sum_{\gamma \in S_4} A(\vec{G}_{(\phi,\gamma)}) \otimes \rho_2(\gamma) \Big). \tag{3}$$

For example, for the diamond graph G which is the complete graph K_4 minus an edge, one can see that

$$\Phi(G, \lambda) = \lambda(\lambda + 1)(\lambda^2 - \lambda - 4).$$

Consider a voltage assignment ϕ which is defined as in Figure 1.

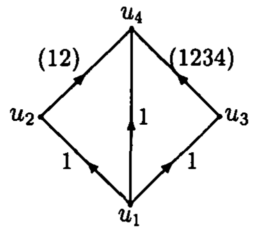

Figure 1: An S_4-voltage assignment ϕ on the diamond graph

From equation (3), one can get the characteristic polynomial of the graph G^ϕ as

$$\Phi(G^\phi;\lambda) = \Phi(G;\lambda)\lambda^2(\lambda^{10} - 12\lambda^8 + 2\lambda^7 + 51\lambda^6 - 22\lambda^5 - 87\lambda^4 + 66\lambda^3 + 39\lambda^2 - 54\lambda + 12).$$

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