III

Sussmann's orbits and unique continuation

In this chapter we will present various results on unique continuation for solutions and approximate solutions of locally integrable structures. Our main focus will be on those results where Sussmann's orbits have played a decisive role. We will begin with some general discussion of these orbits, taken mainly from [**Su**] and [**BM**].

III.1 Sussmann's orbits

Let $\mathcal{M}$ be a C^∞, paracompact manifold. Let D be a set of locally defined, smooth real vector fields. That is, each X in D is defined on some open subset of $\mathcal{M}$ and it is smooth there. Assume that the union of the domains of the elements of D equals $\mathcal{M}$. We define an equivalence relation on $\mathcal{M}$ as follows: two points p and q are related if there is a curve $\gamma : [0, T] \longrightarrow \mathcal{M}$ such that

(1) $\gamma(0) = p$, $\gamma(T) = q$;
(2) there exist $t_0 = 0 < t_1 < \cdots < t_n = T$ and vector fields $X_i \in D$ ($i = 1, \ldots, n$) such that for each i, the restriction $\gamma : [t_{i-1}, t_i] \longrightarrow \mathcal{M}$ is an integral curve of X_i or $-X_i$.

The equivalence classes of this relation will be called the *orbits* of D. In [**Su**], Sussmann showed that these orbits can be equipped with a natural topology and differentiable structure which makes them immersed submanifolds of $\mathcal{M}$. We will next briefly describe the orbit topology and C^∞ structure (the reader is referred to [**Su**] and [**BER**] for more details). If $X \in D$ is defined near p in $\mathcal{M}$, let $\Phi_t^X(p)$ denote the integral curve of X which at $t = 0$ equals p and is defined on a maximal interval. If $Y = (X_1, \ldots, X_m) \in D^m$ (i.e., each $X_i \in D$),

$s = (t_1, \ldots, t_m) \in \mathbb{R}^m$, and $p \in \mathcal{M}$, we write

$$\Phi_s^Y(p) = \Phi_{t_1}^{X_1}(\Phi_{t_2}^{X_2}(\ldots \Phi_{t_m}^{X_m}(p)\ldots)),$$

and let $\Omega(Y)$ denote the open subset of $\mathbb{R}^m \times \mathcal{M}$ consisting of the points (s, p) where $\Phi_s^Y(p)$ is defined. For $p \in \mathcal{M}$ and $Y \in D^n$, let $\Phi^Y(p)$ denote the map $s \longmapsto \Phi_s^Y(p)$, and let $\Omega(Y, p)$ be its domain. Note that $\Omega(Y, p)$ is a subset of $\mathbb{R}^n$. Suppose that $\mathcal{L} = \mathcal{L}_x$ is an orbit of D through a point x. Observe that $\mathcal{L}$ is the union of the sets $\Phi^Y(x)(\Omega(Y, x))$, where $Y \in D^n$ for $n = 1, 2, \ldots$ The orbit $\mathcal{L}$ is topologized by giving it the strongest topology that makes all the $\Phi^Y(x)$ continuous (for all n, and for all $Y \in D^n$). Note that since each $\Phi^Y(x) : \Omega(Y, x) \longrightarrow \mathcal{M}$ is continuous, it follows that the topology of $\mathcal{L}$ is finer than the subspace topology. Equivalently, the inclusion map from $\mathcal{L}$ into $\mathcal{M}$ is continuous. As the examples below will show, in general, this inclusion won't be a homeomorphism. For the independence of the topology of $\mathcal{L}$ on the point x, we refer the reader to [**Su**, page 176]. We will briefly recall the differentiable structure on $\mathcal{L}$ by describing the coordinate charts. Let $\Gamma(D)$ be the smallest set of locally defined C^∞ vector fields on $\mathcal{M}$ satisfying:

(1) $D \subseteq \Gamma(D)$, and
(2) for any $p \in \mathcal{M}$, $\{X_p : X \in \Gamma(D)\}$ is a subspace of $T_p\mathcal{M}$.

We will use $\widehat{\Gamma(D)}$ to denote the smallest set of locally defined, smooth vector fields which contains $\Gamma(D)$ and is invariant under the group of local diffeomorphisms generated by $\Gamma(D)$. It is not hard to see that the dimension of the fibers $\widehat{\Gamma(D)}_x$ is constant as x varies in the orbit $\mathcal{L}$. Suppose now $q \in \mathcal{L}$. By lemmas 5.1 and 5.2 in [**Su**], there exist $Y \in D^n$ for some n, $q' \in \mathcal{L}$ and $s \in \Omega(Y, q')$ such that

$$\Phi^Y(q')(s) = q,$$

and the rank k of the differential of

$$\Phi^Y(q') : \Omega(Y, q') \longrightarrow \mathcal{M}$$

at the point s is maximal, and that in fact, this rank equals $\dim \widehat{\Gamma(D)}_x$ for any $x \in \mathcal{L}$. By the rank theorem, we can find neighborhoods U of s in $\mathbb{R}^n$, V of q in $\mathcal{M}$, diffeomorphisms F from U onto C^n, G from V onto C^N ($N =$ dimension of $\mathcal{M}$) such that

$$G \circ \Phi^Y(q') \circ F^{-1}(x_1, \ldots, x_n) = (x_1, \ldots, x_k, 0, \ldots, 0).$$

Here C^l denotes the cube

$$\{(x_1, \ldots, x_l) \in \mathbb{R}^l : |x_i| < 1 \quad \forall i\}.$$

Let $\Lambda = \Phi^Y(q')(U)$. Λ is an open subset of $\mathcal{L}$ (see [**Su**]). Moreover, Λ is a submanifold of $\mathcal{M}$ since

$$G(\Lambda) = \{(x_1, \ldots, x_k, 0, \ldots, 0)\}.$$

The differentiable structure on the orbit $\mathcal{L}$ is defined by taking the pairs $\{(\Lambda, G|_\Lambda)\}$ as charts. One of the main results proved by Sussmann may be stated as follows:

Theorem III.1.1 (Theorem 4.1 in [**Su**]). *Let $\mathcal{M}$ be a C^∞ manifold, and let D be a set of locally defined, smooth vector fields such that the union of the domains of the elements of D is $\mathcal{M}$. Then*

(1) *If $\mathcal{L}$ is an orbit of D then $\mathcal{L}$ (with the topology described above) admits a unique differentiable structure such that $\mathcal{L}$ is a submanifold of $\mathcal{M}$.*
(2) *With the topology and differentiable structure as above, every orbit of D is a maximal integral submanifold of $\widehat{\Gamma(D)}$.*

We will next present several examples.

Example III.1.2. Let $\mathcal{M}$ be a manifold and suppose P is a sub-bundle of the tangent bundle $T\mathcal{M}$ of dimension k. That is, for each $x \in \mathcal{M}$, the fiber P_x is a k-dimensional subspace of $T_x\mathcal{M}$, and for each $y \in \mathcal{M}$, there exists a neighborhood U of y and smooth vector fields $X^1, X^2, \ldots, X^k$ on U such that $\{X^j(x) : 1 \le j \le k\}$ is a basis of P_x for each $x \in U$. We assume that P is closed under Lie brackets. Then by the Frobenius theorem, the manifold $\mathcal{M}$ is foliated by leaves each of which is an integral manifold of P. If we set D to be equal to the set of smooth local sections of P, then these leaves are precisely the orbits of D. Note that in this example, the orbits have the same dimension. Thus the concept of Sussmann's orbits may be viewed as a generalization of Frobenius foliations. For a concrete example of this kind, consider the 2-torus $\mathbb{T}^2 = S^1 \times S^1$. Use the angles (θ_1, θ_2) as coordinates for points in $\mathbb{T}^2$, so θ_1 and θ_2 are determined modulo integral multiples of 2π. Pick two real numbers a and b, not both equal to zero, and consider the sub-bundle of the tangent bundle of $\mathbb{T}^2$ generated by the vector field

$$L = a\frac{\partial}{\partial\theta_1} + b\frac{\partial}{\partial\theta_2}.$$

The orbits are the integral curves of L. If a and b are linearly dependent over the rational numbers, then each orbit is diffeomorphic to S^1. In this case, an orbit is an embedded submanifold of $\mathbb{T}^2$ and so its orbit topology agrees with the induced subspace topology. If a and b are linearly independent over the rational numbers, each orbit is diffeomorphic to the real line. In this case the orbits are dense in $\mathbb{T}^2$, and hence are not embedded submanifolds.

Example III.1.3. Let $\mathcal{M}_1 = \mathbb{R}^2$ and $\mathcal{M}_2 = \{x \in \mathcal{M}_1 : \|x\| < 1\}$. Let $g(t) \in C^\infty(\mathbb{R})$, $g > 0$ on $(1, 2)$ and $g \equiv 0$ outside $(1, 2)$. Let $D = \left\{\frac{\partial}{\partial x_1}, g(x_1)\frac{\partial}{\partial x_2}\right\}$. $\mathcal{M}_1$ is the only orbit for D. However, if we consider D on $\mathcal{M}_2$, the orbits are the horizontal segments in $\mathcal{M}_2$. Notice also that the tangent space of the orbit $\mathcal{M}_1$ at points (x_1, x_2) with $x_1 \notin (1, 2)$ does not coincide with the fiber of the Lie algebra generated by $\dfrac{\partial}{\partial x_1}$ and $g(x_1)\dfrac{\partial}{\partial x_2}$.

Example III.1.4. Consider the orbits of $\left\{\frac{\partial}{\partial x_2}, x_1\frac{\partial}{\partial x_1}\right\}$ in $\mathbb{R}^2$. There are three orbits: $\{x_1 > 0\}$, $\{x_1 < 0\}$, and $\{x_1 = 0\}$. Thus the dimension of orbits is not locally constant. In general, if $d(x) =$ the dimension of the orbit through x, then $d(x)$ is a lower semicontinuous function.

Example III.1.5. The analytic case: suppose $\mathcal{M}$ is a real-analytic manifold and D is a set of real-analytic vector fields on $\mathcal{M}$. Let D^* be the smallest Lie algebra (under brackets) of real-analytic vector fields that contains D. It is well known (see [**Su**], for example) that if $p \in \mathcal{M}$, then there are a finite number of elements $X_1, \ldots, X_k$ of D^* such that every $X \in D^*$ can be expressed in a neighborhood of p as

$$\sum_{j=1}^{k} f_j X_j$$

for some real-analytic functions f_j. Moreover, in this case, if $\mathcal{L}$ is an orbit of D and $p \in \mathcal{L}$, then its tangent space

$$T_p\mathcal{L} = D_p^* \quad \text{where} \quad D_p^* = \{X(p) : X \in D^*\}.$$

This makes it easier to compute the dimensions of orbits in the analytic case. The concept of orbits in the analytic case dates back to Nagano's paper ([**Na**]). Orbits arise in a locally integrable structure $(\mathcal{M}, \mathcal{V})$ by taking D as the collection of the real parts of smooth, local sections of $\mathcal{V}$. Below we will give an example of orbits arising from the CR structure of a hypersurface in $\mathbb{C}^2$. More examples will be given in the rest of the sections.

Example III.1.6. Let $z = x + iy$, $w = s + it$ denote the variables in $\mathbb{C}^2$ and suppose $g = g(x, y)$ is a real-valued, real-analytic function defined on the plane such that

(1) $g(0,0) = 0$, $g(x,y) > 0$ for $(x,y) \neq (0,0)$; and

(2) $\Delta g < 2\dfrac{|\nabla g|^2}{g}$.

Define

$$\rho(z,w) = s^2 + (t - g(x,y))^2 - g(x,y)^2$$

and let

$$\mathcal{M} = \{(z,w) \in \mathbb{C}^2 \setminus \{0\} : \rho(z,w) = 0\}.$$

Notice that since $\mathrm{d}\rho \neq 0$ on $\mathcal{M}$, $\mathcal{M}$ is a real-analytic hypersurface. We consider the orbits arising from the CR structure of $\mathcal{M}$. Observe first that the complex line $\Sigma = \mathbb{C}\setminus\{0\} \times \{0\} \subset \mathcal{M}$. Since the bundle $\mathcal{V}$ is tangent to Σ, the bracket $[X, Y]$ of any two smooth sections X and Y of $\Re\mathcal{V}$ is also tangent to Σ. Hence by the remarks in Example III.1.5, Σ is an orbit. We will next show that $\mathcal{M}\setminus\Sigma$ is strictly pseudo-convex. For any $a = (a_1, a_2) \in \mathbb{C}^2$, we have

$$\langle \partial\bar{\partial}\rho a, a\rangle = i[(w - \bar{w})g_{z\bar{z}}|a_1|^2 - g_z a_1 \bar{a}_2 + g_{\bar{z}} a_2 \bar{a}_1 + i|a_2|^2].$$

On the manifold $\mathcal{M}$, $|\bar{w} + ig|^2 = g^2 \neq 0$ and so if for $a = (a_1, a_2)$, $\langle \partial\rho, a\rangle = 0$ at a point of $\mathcal{M}$, then

$$a_2 = \left(\frac{i(\bar{w} - w)g_z}{\bar{w} + ig}\right) a_1.$$

It follows that if $\langle \partial\rho, a\rangle = 0$, then

$$\langle \partial\bar{\partial}\rho a, a\rangle = i|a_1|^2(w - \bar{w})\left(g_{z\bar{z}} - \frac{2|g_z|^2}{g}\right).$$

The latter, together with the assumptions on g, show that $\mathcal{M}\setminus\Sigma$ is strictly pseudo-convex. Thus $\mathcal{M}$ has one orbit of dimension 2, and all other orbits are of dimension 3. If we make a further assumption on g, say for example, $g(z,\bar{z}) = g(|z|)$, then $\mathcal{M}\setminus\Sigma$ is connected, and hence a single open orbit. When $g(x,y) = x^2 + y^2$, this example appeared in [**BM**]. Our next objective is to analyze the extent to which orbits behave like embedded submanifolds. We begin with:

LEMMA III.1.7. *Let $\mathcal{L}$ be an orbit through p_0 of dimension k, dimension $\mathcal{M} = n$. Then there exists a local chart $(T \times V, \psi)$ on $\mathcal{M}$ about p_0 with T and V neighborhoods of* 0 *in $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$ respectively, such that*

$$\mathcal{L} \cap \psi(T \times V) = \psi(T \times P_{\mathcal{L}}), \ \textit{where}$$

$$P_{\mathcal{L}} = \{v \in V : \psi(0, v) \in \mathcal{L}\}.$$

PROOF. Let S be a submanifold of $\mathcal{M}$ through p_0 of dimension $n-k$ such that

$$T_{p_0}\mathcal{M} = T_{p_0}S + T_{p_0}\mathcal{L},$$

where we view $T_{p_0}\mathcal{L}$ as a subspace of $T_{p_0}\mathcal{M}$. Let $X_1, \dots, X_k$ be locally defined vector fields in $\widehat{\Gamma}$ spanning $T_{p_0}\mathcal{L}$ at p_0. After contracting S about p_0 if necessary, we can find a neighborhood T of 0 in $\mathbb{R}^k$ and a neighborhood U of p_0 in $\mathcal{M}$ such that the map

$$F : T \times S \longrightarrow U$$

given by

$$F(t_1, \dots, t_k; p) = \Phi_{t_1}^{X_1}(\Phi_{t_2}^{X_2}(\dots \Phi_{t_k}^{X_k}(p)\dots))$$

is a diffeomorphism. Suppose now that $q \in \mathcal{L} \cap U$. Then $q = F(t, s)$ for a unique $(t, s) \in T \times S$. Hence,

$$F(T \times \{s\}) \subset \mathcal{L} \cap U.$$

Therefore, $\mathcal{L} \cap U = F(T \times P_{\mathcal{L}})$ where $P_{\mathcal{L}} = \mathcal{L} \cap S$. After introducing a chart on S about p, we get the lemma. □

Observe that if an orbit $\mathcal{L}$ is an embedded submanifold, then the sets T and V in Lemma III.1.7 can be chosen so that $P_{\mathcal{L}}$ is a single point. For a general orbit, we will next show that $P_{\mathcal{L}}$ can be chosen to be a countable set. This will follow from:

LEMMA III.1.8. *The topology on an orbit $\mathcal{L}$ is second countable.*

PROOF. For $p \in \mathcal{L}$ we will consider the charts $(T \times V, \psi)$ of Lemma III.1.7. The discussion on the differentiable structure of $\mathcal{L}$ shows that $\psi(T \times V)$ is an open set in $\mathcal{L}$. Since $\mathcal{M}$ is second countable, the subspace topology on $\mathcal{L}$ is second countable. Hence we can get a locally finite open cover for $\mathcal{L}$ of the form

$$\{U_j = \psi(T_j \times V_j)\}_{j=1}^{\infty}.$$

Recall that for each j, $\mathcal{L} \cap \psi(T_j \times V_j) = \psi(T_j \times P_j)$ where $P_j = \{v \in V_j : \psi(0, v) \in \mathcal{L}\}$. If $q \in P_j$, we will call the set $\psi(T \times \{q\})$ a slice of $\mathcal{L}$ in U_j. Fix $p_0 \in U_{j_0} \cap \mathcal{L}$ for some j_0, and hence $p_0 \in \psi(T_{j_0} \times \{p'\}) \subseteq \mathcal{L}$ for some $p' \in V_{j_0}$. For every finite tuple $i = (i_1, \dots, i_m)$, let A_i be the set of points x in $\mathcal{L}$ such that x can be joined to p_0 by a curve γ consisting of m pieces γ_l where each γ_l lies in U_{i_l}, $l = 1, \dots, m$. From the definition, it is clear that each A_i is a union of slices in U_{i_m}. The family $\{A_i\}$ where i varies over all finite tuples of positive integers is a countable collection of open subsets of $\mathcal{L}$ which form a basis for the topology of $\mathcal{L}$. Hence we only need to show that each A_i consists of a countable number of slices in U_{i_m}. We will do this by induction on m. When $m = 1$, A_{i_1} contains at most one slice. Suppose the result holds for all tuples $j = (i_1, \ldots, i_{k-1})$ of length $k-1$. Then A_j is the union of countably many slices in $U_{i_{k-1}}$. Fix a slice Σ in A_j. Since slices are open sets in $\mathcal{L}$, the intersection of Σ with each slice in U_{i_k} is an open set. Moreover, since the slices in U_{i_k} are pairwise disjoint, and Σ is homeomorphic to an open set in $\mathbb{R}^d$, it follows that Σ can intersect only a countable number of slices in U_{i_k}. Thus each A_i is the union of a countable number of slices and therefore $\mathcal{L}$ is second countable. $\square$

The preceding lemma can be used to show that orbits possess properties not shared by a general immersed submanifold. To see one such property, call an immersed submanifold N of a manifold $\mathcal{M}$ *weakly embedded* if whenever A is a manifold and $f: A \longrightarrow \mathcal{M}$ is smooth with $f(A) \subseteq N$ then $f: A \longrightarrow N$ is smooth. This notion was introduced by Pradines in [**Pr**]. For an example of an immersed submanifold that is not weakly embedded, see remark 6.8 in [**Boo**].

Proposition III.1.9. *An orbit $\mathcal{L}$ in a manifold $\mathcal{M}$ is weakly embedded.*

Proof. Suppose $f: A \longrightarrow \mathcal{M}$ is C^∞ and $f(A) \subseteq \mathcal{L}$. Let $q \in A$ and $p = f(q)$. Let $\dim \mathcal{L} = k$, $\dim \mathcal{M} = n$, and suppose $(T \times V, \psi)$ is a chart on $\mathcal{M}$ about p as in Lemma III.1.7 with T and V cubes centered about 0 in $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$. Since $f: A \longrightarrow \mathcal{M}$ is C^∞, we can choose a connected neighborhood W of q such that

$$f(W) \subseteq \psi(T \times V).$$

Recall from Lemma III.1.7 and Lemma III.1.8 that

$$\mathcal{L} \cap \psi(T \times V) = \bigcup_{v \in P} \psi(T \times \{v\}),$$

where $P \subseteq V$ is a countable set. The map $\psi^{-1} \circ f: W \longrightarrow T \times V$ is C^∞ and $\psi^{-1} \circ f(W) \subseteq \bigcup_{v \in P} T \times \{v\}$. Since W is connected, there exists a unique $v \in P$ such that $\psi^{-1} \circ f(W) \subseteq T \times \{v\}$. Hence $f: W \longrightarrow \psi(T \times \{v\}) \subseteq \mathcal{L}$ is C^∞. $\square$

Corollary III.1.10. *If $\mathcal{L}$ is an orbit of $\mathcal{M}$, then when topologized with its orbit topology, it has a unique differentiable structure that makes it an immersed submanifold of $\mathcal{M}$.*

Another property of orbits not shared by a general immersed submanifold concerns the propagation of embeddedness. More precisely, we have

Proposition III.1.11. *Let $\mathcal{L}$ be an orbit in $\mathcal{M}$ and suppose for a point p in $\mathcal{L}$, there is a neighborhood W in $\mathcal{M}$ such that $W \cap \mathcal{L}$ is an embedded submanifold of W. Then $\mathcal{L}$ is an embedded submanifold of $\mathcal{M}$.*

Proof. Let $q \in \mathcal{L}$ and assume $q = \Phi_t^X(p)$ for some $X \in \Gamma$ and $t \in \mathbb{R}$. Set $W' = \Phi_t^X(W)$. Here we may assume W has been contracted enough to lie in the domain of the flow of Φ_t^X. Since $\Phi_t^X : W \longrightarrow W'$ is a diffeomorphism, the submanifold $\Phi_t^X(W \cap \mathcal{L})$ is an embedded submanifold of W'. It is also easy to see that

$$\Phi_t^X(W \cap \mathcal{L}) = W' \cap \mathcal{L}.$$

Hence $\mathcal{L}$ is an embedded submanifold of $\mathcal{M}$. □

Corollary III.1.12. *If an orbit $\mathcal{L}$ is a closed subset of $\mathcal{M}$, then it is an embedded submanifold.*

Proof. Let $p \in \mathcal{L}$. Choose a chart $(T \times V, \psi)$ about p as in Lemma III.1.7. If such a chart can be selected so that $P_{\mathcal{L}}$ is a finite subset of V, then by Proposition III.1.11, $\mathcal{L}$ is embedded. Otherwise, such a selection is not possible for any point in $\mathcal{L}$. In particular, this means that for any $v \in P_{\mathcal{L}}$, the point $\psi(0, v)$ is an accumulation point of the set $\psi(0 \times P_{\mathcal{L}})$. Hence $v \in P_{\mathcal{L}}$ is an accumulation point of $P_{\mathcal{L}}$. Moreover, since $\mathcal{L}$ is closed, $P_{\mathcal{L}}$ is a closed subset of V. It follows that $P_{\mathcal{L}}$ is a perfect set and hence it is uncountable. This gives rise to the pairwise disjoint, uncountable family of open subsets $\{\psi(T \times \{v\}) : v \in P_{\mathcal{L}}\}$ of $\mathcal{L}$, contradicting the second countability of $\mathcal{L}$. □

III.2 Propagation of support and global unique continuation

This section discusses the relevance of orbits to a variety of global questions of unique continuation in involutive structures. Suppose $\mathcal{V}$ is an involutive structure on $\mathcal{M}$ for which uniqueness for solutions in the (noncharacteristic) Cauchy problem holds, i.e., every solution defined in a neighborhood of a noncharacteristic (with respect to $\mathcal{V}$) hypersurface Σ and whose trace on Σ is zero vanishes in a neighborhood of Σ. The uniqueness results of Chapter II show that an example of such a $\mathcal{V}$ is provided by a locally integrable structure. Our first goal is to present another proof of Corollary II.4.7, which is a result on the propagation of the support of a solution along orbits. Special cases of this theorem were proved by several authors (see the notes). The result stated here is due to Treves ([**T4**]), but the proof is taken from [**BM**].

THEOREM III.2.1. *Assume that $\mathcal{V}$ is an involutive structure for which uniqueness in the Cauchy problem holds. If u is a solution, then the support of u is a union of orbits.*

Before we provide the proof, we will recall some definitions and results from a paper of Bony ([**Bo**]).

DEFINITION III.2.2. *Let Ω be an open subset of $\mathbb{R}^n$ and F a closed subset of Ω. A vector v is said to be normal to F at $x_0 \in F$ if there is an open ball $B \subseteq \Omega \backslash F$ centered at x such that $x_0 \in \partial B$ and $v = \lambda(x - x_0)$ for some $\lambda > 0$.*

REMARK III.2.3. By considering cones of varying apertures, it is easy to see that a closed set may have no normals or many normals at a boundary point.

DEFINITION III.2.4. *Suppose Ω is open in $\mathbb{R}^n$ and $F \subseteq \Omega$ is closed. A vector field $X(x)$ is tangent to F if whenever v is normal to x_0 in F, the vector $X(x_0)$ is orthogonal to v.*

In [**Bo**], Bony proved the following:

THEOREM III.2.5. *Suppose Ω is open in $\mathbb{R}^n$ and F a closed subset of Ω. Let $X(x)$ be a Lipschitz vector field in Ω which is tangent to F. If an integral curve of X intersects F at a point, then it is entirely contained in F.*

PROOF OF THEOREM III.2.1. Let π denote the projection map from $T^*\mathcal{M}$ onto $\mathcal{M}$. Suppose u is a solution on $\mathcal{M}$ and F denotes the support of u. Let $\Omega = \mathcal{M} \backslash F$. Define $N(F)$ to be the set of $\nu \in T^*\mathcal{M} \backslash \{0\}$ over points in F such that there exists f real-valued, smooth, defined near $p = \pi(\nu)$ and such that $f(p) = 0$, $\mathrm{d}f(p) = \nu$ and $f \leq 0$ on F near p. Fix $p \in F$ and suppose $\nu \in N(F)$ with $\pi(\nu) = p$. Suppose we show that for any $X = \Re L$ (for some smooth section L of $\mathcal{V}$) defined near p, $\langle \nu, X \rangle = 0$. Then by Bony's theorem, the integral curve of X through p will lie in F, thus proving the theorem. Let f be chosen as above with $\mathrm{d}f(p) = \nu$. Note that near p, the zero set of f is a smooth hypersurface, and $u \equiv 0$ on a side of this hypersurface. Since $p \in F =$ supp u, by the uniqueness in the Cauchy problem, $\mathcal{V}$ has to be characteristic to $\{f = 0\}$ at p. Hence, $\langle \nu, X \rangle = 0$. □

We note that if $\mathcal{V}$ is an involutive structure for which uniqueness in the Cauchy problem is not valid, then the support of a solution may not be a union of orbits, as demonstrated by Cohen's celebrated example ([**Co**]).

DEFINITION III.2.6. *A formally integrable structure $(\mathcal{M}, \mathcal{V})$ satisfies the global unique continuation property if every solution that vanishes on an open subset vanishes everywhere on $\mathcal{M}$.*

According to Theorem III.2.1, global unique continuation holds in a locally integrable structure $(\mathcal{M}, \mathcal{V})$ whenever $\mathcal{M}$ is a single orbit for $\mathcal{V}$. However, global unique continuation may hold even when $\mathcal{M}$ is not a single orbit, as shown by the structure on the 2-torus generated by a real vector field each of whose integral curves is dense. The obstruction to the validity of global unique continuation is the presence of proper, closed subsets of $\mathcal{M}$ which are unions of orbits, since by Theorem III.2.1, such sets can potentially be the supports of solutions. We will refer to sets that are unions of orbits as *invariant sets*. In order to check the validity of global unique continuation, one needs to understand when a given proper, closed, invariant set equals the support of a solution. It turns out that in a general locally integrable structure, a proper, closed orbit may not be the support of a solution. This is illustrated by examples below. Some sufficient conditions for the existence of a solution supported on a proper, closed orbit were studied in the work [**BM**]. In particular, the following theorem was proved (see also Theorem III.2.12 below):

Theorem III.2.7 (Theorem 5.8 in [**BM**].). *Suppose $\mathcal{M}$ is an orientable, connected analytic hypersurface in $\mathbb{C}^n$. If $\mathcal{M}$ is not Levi flat and has a codimension one orbit $\mathcal{L}$, then there is a solution supported on $\mathcal{L}$. Thus, on an analytic, non-Levi flat hypersurface in $\mathbb{C}^n$, the global unique continuation property holds if and only if there is only one orbit.*

Example III.2.8. We consider real-analytic vector fields L in the plane that are rotation-invariant. That is, if $\mathcal{V}$ is the bundle generated by L, then

$$\mathrm{d}R_\alpha(\mathcal{V}) = \mathcal{V}$$

for every rotation R_α (with angle α) of $\mathbb{R}^2$. In polar coordinates, such an L takes the form (see [**BMe**])

$$L = g(r,\theta)\left(rY(r)\frac{\partial}{\partial r} + iX(r)\frac{\partial}{\partial \theta}\right)$$

where g, X, Y are real-analytic functions, $\dfrac{X}{Y}$ is even in r away from the zeros of Y and we may assume that $X(0) = Y(0) = 1$. The characteristic set $\Sigma = \{\Re(X(r)\overline{Y}(r)) = 0\}$ is a union of circles centered at 0 and $0 \notin \Sigma$. Assume $\Sigma = \{r = 1\}$. If L is of finite type at a point p in Σ, then it is of the same type at every point p in Σ and in this case, $\mathcal{V}$ has only one orbit. Suppose now L is of ∞ type at some and hence every point of Σ. Then (see [**BMe**]) it can be shown that $\mathcal{V}$ is generated by

$$L = \frac{\partial}{\partial \theta} - \sqrt{-1}\,rY(r)\frac{\partial}{\partial r},$$

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where $Y(r) = (1-r^2)^N h(r)$, h is real-analytic, $h(r) \neq 0$, and $h(0) \in \{\pm 1\}$. Without loss of generality, assume $h(0) = 1$. Then, $\mathcal{V}$ has three orbits: $\{r < 1\}$, $\{r > 1\}$, and $\Sigma = \{r = 1\}$. We consider next whether Σ can be the support of a distribution solution. When $N \geq 2$, the distribution

$$\langle u, \psi(r,\theta)\rangle = \int_0^{2\pi} \psi(1,\theta)\,\mathrm{d}\theta$$

is a solution supported on Σ. Assume $N = 1$. In this case, such a u exists if and only if $h(1)$ is a rational number ([**BMe**]).

Indeed, suppose $Lu = 0$ and u is supported on Σ. Then there exist an integer $k \geq 0$ and $a_j(\theta) \in \mathcal{D}'(S^1) \quad (0 \leq j \leq k)$ such that

$$\langle u, \psi(r,\theta)\rangle = \sum_{m=0}^{k} \int_0^{2\pi} a_m(\theta) \left(\frac{\partial}{\partial r}\right)^m \psi(1,\theta)\,\mathrm{d}\theta.$$

Since L is in the tangential direction on Σ, each $a_j(\theta) \in C^\infty(\Sigma)$. Let $\psi_{j,n}(r,\theta) = f_j(r)\,\mathrm{e}^{in\theta}$, where $f_j(r)$ is C^∞ and $f_j^{(l)}(1) = \delta_{jl}$ for $0 \leq j \leq k$. Note that the transpose of L is given by

$${}^t Lw = -\frac{\partial w}{\partial \theta} + irY(r)\frac{\partial w}{\partial r} + i(2Y(r) + rY'(r))w$$

and so

$${}^t L\psi_{k,n} = i\mathrm{e}^{in\theta}[rY(r)f_k'(r) + (2Y(r) + rY'(r) - n)f_k].$$

Moreover,

$$\left(\frac{\partial}{\partial r}\right)^m (rY(r)f_k'(r))|_{r=1} = \begin{cases} 0, & m < k \\ kY'(1), & m = k \end{cases}$$

and

$$\left(\frac{\partial}{\partial r}\right)^m [(2Y(r) + rY'(r) - n)f_k(r)]|_{r=1} = \begin{cases} 0, & m < k \\ Y'(1) - n, & m = k. \end{cases}$$

Thus, we get:

$$\begin{aligned} 0 &= \langle Lu, \psi_{k,n}\rangle \\ &= \langle u, {}^t L\psi_{k,n}\rangle \\ &= \left(\int_0^{2\pi} a_k(\theta)\,\mathrm{e}^{in\theta}\,\mathrm{d}\theta\right)[(k+1)Y'(1) - n]. \end{aligned} \tag{III.1}$$

Since we may assume that $a_k(\theta)$ does not vanish identically, there is an integer M for which

$$\int_0^{2\pi} a_k(\theta)\,\mathrm{e}^{iM\theta}\,\mathrm{d}\theta \neq 0. \tag{III.2}$$

From (III.1) and (III.2) it follows that

$$h(1) = \frac{-M}{2(k+1)} \in \mathbb{Q}, \text{ and}$$

$$a_k(\theta) = c\mathrm{e}^{-iM\theta}$$

for some $c \neq 0$. Conversely, suppose $k \geq 0$ and M are integers satisfying

$$2(k+1)h(1) = -M.$$

We will seek a solution u of the form

$$\langle u, \psi(r,\theta)\rangle = \sum_{m=0}^{k} \int_0^{2\pi} b_m(\theta) \left(\frac{\partial}{\partial r}\right)^m \psi(1,\theta)\,\mathrm{d}\theta.$$

Set $b_k(\theta) = \mathrm{e}^{-iM\theta}$. Each $b_j(\theta)$ can be determined from the equation $\langle Lu, \psi_{j,M}\rangle = 0$. To see this, note that $\langle Lu, \psi_{k-1,M}\rangle = 0$ is equivalent to

$$0 = 2\pi \left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^k [rYf'_{k-1} + (rY' + 2Y - M)f_{k-1}](1) + \left(\int_0^{2\pi} b_{k-1}(\theta)\,\mathrm{e}^{iM\theta}\mathrm{d}\theta\right)$$
$$\times \left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^{k-1} [rYf'_{k-1} + (rY' + 2Y - M)f_{k-1}](1).$$

The coefficient of $\int_0^{2\pi} b_{k-1}(\theta)\mathrm{e}^{iM\theta}\mathrm{d}\theta$ in the latter equation is $-2kh(1) - M = 2h(1) \neq 0$, and hence we can get a constant c_{k-1} such that if we set

$$b_{k-1}(\theta) = c_{k-1}\mathrm{e}^{-iM\theta}, \text{ then } \quad \langle Lu, \psi_{k-1,M}(r,\theta)\rangle = 0.$$

In general, we can determine $b_l(\theta)$ from $\langle Lu, \psi_{l,M}\rangle = 0$. This leads to $b_l(\theta) = c_l\mathrm{e}^{-iM\theta}$ for some constant c_l since

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^l [rYf'_l + (2Y + rY' - M)f_l](1) = 2(k-l)h(1) \neq 0.$$

Thus $\langle Lu, \psi_{j,m}\rangle = 0$ for all $m \in \mathbb{Z}$, and all $j = 0, \ldots, k$. Since Lu is a distribution of order k, it follows that $Lu = 0$.

Example III.2.9. (See [**BM**].) We denote the coordinates in $\mathbb{R}^3$ by (x, y, s) and we will write $\mathbb{R}^3 = \mathbb{R}_x \times \mathbb{R}_y \times \mathbb{R}_s$. Let $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ be a smooth, 2π-periodic function, $\phi \geq 0$ and ϕ not identically 0. Define

$$L = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} + \phi(x)\sin(s)\frac{\partial}{\partial s} = X + iY.$$

The coefficients of L are 2π-periodic and so L induces a vector field $\widetilde{L}$ on $\mathbb{T}^3 = S^1 \times S^1 \times S^1$. The involutive structure generated by $\widetilde{L}$ is a Levi flat, locally integrable CR structure. We will show the following:

(1) The orbits of $\widetilde{L}$ through $p_1 = (1, 1, 1)$ and $p_2 = (1, 1, -1)$ are compact but all other orbits are noncompact.
(2) Depending on the value of

$$\int_0^{2\pi} \phi(x)\mathrm{d}x,$$

there may not be any solution supported on either of the compact orbits.
(3) Global unique continuation is valid for continuous solutions.

Let $F : \mathbb{R}^3 \longrightarrow \mathbb{T}^3$ be given by

$$F(x, y, s) = (\mathrm{e}^{ix}, \mathrm{e}^{iy}, \mathrm{e}^{is}).$$

Consider the orbit $\mathcal{L}_1$ through the point $p_1 = (1, 1, 1)$. $F(0, 0, 0) = p_1$ and the orbit in $\mathbb{R}^3$ of $\{X, Y\}$ through $(0, 0, 0)$ is $\mathbb{R}_x \times \mathbb{R}_y \times \{0\}$. Therefore, $\mathcal{L}_1 = S^1 \times S^1 \times \{1\}$. Likewise, for the point $p_2 = (1, 1, -1)$, the orbit $\mathcal{L}_2 = S^1 \times S^1 \times \{-1\}$. Consider now a typical point $p = (1, 1, \mathrm{e}^{is_0})$ for some $0 < s_0 < \pi$. If $\gamma(t) = (x(t), s(t))$ is the integral curve of X with $\gamma(0) = (0, s_0)$, we will see that the orbit through p is given by

$$\mathcal{L} = \{(\mathrm{e}^{it}, \mathrm{e}^{iy}, \mathrm{e}^{is(t)}) : t, y \in \mathbb{R}\}.$$

Indeed, $x(t) = t$ and $s'(t) = \phi(t)\sin(s(t))$, $s(0) = s_0$. If for some t_0, $s(t_0) = \pi$, then the curves $\gamma(t)$ and $\gamma_1(t) = (t, \pi)$ will both be integral curves of X passing through (t_0, π) at $t = t_0$. This implies that $s(t) \equiv \pi$, contradicting the assumption that $s(0) = s_0 < \pi$. Likewise, $s(t)$ can never equal zero. Thus, $0 \leq s(t) \leq \pi$ and $s'(t) \geq 0$. Suppose

$$\lim_{t\to\infty} s(t) = a < \pi.$$

Then $s_0 \le s(t) \le a$ for all $t \ge 0$. Therefore, $s'(t) \ge c\phi(t)$ for some $c > 0$, which in turn leads to

$$\lim_{t\to\infty} s(t) = \infty.$$

Hence,

$$\lim_{t\to\infty} s(t) = \pi$$

and by a similar reasoning,

$$\lim_{t\to-\infty} s(t) = 0.$$

Thus the closure of $\mathcal{L} = \mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2$.

We consider now the question of existence of a solution supported on a compact orbit, say $\mathcal{L}_1$. Since L is tangent to $\mathcal{L}_1$ and defines a complex structure there, any distribution solution u supported on this orbit has the form

$$u(x, y) = \sum_{l=0}^{N} u_l(x, y)\sigma_l$$

where the u_l are C^∞ on $S^1 \times S^1$ and

$$\langle \sigma_k, g(x, y, s)\rangle = \int_0^{2\pi} \int_0^{2\pi} \partial_s^k g(x, y, 0)\, \mathrm{d}x\mathrm{d}y.$$

We have

$$\begin{aligned}\langle L\sigma_k, g(x, y, s)\rangle &= \langle \sigma_k, {}^tLg(x, y, s)\rangle \\ &= -\int_0^{2\pi} \int_0^{2\pi} \phi(x) \left(\frac{\partial}{\partial s}\right)^k \left[(\sin s)\frac{\partial g}{\partial s} + (\cos s)g\right](x, y, 0)\, \mathrm{d}x\mathrm{d}y \\ &= -\int_0^{2\pi} \int_0^{2\pi} \phi(x) \left(\frac{\partial}{\partial s}\right)^{k+1} [(\sin s)g](x, y, 0)\, \mathrm{d}x\mathrm{d}y \\ &= -\sum_{l=0}^{k+1} \int_0^{2\pi} \int_0^{2\pi} \binom{k+1}{l} \left(\frac{\partial}{\partial s}\right)^{k-l} \cos s \left(\frac{\partial}{\partial s}\right)^{l} g(x, y, 0)\, \mathrm{d}x\mathrm{d}y.\end{aligned}$$

Thus,

$$L\sigma_k = -(k+1)\phi(x)\sigma_k - \phi(x) \sum_{l=0}^{k-2} \binom{k+1}{l} \partial_s^{k-l} \cos s(0)\sigma_l.$$

Let $M_k = \dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y} - (k+1)\phi(x)$, for $k = 0, 1, 2, \ldots$ If $v(x, y) \in C^\infty(S^1 \times S^1)$, it follows that

$$L(v\sigma_k) = (M_k v)\sigma_k - v\phi \sum_{l=0}^{k-2} \binom{k+1}{l} \left(\partial_s^{k-l} \cos s\right)(0)\sigma_l.$$

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Suppose now

$$u = \sum_{k=0}^{N} u_k(x, y)\sigma_k$$

is a solution. Then

$$Lu = \sum_{l=0}^{N}(M_l u_l)\sigma_l - \phi \sum_{l=0}^{N-2} \sum_{k=l+2}^{N} \binom{k+1}{l} u_k \left(\partial_s^{k-l} \cos s\right)(0)\sigma_l.$$

Let

$$\phi_0 = \frac{1}{2\pi}\int_0^{2\pi} \phi(x)\,\mathrm{d}x$$

and define $\phi_1 = \phi - \phi_0$. Since $Lu = 0$ and u_N may be assumed nontrivial, we must have $(N+1)\phi_0 \in \mathbb{Z}$. Thus if ϕ_0 is not a rational number, there are no solutions supported on the orbit $\mathcal{L}_1$. If ϕ_0 is rational, with $\phi_0 = p/q$ where p and q are relatively prime, then $N = q-1$ is the lowest possible transversal order of a nontrivial solution supported on $\mathcal{L}_1$. This follows from the injectivity of M_l for $l < N$ and the fact that M_N has a nontrivial kernel. Since M_l is also surjective for $l < N$ (as is easily seen using Fourier series), one can correct the 'errors' to obtain a solution u iteratively.

Finally, we remark that there are solutions supported on the closure of any noncompact orbit. This will follow from Theorem III.2.12 as stated below, or can be constructed explicitly as in [**BM**]. Thus, global unique continuation is not valid for distribution solutions. However, it is valid for continuous solutions.

We will now place these two examples in a more general context following [**BM**]. Given a locally integrable structure $(\mathcal{M}, \mathcal{V})$, let Σ be an orbit such that $\dim \Sigma < \dim \mathcal{M} = m+n$, where n is the rank of $\mathcal{V}$. Assume that Σ is an embedded submanifold of $\mathcal{M}$. Fix $p \in \Sigma$ and let $\{Z_1, \ldots, Z_m\}$ be a complete set of first integrals defined in a neighborhood U in $\mathcal{M}$ of p. Let $\{L_1, \ldots, L_n\}$ be smooth, local generators of $\mathcal{V}$ in U such that the brackets $[L_i, L_j] = 0$ for all i, j. Complete this to a basis

$$\{L_1, \ldots, L_n, M_1, \ldots, M_m\}$$

of $\mathbb{C}T\mathcal{M}$ in U such that

(1) $[L_i, M_k] = 0$, and
(2) $M_k Z_i = \delta_{ik}$.

Let $\{\omega_1, \ldots, \omega_n\}$ be smooth, exact one-forms in U such that

$$\{\omega_1, \ldots, \omega_n, \mathrm{d}Z_1, \ldots, \mathrm{d}Z_m\}$$

is a dual basis to $\{L_1, \dots, L_n, M_1, \dots, M_m\}$.

If $\mathcal{V}_\Sigma$ denotes the restriction of $\mathcal{V}$ to Σ, then $\mathcal{V}_\Sigma$ also has rank n. Hence if $\dim \Sigma = k+n$, then after shrinking U about p, the restrictions of exactly k of $\{Z_1, \dots, Z_m\}$ have linearly independent differentials along Σ. Without loss of generality, assume that $\{Z_1, \dots, Z_k\}$ have this latter property. It follows that $\{M_{k+1}, \dots, M_m\}$ is a basis of the complexified normal bundle of Σ in U.

Fix orientations in U and in $U \cap \Sigma$ so that distributions in U (resp. in $U \cap \Sigma$) may be viewed as acting on forms of top degree. We wish to describe all solutions in U that are supported on $U \cap \Sigma$.

Let $M'' = (M_{k+1}, \dots, M_m)$. If u is any distribution in U that is supported on $U \cap \Sigma$, it is well known that there is an integer N and distributions u_α on $U \cap \Sigma$ for $|\alpha| \le N$, such that for any $\phi \in C_c^\infty(U)$,

$$\langle u, \phi\, \omega \wedge \mathrm{d}Z \rangle = \sum_{|\alpha| \le N} \langle u_\alpha, (M'')^\alpha \phi\, \omega \wedge \mathrm{d}Z' \rangle$$

where $\omega = \omega_1 \wedge \cdots \wedge \omega_n$, $\mathrm{d}Z = \mathrm{d}Z_1 \wedge \dots \mathrm{d}Z_m$, $\mathrm{d}Z' = \mathrm{d}Z_1 \wedge \cdots \wedge \mathrm{d}Z_k$ and $u_\alpha \neq 0$ for some α, $|\alpha| = N$. Here and in what follows, by abusing notations, we are denoting by $\omega \wedge \mathrm{d}Z'$ the pullback to Σ.

Observe now that if $h \in C^1(U)$, then

$$\mathrm{d}h = \sum_{i=1}^{m} M_i h \mathrm{d}Z_i + \sum_{j=1}^{n} L_j h \omega_j$$

as can be seen by applying both sides of the equation to the basis

$$\{L_1, \dots, L_n, M_1, \dots, M_m\}.$$

Hence if $h \in C^\infty(U)$ and $\phi \in C_c^\infty(U)$, then

$$\begin{aligned}
\langle L_j h, \phi\, \omega \wedge \mathrm{d}Z \rangle &= \int_U (L_j h) \phi\, \omega \wedge \mathrm{d}Z \\
&= (-1)^j \int_U \mathrm{d}(h\phi\, \omega_1 \wedge \cdots \wedge \hat{\omega}_j \wedge \cdots \wedge \omega_n \wedge \mathrm{d}Z) \\
&\quad - \int_U h(L_j \phi)\, \omega \wedge \mathrm{d}Z \\
&= -\int_U h(L_j \phi)\, \omega \wedge \mathrm{d}Z \\
&= -\langle h, L_j \phi\, \omega \wedge \mathrm{d}Z \rangle \qquad \forall j = 1, \dots, n.
\end{aligned}$$

It follows that for the distribution u supported on $U \cap \Sigma$ as before, if $\phi \in C_c^\infty(U)$, we have:

$$\langle L_j u, \phi\, \omega \wedge \mathrm{d}Z \rangle = -\langle u, L_j \phi \omega \wedge \mathrm{d}Z \rangle$$

$$= -\sum_{|\alpha|\le N} \langle u_\alpha, (M'')^\alpha (L_j\phi)\,\omega\wedge \mathrm{d}Z'\rangle \qquad \forall j=1,\dots,n.$$

Assume now that u is also a solution. We will next show that each u_α is a solution of the induced structure $\mathcal{V}_\Sigma$. Fix a point $q\in U\cap\Sigma$. The restrictions of Z_l ($l=k+1,\dots,m$) to Σ are solutions of $\mathcal{V}_\Sigma$. By the Baouendi–Treves approximation theorem, for each such Z_l, there is a sequence $\{P_i^l\}_{i=1}^\infty$ of holomorphic polynomials such that

$$Z_l = \lim_{i\to\infty} P_i^l(Z_1,\dots,Z_k)$$

in $C^\infty(V\cap\Sigma)$ for some neighborhood V of q in $\mathcal{M}$. For each $l=k+1,\dots,m$, define the sequence $\{f_i^l\}_{i=1}^\infty$ by

$$f_i^l = Z_l - P_i^l(Z_1,\dots,Z_k).$$

Each $f_i^l\in C^\infty(V)$ and for every l,

$$\lim_{i\to\infty} f_i^l = 0$$

in $C^\infty(V\cap\Sigma)$. Let $f_i=(f_i^{k+1},\dots,f_i^m)$ for $i=1,2,\dots$ Fix a multi-index β in $\mathbb{N}^{m-k}$ such that $|\beta|=N$. For any $\phi\in C_c^\infty(V)$ and any $j=1,\dots,n$,

$$\begin{aligned}
0 &= \langle L_j u, f_i^\beta \phi\,\omega\wedge \mathrm{d}Z\rangle\\
&= -\langle u, L_j(f_i^\beta\phi)\,\omega\wedge \mathrm{d}Z\rangle\\
&= -\sum_{|\alpha|\le N}\langle u_\alpha, (M'')^\alpha (L_j(f_i^\beta\phi))\,\omega\wedge \mathrm{d}Z'\rangle\\
&= -\sum_{|\alpha|\le N}\langle u_\alpha, L_j(M'')^\alpha (f_i^\beta\phi)\,\omega\wedge \mathrm{d}Z'\rangle\\
&= -\sum_{|\alpha|\le N}\langle L_j u_\alpha, (M'')^\alpha (f_i^\beta\phi)\,\omega\wedge \mathrm{d}Z'\rangle\\
&= -\langle L_j u_\beta, \phi\,\omega\wedge \mathrm{d}Z'\rangle + E_i,
\end{aligned}$$

since $E_i\longrightarrow 0$ on $V\cap\Sigma$ as $i\longrightarrow\infty$ and $M_s f_i^l=\delta_{sl}$. Hence $L_j u_\beta = 0$ whenever $|\beta|=N$. Thus

$$\begin{aligned}
0 &= \langle L_j u, \psi\,\omega\wedge \mathrm{d}Z\rangle\\
&= \sum_{|\alpha|\le N-1}\langle L_j u_\alpha, (M'')^\alpha(\psi)\,\omega\wedge \mathrm{d}Z'\rangle
\end{aligned}$$

for any $\psi\in C_c^\infty(U)$. Plugging $\psi=f_i^\beta\phi$ with $|\beta|=N-1$ and $\phi\in C_c^\infty(V)$ in these latter equations will likewise lead to

$$L_j u_\beta = 0 \text{ whenever } |\beta| = N-1.$$

Continuing this way, we conclude: $L_j u_\alpha = 0 \quad \forall j, \quad \forall \alpha.$

Conversely, it is easy to see that if u has the form

$$\langle u, \phi\, \omega \wedge \mathrm{d}Z\rangle = \sum_{|\alpha|\le N} \langle u_\alpha, (M'')^\alpha \phi\, \omega \wedge \mathrm{d}Z'\rangle$$

where each u_α is a solution of $\mathcal{V}_\Sigma$ and some u_α is nontrivial, then u is a solution in U supported on $U \cap \Sigma$. In particular, the distributions σ_α ($\alpha \in \mathbb{N}^{m-k}$) defined by

$$\langle \sigma_\alpha, \phi\, \omega \wedge \mathrm{d}Z\rangle = \int_\Sigma (M'')^\alpha \phi\, \omega \wedge \mathrm{d}Z'$$

are solutions in U supported on $U \cap \Sigma$. Observe that

$$\sigma_\alpha = (M'')^\alpha \sigma_0.$$

Heuristically speaking then, we may say that each solution u in U supported on $U \cap \Sigma$ can be expressed as

$$u = \sum_{|\alpha|\le N} u_\alpha \sigma_\alpha$$

where the u_α are solutions of $\mathcal{V}_\Sigma$ in $U \cap \Sigma$.

The distribution σ_0 was introduced by Treves ([**T5**]). The existence of local solutions such as σ_0 supported on a nonopen orbit had previously been established by Baouendi and Rothschild in their proof of the necessity of Tumanov's minimality condition for the holomorphic extension of CR functions into wedges (see Section III.3). We have proved:

THEOREM III.2.10. *Let $p \in \Sigma$, U, $Z_1, \ldots, Z_m$, $\omega_1, \ldots, \omega_n$ and $M'' = (M_{k+1}, \ldots, M_m)$ be chosen as above. Then, u is a solution in U supported on $U \cap \Sigma$ if and only if u can be expressed as*

$$\langle u, \phi\, \omega \wedge \mathrm{d}Z\rangle = \sum_{|\alpha|\le N} \langle u_\alpha, (M'')^\alpha \phi\, \omega \wedge \mathrm{d}Z'\rangle,$$

where the u_α are solutions of $\mathcal{V}_\Sigma$ and u_α is nontrivial for some $|\alpha| = N$.

Suppose now u is a distribution supported on Σ. In a chart U about $p \in \Sigma$, write as before

$$\langle u, \phi\, \omega \wedge \mathrm{d}Z\rangle = \sum_{|\alpha|\le N} \langle u_\alpha, (M'')^\alpha \phi\, \omega \wedge \mathrm{d}Z'\rangle.$$

Let $N = N(p)$ be the minimum integer for which such a representation is possible. We will call $N(p)$ the *transversal order* of u at p. When u is also a solution, we have:

THEOREM III.2.11. *If u is a solution supported on Σ, the transversal order $N(p)$, $p \in \Sigma$ is constant.*

PROOF. Let $p \in \Sigma$. Choose a chart U as before such that $U \cap \Sigma$ is connected and

$$\langle u, \phi\,\omega \wedge \mathrm{d}Z\rangle = \sum_{|\alpha|\leq N} \langle u_\alpha, (M'')^\alpha \phi\,\omega \wedge \mathrm{d}Z'\rangle$$

where $N = N(p)$. Let $\gamma : [0, 1] \longrightarrow \Sigma$ be an integral curve of X for some smooth section of $\Re\mathcal{V}$ such that $\gamma(0) = p$. We consider $N(\gamma(t))$ for those t for which $\gamma(t) \in U$. In any neighborhood of such a $\gamma(t)$, u has the representation above. Moreover, if each u_α for $|\alpha| = N$ vanishes in a neighborhood of such $\gamma(t)$, then since the u_α are solutions for $\mathcal{V}_\Sigma$, by Theorem III.2.1 the u_α will vanish identically in a neighborhood of p in Σ (for $|\alpha| = N$), leading to the contradiction that $N(p) < N$. Thus whenever $\gamma(t) \in U$, then $N(\gamma(t)) = N(p)$. This argument shows that the set $\{t \in [0, 1] : N(\gamma(t)) = N(p)\}$ is both closed and open, and hence $N(\gamma(1)) = N(p)$. Since any two points of Σ can be joined by a finite number of such γ's, the theorem follows. □

We will continue to assume that the orbit Σ is an embedded orbit. Let $\{U_\alpha\}_\alpha$ be a covering of Σ by open sets in $\mathcal{M}$ such that in each U_α we have a basis $\{L_1^\alpha, \ldots, L_n^\alpha\}$ of $\mathcal{V}$, a basis $\{L_1^\alpha, \ldots, L_n^\alpha, M_1^\alpha, \ldots, M_m^\alpha\}$ of $\mathbb{C}TU_\alpha$, a dual basis

$$\{\omega_1^\alpha, \ldots, \omega_n^\alpha, \mathrm{d}Z_1^\alpha, \ldots, \mathrm{d}Z_m^\alpha\}$$

where the ω_i^α are exact and $\{Z_1^\alpha, \ldots, Z_m^\alpha\}$ is a complete set of first integrals. We will assume that the restrictions of $\{Z_1^\alpha, \ldots, Z_k^\alpha\}$ to $U_\alpha \cap \Sigma$ form a complete set of first integrals for $\mathcal{V}_\Sigma$. If u is a solution supported on Σ of transversal order zero, then we know that it is given by distributions u_α in $U_\alpha \cap \Sigma$ in the sense that for any $\phi \in C_c^\infty(U_\alpha)$,

$$\langle u, \phi\,\mathrm{d}Z^\alpha \wedge \omega^\alpha\rangle = \langle u_\alpha, \phi\,\mathrm{d}(Z^\alpha)' \wedge \omega^\alpha\rangle$$

where in the right-hand side we mean the pullback of the form on Σ. Let $V_\alpha = U_\alpha \cap \Sigma$ and whenever $V_\alpha \cap V_\beta \neq \emptyset$, let $g_{\alpha\beta} \in C^\infty(V_\alpha \cap V_\beta)$ satisfy

$$i^*(\mathrm{d}Z_1^\alpha \wedge \cdots \wedge \mathrm{d}Z_k^\alpha \wedge \omega^\alpha) = g_{\alpha\beta}\, i^*(\mathrm{d}Z_1^\beta \wedge \cdots \wedge \mathrm{d}Z_k^\beta \wedge \omega^\beta),$$

where for a form θ in $\mathcal{M}$, $i^*\theta$ denotes the pullback to Σ. Note that the $g_{\alpha\beta}$ are nonvanishing and on $V_\alpha \cap V_\beta$, $g_{\alpha\beta}u_\alpha = u_\beta$. Therefore, $0 = L_j u_\beta = (L_j g_{\alpha\beta})u_\alpha$. If $L_j g_{\alpha\beta}$ is not zero on an open set, then u_α will be zero there. But then u will vanish on this open set and hence on Σ, contradicting the nontriviality of u. Hence the $g_{\alpha\beta}$ are solutions on $V_\alpha \cap V_\beta$. Thus Σ is covered by $\{V_\alpha\}$ and whenever $V_\alpha \cap V_\beta \neq \emptyset$, we have a nonvanishing, smooth solution

$$g_{\alpha\beta} : V_\alpha \cap V_\beta \longrightarrow \mathbb{C}.$$

It follows that we can construct a line bundle $\pi : E \longrightarrow \Sigma$ having the $g_{\alpha\beta}$ as transition functions. In particular, if $(\Sigma, \mathcal{V}_\Sigma)$ is a complex structure, the bundle E becomes a holomorphic line bundle and solutions of $\mathcal{V}$ supported on Σ of transversal order zero correspond to nontrivial holomorphic sections of this bundle. In the situation where Σ is a Stein manifold, it is well known that a holomorphic bundle always has a nontrivial holomorphic section. In other words, we have:

THEOREM III.2.12. *Suppose Σ is an embedded orbit of $\mathcal{V}$ and $(\Sigma, \mathcal{V}_\Sigma)$ is a complex structure. If Σ is a Stein manifold, there are solutions supported on Σ of transversal order* 0.

III.3 The strong uniqueness property for locally integrable solutions

In this section we will consider locally integrable structures $\mathcal{V}$ on an open domain Ω in $\mathbb{R}^N$. The solutions we study will be assumed to be elements of the space L^1_{loc} of locally integrable functions with respect to Lebesgue measure.

DEFINITION III.3.1. *The structure $(\Omega, \mathcal{V})$ satisfies the strong uniqueness property if every solution $u \in L^1_{\text{loc}}(\Omega)$ that is zero on a set of positive measure vanishes identically.*

EXAMPLE III.3.2. Let $\mathcal{V}$ be the structure generated by the Cauchy–Riemann vector fields $\dfrac{\partial}{\partial \bar{z}_j}$ $(1 \le j \le n)$ on a domain Ω in $\mathbb{C}^n$. Then $(\Omega, \mathcal{V})$ satisfies the strong uniqueness property.

EXAMPLE III.3.3. Let $\mathcal{V}$ be the structure generated by a real-analytic vector field L on a domain Ω in the plane. Assume that there is only one orbit. Then $(\Omega, \mathcal{V})$ satisfies the strong uniqueness property. Indeed, suppose $u \in L^1_{\text{loc}}(\Omega)$, $Lu = 0$, and u vanishes on a set E of positive measure. Since there is only one orbit, it follows that there is an open set Ω' where L is elliptic and a subset $E' \subseteq \Omega'$ of positive measure where u vanishes. By Corollary I.13.4, the ellipticity of L implies that locally, coordinates can be found in which L

becomes a nonvanishing multiple of the Cauchy–Riemann operator. Hence u vanishes on Ω'. By Theorem III.2.1, u has to vanish on Ω.

EXAMPLE III.3.4. Let $\mathcal{V}$ be the structure generated by $\dfrac{\partial}{\partial x_j}$ $(1 \leq j \leq n)$ on a domain $\Omega \subseteq \mathbb{R}^N$. It is easy to see that if $n < N$, $(\Omega, \mathcal{V})$ will not satisfy the strong uniqueness property.

It turns out that orbits play a role in the validity of the strong uniqueness property. Before stating the main results, we need to introduce refinements of the concept of an orbit.

DEFINITION III.3.5. *The bundle $\mathcal{V}$ is called minimal at $p \in \Omega$ if, given an open set $p \in U \subseteq \Omega$, there exists a smaller open set $p \in U' \subseteq U$ such that every point in U' can be reached from p by a finite number of integral curves of sections of $\Re\mathcal{V}$ and each integral curve lies in U.*

EXAMPLE III.3.6. If $\mathcal{V}$ is real-analytic and has an open orbit $\mathcal{O}$, then $\mathcal{V}$ is minimal at every $p \in \mathcal{O}$. In this case, we can take $U' = U$.

EXAMPLE III.3.7. Let $\mathcal{V}$ be the structure generated by the vector field

$$L = \frac{\partial}{\partial x_1} + ig(x_1)\frac{\partial}{\partial x_2}$$

where $g \in C^\infty(\mathbb{R})$, $g > 0$ on $(1,2)$ and $g \equiv 0$ outside $(1,2)$. Observe that there is only one orbit in the plane. However, $\mathcal{V}$ is minimal at a point $p = (x_1, x_2)$ if and only if $x_1 \in [1,2]$.

If $\mathcal{M}$ is a real hypersurface in $\mathbb{C}^n$ with the standard CR structure which is a single orbit, it always has minimal points. This follows from the fact that if there are no minimal points in $\mathcal{M}$, then $\Re\mathcal{V}$ will be closed under Lie brackets leading to a Frobenius foliation of $\mathcal{M}$ by orbits each of dimension $2n-2$. Each of these orbits is a complex hypersurface. Indeed, the CR bundle $\mathcal{V}$ induces on each orbit $\mathcal{O}$ a locally integrable structure that is CR and elliptic. By Theorem I.10.1, near each $p \in \mathcal{O}$, we can find coordinates $\{x_1, \ldots, x_m, y_1, \ldots, y_m\}$ $(m = n-1)$ such that the induced structure on $\mathcal{O}$ is generated by $\frac{\partial}{\partial \bar{z}_j}$, $j = 1, \ldots, m$. In particular, any solution on $\mathcal{O}$ is a holomorphic function of the first integrals $(Z_1, \ldots, Z_m)$, $Z_j = x_j + iy_j$. Going back to the complex coordinates $(z_1, \ldots, z_n)$ of $\mathbb{C}^n$, it follows that the restriction to $\mathcal{O}$ of one of these coordinates is a holomorphic function of the remaining coordinates. In other words, $\mathcal{O}$ is a complex hypersurface—contradicting the fact that $\mathcal{M}$ is a single orbit. However, there are CR manifolds in $\mathbb{C}^n$ consisting of a single orbit with no minimal points. Examples of such are provided by the following, which appeared in [**Jo1**]:

Example III.3.8. Let $\mathcal{M} \subseteq \mathbb{C}^3$ be given by

$$\mathcal{M} = \{(x_1 + iy_1, x_2 + iy_2, x_3 + iy_3) : x_1 = h_1(x_3), x_2 = h_2(x_3)\},$$

where $h_1 \equiv 0$ for $x_3 \geq -\frac{1}{2}$ and h_1 is strictly convex for $x_3 < -\frac{1}{2}$, $h_2 \equiv 0$ for $x_3 \leq \frac{1}{2}$ and h_2 is strictly convex for $x_3 > \frac{1}{2}$. $\mathcal{M}$ is a CR submanifold of codimension 2. It consists of a single orbit but has no minimal points.

The concept of minimality appeared in Tumanov's theorem on the holomorphic extension of CR functions into wedges. Minimality is a necessary and sufficient geometric condition for the holomorphic extension of all CR functions into wedges. In [**Tu1**] Tumanov proved:

Theorem III.3.9. *Let $\mathcal{M}$ be a generic CR submanifold of $\mathbb{C}^N$ and $p \in \mathcal{M}$. If $\mathcal{M}$ is minimal at p, then for every neighborhood U of p in $\mathcal{M}$ there exists a wedge $\mathcal{W}$ with edge $\mathcal{M}$ centered at p such that every continuous CR function in U extends holomorphically to the wedge $\mathcal{W}$.*

Conversely, if $\mathcal{M}$ is not minimal at p, Baouendi and Rothschild ([**BR**]) proved that there exists a continuous CR function defined in a neighborhood of p in $\mathcal{M}$ which does not extend holomorphically to any wedge of edge $\mathcal{M}$ centered at p.

Tumanov's original definition of minimality was stated differently. He called a CR submanifold of $\mathbb{C}^N$ minimal at p if it contains no proper (i.e., of smaller dimension) CR submanifold of the same CR dimension through p. For the equivalence of the two definitions, we refer the reader to Marson's paper ([**Ma**]).

Definition III.3.10. *Given an involutive structure $\mathcal{V}$ on an open subset Ω of $\mathbb{R}^N$, we say that an orbit $\mathcal{O}$ is a.e. minimal if $\mathcal{V}$ is minimal at p for almost every $p \in \mathcal{O}$ in the sense of Lebesgue measure in $\mathbb{R}^N$.*

Note that if an orbit $\mathcal{O}$ is a.e. minimal, then it is an open orbit.

Example III.3.11. If $\mathcal{V}$ is real-analytic and $\mathcal{O}$ is an open orbit, then $\mathcal{O}$ is a.e. minimal since $\mathcal{V}$ is minimal at every $p \in \mathcal{O}$.

Here is a simple example of an a.e. minimal orbit which is not minimal everywhere:

EXAMPLE III.3.12. Let $\mathcal{M} = \mathbb{R}^2$ and $\mathcal{V}$ be the structure generated by

$$L = \frac{\partial}{\partial x} + ib(x, y)\frac{\partial}{\partial y}$$

where $b(x, y)$ is smooth, real-valued, and $b = 0$ only on $(-1, 1) \times \{0\}$. Then $\mathcal{M}$ is minimal exactly at the points in $\mathcal{M}\setminus((-1, 1) \times \{0\})$.

We can now state the main result on strong uniqueness:

THEOREM III.3.13. *Let $\mathcal{V}$ be a locally integrable structure defined on a connected open set Ω in $\mathbb{R}^N$. Assume that $\Omega = \mathcal{O} \cup F$ where $\mathcal{O}$ is an open a.e. minimal orbit of $\mathcal{V}$ and F is a set of measure zero. Then any solution $u \in L^1_{\mathrm{loc}}(\Omega)$ that vanishes on a set of positive measure must vanish identically.*

Theorem III.3.13 was proved in [**BH2**]. According to the theorem, if Ω satisfies the hypotheses, then almost every point $p \in \Omega$ can be reached from a fixed point $q \in \mathcal{O}$ by a piecewise smooth curve consisting of integral curves of smooth sections of $\Re\mathcal{V}$. We may say that Ω has an a.e. reachability property with respect to $\mathcal{V}$. Thus Ω satisfies the a.e. reachability property if and only if Ω admits a trivial decomposition, that is, if it can be expressed as the union of an open orbit and a set of measure zero. We note, however, that this a.e. reachability condition is not necessary for the conclusion of the theorem. For example, the structure $\mathcal{V}$ generated on the 2-torus $\mathbb{T}^2$ by a real globally hypoelliptic vector field L has the strong uniqueness property although the torus does not admit a trivial decomposition. However, local a.e. reachability is necessary if the conclusion of Theorem III.3.13 is to hold on any base of connected neighborhoods of a given point. Indeed, we have the following [**BH2**] partial converse to Theorem III.3.13:

THEOREM III.3.14. *Let $\mathcal{V}$ be a sub-bundle of $\mathbb{C}T\Omega$ where $\Omega \subseteq \mathbb{R}^N$ is open. Assume there is a base $\{\Omega_j\}_{j=1}^{\infty}$ of connected neighborhoods of p which do not admit a trivial decomposition. Then there is a base of connected neighborhoods $U_k \subseteq \Omega_k$ of p and nontrivial solutions $u_k \in L^1(U_k)$ for which the sets $\{u_k = 0\}$ all have positive measure.*

We remark that in Theorem III.3.14, $\mathcal{V}$ is not assumed to be locally integrable. It is not even assumed that it is involutive. Thus for analytic involutive structures $\mathcal{V}$ (which are always locally integrable), Theorems III.3.13 and III.3.14 establish the local equivalence between a.e. reachability and the uniqueness property that local solutions are determined on sets of positive measure.

We will prove Theorem III.3.13 in the important situation where $\mathcal{V}$ is the tangential Cauchy–Riemann bundle of a CR manifold embedded in $\mathbb{C}^N$

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(see Theorem III.3.15 below). In fact, by using Marson's ([**Ma**]) trick of embedding a general locally integrable structure $\mathcal{V}$ into a CR structure, one can deduce Theorem III.3.13 from Theorem III.3.15 (see [**BH2**] for the details). Theorem III.3.15 states that the strong uniqueness property that holomorphic functions have—that of being determined on any domain by their values on any subset of positive measure or, equivalently, that their zero sets have measure zero except in a trivial case—is inherited by their boundary values at the edge of the wedge where they are defined. In the particular classical case of a holomorphic function of one variable defined on a disk, this principle is well known and is attributed to Priwaloff and Riesz. Thus Theorem III.3.15 is, to a certain extent, a higher-dimensional version of the theorem of Priwaloff and Riesz.

THEOREM III.3.15. *Let $\mathcal{M} \subseteq \mathbb{C}^N$ be a generic CR manifold of codimension d ($N = n + d$). Assume that $\mathcal{W} \subseteq \mathbb{C}^N$ is a wedge with edge $\mathcal{M}$. Suppose F is a holomorphic function of tempered growth on $\mathcal{W}$ with distribution boundary value $f \in L^1_{\mathrm{loc}}(\mathcal{M})$. If f vanishes on a subset E of positive measure, then $f \equiv 0$ in a neighborhood of any Lebesgue density point of E.*

In the proof of Theorem III.3.15, we will use the following lemma where Σ is a smooth hypersurface in $\mathbb{C}^n$, f is a CR function on Σ, and $f \in L^p(\Sigma)$ for some $1 \le p \le \infty$. Suppose also that f extends to a holomorphic function F on a side Σ^+, that is, f is the boundary value of F in the distribution sense. Then we have:

LEMMA III.3.16. *For any $\Sigma' \subset\subset \Sigma$, and a sufficiently small ball B in $\mathbb{C}^n$ containing zero, the restrictions of F to the hypersurfaces $\{z \in B : \mathrm{dist}\,(z, \Sigma') = t\}$ have uniformly bounded L^p norms. In particular, $F \in L^p(B \cap \Sigma^+)$.*

PROOF. Without loss of generality, we may assume that Σ is part of the boundary of a bounded open set D with smooth boundary such that $D \subseteq \Sigma^+$. Let H be harmonic in D with boundary value f on Σ and 0 off Σ. By the classical h^p theory for harmonic functions, the restrictions H_t of H to the hypersurfaces $S_t = \{z \in D : \mathrm{dist}\,(z, \partial D) = t\}$ (t small) are all in L^p and $\|H_t\|_{L^p(S_t)} \le \|f\|_{L^p(\Sigma)}$. Moreover, it is well known that '$\mathrm{dist}\,(z, \partial D)$' can be replaced by any defining function for ∂D. Since F is holomorphic in Σ^+ and has a boundary value on Σ, there exist $C, k > 0$ such that for any $z \in D$,

$$|F(z)| \le C\,\mathrm{dist}\,(z, \Sigma)^{-k}.$$

This may require contracting Σ. It follows that F has a boundary value which is a distribution on ∂D. Let

$$u = F - H.$$

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u is harmonic in D, has a distributional boundary value bu on ∂D which is 0 on the piece Σ. We wish to show u is smooth up to Σ. Let $G(x, y)$ be the Green's function for D and $P(x, y)$ its Poisson kernel. We recall that

$$P(x, y) = -N_y G(x, y) \text{ for } x \in D, y \in \partial D$$

where $N_y =$ the unit outer normal to D at y. Fix $x \in D$. The function $y \longmapsto G(x, y)$ is 0 on ∂D and positive on $D \backslash \{x\}$. By Hopf's lemma, $N_y G(x, y) \neq 0$ for all $y \in \partial D$. Hence for ϵ small enough, the open sets

$$D_\epsilon = \{y \in D : G(x, y) > \epsilon\}$$

have smooth boundaries. Observe that if $\widetilde{G}_\epsilon(z, y)$ is the Green's function for D_ϵ, then

$$\widetilde{G}_\epsilon(x, y) = G(x, y) - \epsilon.$$

Hence the Poisson kernel $P_\epsilon(z, y)$ for D_ϵ satisfies

$$P_\epsilon(x, y) = -N_y^\epsilon G(x, y),$$

where N_y^ϵ is the unit outer normal to D_ϵ at y. We thus have

$$\begin{aligned} u(x) &= \int_{\partial D_\epsilon} P_\epsilon(x, y) u(y)\, \mathrm{d}\sigma_\epsilon(y) \\ &= \int_{\partial D} P_\epsilon(x, \Pi_\epsilon^{-1}(y)) u(\Pi_\epsilon^{-1}(y)) J_\epsilon(y)\, \mathrm{d}\sigma(y), \end{aligned}$$

where $\Pi_\epsilon : \partial D_\epsilon \longrightarrow \partial D$ is the normal projection map and J_ϵ is the Jacobian of Π_ϵ^{-1}. Since $P_\epsilon(x, y) = -N_y^\epsilon G(x, y)$, as $\epsilon \longrightarrow 0^+$,

$$P_\epsilon(x, \Pi_\epsilon^{-1}(y)) J_\epsilon(y) \longrightarrow P(x, y)$$

in $C^\infty(\partial D)$. It follows that for any $x \in D$,

$$u(x) = \langle bu, P(x, .)\rangle.$$

This latter formula, together with the vanishing of bu on Σ, tells us that u is C^∞ up to the boundary piece Σ. Since $F = H + u$ and $H \in h^p(D)$, the assertions of the theorem follow. □

Corollary III.3.17. [Nontangential Convergence] *Let f and F be as in the lemma and D be as in the proof of the lemma. For $\alpha > 1$ and $A \in \Sigma$, define*

$$\Gamma_\alpha(A) = \{z \in D : |z - A| < \alpha\delta(z)\},$$

where $\delta(z) = \operatorname{dist}(z, \partial D)$. Then

$$\lim_{\Gamma_\alpha(A)\ni z\to A} F(z) = f(A)$$

for almost all A in Σ.

Proof. Recall from the proof that $F = H + u$. Since u is smooth up to the piece Σ and bu vanishes on Σ, $\lim_{D\ni z\to A} u(z) = 0$ for all $A \in \Sigma$. The corollary therefore follows from the fact that $H \in h^p(D)$ and that on Σ, $H = f$. □

III.4 Proof of Theorem III.3.15

To prove Theorem III.3.15, we may assume that $0 \in \mathcal{M}$ is a density point of E and that $\mathcal{M}$ near 0 is defined by $\Im w = \phi(x, y, \Re w)$, where $z = x + iy \in \mathbb{C}^n$ and $w \in \mathbb{C}^d$, $N = n + d$. The function ϕ is real-valued, smooth, $\phi(0) = 0$, and $d\phi(0) = 0$. We may also assume that the wedge $\mathcal{W}$ contains a wedge of the form

$$\{(z, w) : w = s + i\phi(x, y, s) + iv, |z| < 2\delta, |s| < 2\delta, |v| < 2\delta, v \in \Gamma\}$$

for some open convex cone $\Gamma \subset \mathbb{R}^d$ and $\delta > 0$. We may suppose that $\|\mathrm{d}\phi(x, y, s)\| < \frac{1}{4}$ for $|x|, |y|, |s| < 2\delta$. Without loss of generality, assume that

$$\Gamma = \{v = (v', v_d) : |v'| < 2\delta v_d\}.$$

Let

$$\widetilde{\Gamma} = \{(y, t) \in \mathbb{R}^{n+d} : |(y, t')| < \delta t_d, t = (t', t_d)\}.$$

For $|y_0| < \delta$, the set

$$\mathcal{W}_{y_0} = \{(x + iy_0 + iy, s + i\phi(x, y_0, s) + it) : (y, t) \in \tilde{\Gamma}, |x|, |y|, |s|, |t| < \delta\}$$

is contained in the wedge $\mathcal{W}$. Indeed, this follows from the definitions of Γ and $\tilde{\Gamma}$ and the assumption on the norm of $\mathrm{d}\phi$. Observe that $\mathcal{W}_{y_0}$ is a wedge in $\mathbb{C}^N$ with a maximally totally real edge

$$\mathcal{M}_{y_0} = \{(x + iy_0, s + i\phi(x, y_0, s) : |x| < \delta, |s| < \delta\}.$$

Fix y_0, $|y_0| < \delta$ such that

$$(x, s) \longmapsto f(x, y_0, s)$$

is in L^1 and the $(n + d)$-dimensional set $\mathcal{M}_{y_0}$ intersects E in a set of positive measure. Note that F is holomorphic and of tempered growth in the wedge $\mathcal{W}_{y_0}$. Hence F has a distribution boundary value bF on $\mathcal{M}_{y_0}$. We will eventually show that bF agrees with f on $\mathcal{M}_{y_0}$ for almost all y_0. Assuming this for

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now, it is clear that Theorem III.3.15 would follow if we show that $F \equiv 0$ on $\mathcal{W}_{y_0}$. This kind of reduction to a maximally totally real manifold also appears in the proof of theorem 7.2.6 in [**BER**].

We are thus led to consider a maximally totally real submanifold Σ of $\mathbb{C}^m$ given in a neighborhood U of $0 \in \mathbb{C}^m$ by

$$t = \phi(s), \quad s \in U,$$

where $w = s + it$ are standard complex coordinates in $\mathbb{C}^m$, ϕ is a smooth $\mathbb{R}^m$-valued function defined near $0 \in \mathbb{R}^m$, and $\phi(0) = \mathrm{d}\phi(0) = 0$. We recall that a $\mathbb{C}^m$-valued analytic disk is a map $A : \overline{\Delta} \to \mathbb{C}^m$ of class $C^{1+\alpha}$ from the closed unit disk of the complex plane which is holomorphic on Δ (here $0 < \alpha < 1$ is fixed once from now on). An analytic disk A is said to be partially attached to Σ at p if (i) $A(\mathrm{e}^{i\theta}) \in \Sigma$ for $|\theta| \leq \frac{\pi}{2}$ and (ii) $A(1) = p$. The Banach space of $\mathbb{C}^m$-valued analytic disks will be denoted by $\mathcal{A}^m$. We recall theorem 7.4.12 of [**BER**] on the existence of analytic disks partially attached to Σ:

THEOREM III.4.1 ([**BER**].). *There exist a neighborhood $U \times V$ of the origin $(0, 0) \in \mathbb{R}^m \times \mathbb{R}^m$ and a smooth map $U \times V \ni (s, v) \mapsto A_{s,v} \in \mathcal{A}^m$ satisfying the following properties for all $(s, v) \in U \times V$:*

(i) $A_{s,v}(1) = s + i\phi(s)$;

(ii) $A_{s,v}(\mathrm{e}^{i\theta}) \in \Sigma$ *for* $|\theta| \leq \pi/2$;

(iii) $\frac{\mathrm{d}}{\mathrm{d}\theta}(A_{s,v})(\mathrm{e}^{i\theta})|_{\theta=0} = v + i\phi'(s) \cdot v.$

(iv) $\frac{\mathrm{d}}{\mathrm{d}r}(A_{s,v})(r)|_{r=1} = iv - \phi'(s) \cdot v.$

Notice that we have included (iv) here since it follows from (iii) and the Cauchy–Riemann equations satisfied by $\zeta \mapsto A_{s,v}(\zeta)$ at $\zeta = 1$. The meaning of (i) and (ii) is that $A_{s,v}$ is partially attached to Σ at $p = (s, \phi(s))$ and (iii) implies that we can choose a neighborhood $\tilde{U} \subset U$ of the origin and a small $\epsilon > 0$ such that for every $p = s_0 + i\phi(s_0)$, $s_0 \in \tilde{U}$, the map

$$(0, \epsilon) \times S^{m-1}(0, \epsilon) \quad \ni \quad (\theta, \omega) \longmapsto A_{s_0,\omega}(\mathrm{e}^{i\theta}) \in \Sigma$$

yields a $C^{1+\alpha}$ local system of polar coordinates centered at p on Σ, where $S^{m-1}(0, \epsilon)$ denotes the sphere of radius ϵ centered at $0 \in \mathbb{R}^m$. In particular, given $v_0 \in \mathbb{R}^m$, $|v_0| = \epsilon$, and $p_1 = s_1 + i\phi(s_1)$, $s_1 \in \tilde{U}$, we may find $s_0 \in U$ and $\theta_0 \in (0, \epsilon)$ such that

$$p_1 = A_{s_0,v_0}(\mathrm{e}^{i\theta_0}).$$

Assume that p_1 is a density point of a measurable set $E \subseteq \Sigma$ (in particular E has positive measure) and let $U_0 \ni s_0$ and $V_0 \ni v_0$ be open sets of diameter $< \epsilon$. Consider the set

$$\widetilde{E}^\epsilon = \left\{ (s, v) \in U_0 \times (S^{m-1}(0, \epsilon) \cap V_0) : \quad \int_0^\epsilon \chi_E(A_{s,v}(\mathrm{e}^{i\theta}))\, \mathrm{d}\theta > 0 \right\}$$

where χ_E denotes the characteristic function of E. We observe that we may assume without loss of generality that $\widetilde{E}^\epsilon$ has positive $(2m-1)$-dimensional measure. Indeed, the function

$$\theta \quad \longmapsto \quad \int_{\substack{|s-s_0|<\epsilon \\ |v-v_0|<2\epsilon}} \chi_E(A_{s,v}(\mathrm{e}^{i\theta}))\, \mathrm{d}s\, \mathrm{d}v$$

is continuous and assumes a positive value at $\theta = 0$ because $A_{s,v}(1) = s + i\phi(s)$. Hence,

$$\int_{\substack{|s-s_0|<\epsilon \\ |v-v_0|<2\epsilon}} \left(\int_0^\epsilon \chi_E(A_{s,v}(\mathrm{e}^{i\theta}))\, \mathrm{d}\theta \right) \mathrm{d}s\, \mathrm{d}v > 0$$

and writing v in polar coordinates we see that for some $0 < \epsilon' < 2\epsilon$ our claim is true for $\widetilde{E}^{\epsilon'}$. We fix such an $\epsilon' > 0$ and, dropping any reference to the dependence on ϵ', simply write $\widetilde{E}^{\epsilon'} = \widetilde{E}$.

Consider now the map

$$U_0 \times (S^{m-1}(0, \epsilon) \cap V_0) \times (1-\epsilon, 1) \quad \ni \quad (s, v, r) \longmapsto A_{s,v}(r) \in \mathbb{C}^m. \quad \text{(III.3)}$$

Taking account of (iv) we note that this map has rank $2m$ for small $\epsilon > 0$ and maps $\{s\} \times (S^{m-1}(0, \epsilon) \cap V_0) \times (1-\epsilon, 1)$ onto $B_p \setminus \{p\}$, where B_p is a $C^{1+\alpha}$-differentiable m-ball that intersects Σ orthogonally at $p = s + i\phi(s)$. Indeed, the respective tangent spaces at p are

$$T_p\Sigma = \{s + i\phi(s) + v + i\phi'(s) \cdot v, \quad v \in \mathbb{R}^m\} \quad \text{and}$$
$$T_pB_p = \{s + i\phi(s) + iv - \phi'(s) \cdot v, \quad v \in \mathbb{R}^m\}.$$

Since the map (III.3) is a local diffeomorphism, it takes $\widetilde{E}$ onto a set of positive measure $\widehat{E}$ which is contained in the union of the disks $\bigcup\{A_{s,v} : \ (s, v) \in \widetilde{E}\}$. We could say that these disks are strongly attached to E in the sense that for any $(s, v) \in \widetilde{E}$ the set of boundary points $\{A_{s,v}(\mathrm{e}^{i\theta}) : \ 0 < \theta < \epsilon\}$ intersects E at a non-negligible set of values of θ. Consider now a holomorphic function F of slow growth defined in a wedge $\mathcal{W} = \Sigma \times \Gamma$ with edge Σ possessing a weak trace $f \in L^p(\Sigma)$ and assume that f vanishes on E. Assume furthermore that $v_0 \in \Gamma$. We will now sketch how we try to prove that F must vanish. First one proves that if $\epsilon > 0$ is small enough and $(s, v) \in \widetilde{E}$ the portion $A^\epsilon_{s,v}$ of the disk $A_{s,v}$ described by the inequalities $-\epsilon < \theta < \epsilon$ and $1-\epsilon < r < 1$ is contained in

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the wedge $\mathcal{W}$. Then the composition $F(A_{s,v}(re^{i\theta}))$ is defined for $-\epsilon < \theta < \epsilon$, $1-\epsilon < r < 1$, is holomorphic and has a weak boundary value which, for a.e. $(s, v) \in \widetilde{E}$, is given by—and the proof of this fact is our second step—$f(A_{s,v}(e^{i\theta}))$. The third step is to prove that for a.e. (s, v), the restriction of f to the curve $(\epsilon/2, \epsilon) \ni \theta \mapsto A_{s,v}(e^{i\theta})$ is in L^p. Hence, by Corollary III.3.17 and the classical theorem of Priwaloff, the holomorphic function of one complex variable $F(A_{s,v}(re^{i\theta}))$ vanishes identically for $-\epsilon < \theta < \epsilon$, $1-\epsilon < r < 1$, in particular for $\theta = 0$. But we know that letting (s, v, r) vary on $\widetilde{E} \times (1-\epsilon, 1)$ and keeping $\theta = 0$, the union of $\{A_{s,v}(re^{i\theta})\}$ covers $\widehat{E}$. Thus, F vanishes a.e. on $\widehat{E}$ and so must vanish identically. The proof of the second step involves a discussion about the trace which will be developed next.

We begin our considerations by looking at the simplest case of a holomorphic function of one complex variable $F(x+iy)$ defined for $|x| < 1$, $0 < y < 2$ which satisfies the inequality

$$|F(x+iy)| \le C\,|\log y|, \quad |x| < 1, \quad 0 < y < 2. \tag{III.4}$$

We assume (III.4) for simplicity but the argument below can be iterated to handle the case $|F(x+iy)| \le C|y|^{-N}$. The standard manner of defining the weak trace f of F as an element of $\mathcal{D}'$ is through the formula

$$\langle f, \psi\rangle = \lim_{\varepsilon \searrow 0} \int F(x+i\varepsilon)\,\psi(x)\,\mathrm{d}x, \quad \psi \in C_c^\infty(-1, 1). \tag{III.5}$$

In formula (III.5) we see that for each fixed x the argument of F describes a straight vertical segment $\varepsilon \mapsto x + i\varepsilon$ that flows toward x as $\varepsilon \to 0$. We wish to see what happens if we change each vertical segment to a curve $\varepsilon \mapsto x + \alpha(x, \varepsilon) + i\varepsilon$. We will assume that $(-1, 1) \times [0, 1) \ni (x, \varepsilon) \mapsto \alpha$ is of class C^2 (we would need class C^{N+2} if we were assuming $|F(x+iy)| \le C|y|^{-N}$ instead of (III.4)) and that $\alpha(x, 0) = 0$, $|x| < 1$. The latter assumption simply means that the curve $\varepsilon \mapsto x + \alpha(x, \varepsilon) + i\varepsilon$ flows toward x as $\varepsilon \to 0$. Thus,

$$\left(\frac{\partial}{\partial x}\right)^j \alpha(x, 0) = 0, \quad j = 0, 1, 2. \tag{III.6}$$

We now define

$$\langle \widetilde{f}, \psi\rangle = \lim_{\varepsilon \searrow 0} \int F(x + \alpha(x, \varepsilon) + i\varepsilon)\,\psi(x)\,\mathrm{d}x, \quad \psi \in C_c^\infty(-1, 1)$$

and wish to prove that $f = \widetilde{f}$. To that end we write

$$F(x+\alpha(x,\varepsilon)+i\varepsilon) = F(x+\alpha(x,\varepsilon)+i) - i\int_\varepsilon^1 F'(x+\alpha(x,\varepsilon)+it)\,\mathrm{d}t.$$

It follows from (III.6) that if x belongs to a compact part of $(-1, 1)$ and ε is small, $|\alpha_x(x, \varepsilon)| < 1/2$. We will assume for simplicity that $|\alpha_x(x, \varepsilon)| < 1/2$ holds everywhere. Then

$$\int F(x+\alpha(x,\varepsilon)+i\varepsilon)\,\psi(x)\,\mathrm{d}x = \int F(x+\alpha(x,\varepsilon)+i)\psi(x)\,\mathrm{d}x$$
$$+i\int\int_{\varepsilon}^{1} F(x+\alpha(x,\varepsilon)+it)\,\mathrm{d}t\,\frac{\partial}{\partial x}\left(\frac{\psi(x)}{1+\alpha_x(x,\varepsilon)}\right)\mathrm{d}x.$$

Letting $\varepsilon \to 0$ and taking account of (III.6), we obtain $\langle f, \psi\rangle = \langle \widetilde{f}, \psi\rangle$ as we wished.

From now on we return to the general situation of a maximally totally real submanifold Σ of $\mathbb{C}^m$ and a holomorphic function F defined on a wedge $\mathcal{W} = \Sigma \times \Gamma$ and possessing a trace $f \in L^p(\Sigma)$. We will now take advantage of two facts:

(1) *The formula*

$$\langle f, \psi\rangle = \lim_{\varepsilon\to 0}\int F(s+\alpha(s,\varepsilon)+i(\phi(s)+\varepsilon v_0+\beta(s,\varepsilon))\,\psi(s)\,\mathrm{d}s$$

is independent of the family of curves

$$\sigma(s,\varepsilon) = (s+\alpha(s,\varepsilon), \varepsilon v_0 + \beta(s,\varepsilon))$$

as long as all curves $\varepsilon \mapsto \sigma(\varepsilon, s)$ are contained in $\mathcal{W}$, they have the right number of bounded derivatives, and $\alpha(s,0) = \beta(s,0) = 0$, $s \in U$ (the assumptions imply that $v_o \in \Gamma$).

(2) *The analytic disks described in theorem 7.4.12 of* **[BER]** *can be taken of class $C^{k+\alpha}$ rather than $C^{1+\alpha}$ where k is a large positive integer.*

The first fact follows from proposition 7.2.22 in [**BER**]. The second fact is true because theorems 6.5.4 and 7.4.12 in **[BER]** are valid with the same proofs if the analytic disks are taken to be in $C^{k,\alpha}$ for a fixed positive integer k. In the proof of theorem 7.4.12, the function γh has to be modified so that one gets a C^k extension.

Set $s' = (s_1, \dots, s_{m-1})$. We will assume without loss of generality that

(i) For any $\epsilon > 0$ the set

$$\{s' : \quad |s'| < \epsilon \text{ and } (s', 0, v_0) \in \widetilde{E}\} \tag{III.7}$$

has positive measure.

(ii) $v_0 = (0, \dots, 0, a)$ for some small $a > 0$.

For $|s'| < \epsilon$, $|\theta| < \epsilon$ consider the map

$$(s', \theta) \longmapsto A_{(s',0),v_0}(\mathrm{e}^{i\theta})$$

which for small ϵ has an injective differential. We consider a family of curves $\sigma(p,\varepsilon)$ defined by

$$p = A_{(s',0),v_0}(\mathrm{e}^{i\theta}), \qquad \sigma(p,\varepsilon) = A_{(s',0),v_0}[(1-\varepsilon)\mathrm{e}^{i\theta}].$$

Observe that $\sigma(p,0) = p$ and that we are implicitly using (s',θ) as local coordinates. For small ε the curves $\varepsilon \mapsto \sigma(p,\varepsilon)$ are contained in $\mathcal{W}$ and it follows from our assumptions that for any test function ψ with small support around $s = 0$,

$$\begin{aligned}\langle f,\psi\rangle &= \lim_{\varepsilon\to 0}\int F(p + \Re\sigma(p,\varepsilon) + i\Im\sigma(p,\varepsilon))\,\psi(p)\,\mathrm{d}s\\ &= \lim_{\varepsilon\to 0}\int F(A_{(s',0),v_0}[(1-\varepsilon)\mathrm{e}^{i\theta}])\,\psi(s',\theta)\,J(s',\theta)\,\mathrm{d}s'\mathrm{d}\theta.\end{aligned}$$

Assuming that $f \in L^p(\Sigma)$ and using Fubini's theorem in the coordinates (s',θ), we see that for a.e. $|s'| < \epsilon$, the function $\theta \mapsto A_{(s',0),v_0}[\mathrm{e}^{i\theta}]$ is in L^p. Fixing such an s' is equivalent to fixing an analytic disk with the property that the restriction of f to a portion of its boundary that is contained in Σ is in L^p. We now take test functions such that ψJ has separated variables, i.e., $\psi(s',\theta)J(s',\theta) = \psi_1(s')\psi_2(\theta)$. Since F has tempered growth, so does the compose $F \circ A_{(s',0),v_0}$ and it follows that

$$\langle \tilde{f}_{s'},\psi_2\rangle = \lim_{\varepsilon\to 0}\int F(A_{(s',0),v_0}[(1-\varepsilon)\mathrm{e}^{i\theta}])\,\psi_2(\theta)\,\mathrm{d}\theta$$

defines a distribution in θ that depends continuously on s' as a parameter (use the usual method to define the trace, integrating by parts with respect to θ). We further have

$$\int \langle \tilde{f}_{s'},\psi_2\rangle\psi_1(s')\,\mathrm{d}s' = \langle f,\psi\rangle.$$

We may now reason as in Lemma II.3.2 to conclude that for a.e. s', $|s'| < \epsilon$, $\tilde{f}_{s'} \in L^p(-\epsilon,\epsilon)$ and $\tilde{f}_{s'}(\theta) = f(s',\theta)$. If s' is in the set (III.7) and $\tilde{f}_{s'}(\theta) = f(s',\theta)$ holds, then $\zeta \mapsto F(A_{(s',0),v_0}(\zeta))$ has an L^p boundary value that vanishes on a set of positive measure which implies that $\zeta \mapsto F(A_{(s',0),v_0}(\zeta))$ vanishes identically. We conclude that for a.e. s' on the set (III.7), $F(A_{(s',0),v_0}(\zeta)) = 0$, or equivalently, that the set

$$E(0,v_0) = \{s' : \quad |s'| < \epsilon \text{ such that } F \circ A_{(s',0),v_0}(\zeta) \equiv 0\}$$

has positive measure. A similar conclusion could have been reached for the set

$$E(s_m,v) = \{s' : \quad |s'| < \epsilon, \quad \text{such that } F \circ A_{(s',s_m),v}(\zeta) \equiv 0\},$$

where s_m is a small number and $|v - v_0|$ is small. Thus, the set $\{(s,v)\}$ such that $F \circ A_{s,v}(\zeta) \equiv 0$ has positive measure and so does the union of the corresponding partially attached disks.

III.5 Uniqueness for approximate solutions

In this and the following sections we will present uniqueness results for the approximate solutions of two structures: locally integrable structures in the plane defined by vector fields which are of a fixed finite type on their characteristic set and real-analytic structures with $m = 1$. The theorems were proved by Cordaro ([**Cor2**]).

Suppose $\mathcal{V}$ is a locally integrable structure defined on a manifold Ω, dim $(\Omega) = N = m + n$, and the fiber dimension of $\mathcal{V}$ over $\mathbb{C}$ equals n. By going to the quotient, the exterior derivative defines a differential operator

$$C^\infty(\Omega) \overset{d_0'}{\to} C^\infty(\Omega, \mathbb{C}T^*\Omega/T')$$

where $T' = \mathcal{V}^\perp$. Equip the manifold $C^\infty(\Omega, \mathbb{C}T^*\Omega/T')$ with a hermitian metric. Observe that a solution for the structure $\mathcal{V}$ is a function or distribution u that satisfies $d_0'u = 0$. If $u \in L^1_{\text{loc}}(\Omega)$, we will say that u is an approximate solution for the structure $\mathcal{V}$ if the coefficients of $d_0'u$ are locally in L^1 and given any $p \in \Omega$, there is a number $M > 0$ such that near p,

$$|d_0'u| \leq M|u| \quad \text{a.e. in } U.$$

One way in which approximate solutions may arise is as follows: suppose $F: \Omega \times \mathbb{C} \to C^\infty(\Omega, \mathbb{C}T^*\Omega/T')$ satisfies $|F(p, z) - F(p, z')| \leq M|z - z'|$ and u and v are two C^1 solutions of the semilinear equation

$$d_0'w(p) = F(p, w(p)).$$

Then the function $u - v$ is an approximate solution for the structure $\mathcal{V}$. Recall next from Corollary I.10.2 that near a point in Ω, coordinates $(x_1, \ldots, x_m, t_1, \ldots, t_n)$ for Ω and local generators $L_1, \ldots, L_n$ for $\mathcal{V}$ can be chosen so that $\mathrm{d}t_j(L_k) = \delta_{jk}$, $j, k = 1, \ldots, n$. With such a choice of coordinates and generators, we can identify the bundle $C^\infty(\Omega, \mathbb{C}T^*\Omega/T')$ with the one spanned by the forms $\mathrm{d}t_1, \ldots, \mathrm{d}t_n$ and the operator d_0' can be realized as

$$Lu = \sum_{j=1}^{n} L_j u \,\mathrm{d}t_j.$$

Before we discuss the uniqueness results, we will present a description of smooth, planar vector fields which have a uniform finite type on their characteristic set.

Proposition III.5.1. *Let L be a C^∞ nonvanishing vector field defined near the origin in $\mathbb{R}^2$ and let Σ denote its characteristic set. If L is of uniform finite type k on Σ, then Σ is contained in a one-dimensional manifold. Moreover, if Σ is a one-dimensional manifold, then L is never tangent to Σ.*

PROOF. We may choose coordinates (x, y) near 0 so that L is a nonvanishing multiple of

$$L' = \frac{\partial}{\partial y} + ib(x, y)\frac{\partial}{\partial x},$$

with b real-valued and C^∞ near 0. Without loss of generality, let $L = L'$. Then

$$\Sigma = \{p : b(p) = 0\}.$$

The uniform type condition implies that

$$\frac{\partial^j b}{\partial y^j} \equiv 0$$

on Σ for $j < k-1$ and

$$\frac{\partial^k b}{\partial y^k} \neq 0$$

on Σ. Hence if

$$f(x, y) = \frac{\partial^{k-1} b}{\partial y^{k-1}}(x, y),$$

then Σ is contained in the manifold $\{f(x, y) = 0\}$ which has a parametrization $\{(x, y(x))\}$ for some smooth $y(x)$. □

PROPOSITION III.5.2. *Suppose L and Σ are as in Proposition III.5.1 and that Σ is a one-dimensional manifold. Assume L is locally integrable in a neighborhood of* 0. *Then we can find coordinates* (s, t) *about* 0 *in which*

$$Z(s, t) = s + i\phi(s, t)$$

is a first integral of L where $\phi(s, t)$ is real-valued and

$$\phi(s, t) = \alpha(s) + t^k \beta(s, t)$$

for some nonvanishing β near 0.

PROOF. We first flatten Σ near the origin so that in coordinates (x, y), $\Sigma = \{(x, 0)\}$. By Proposition III.5.1, L is not tangent to Σ and so if $Z(x, y)$ is a first integral near the origin, then $Z_x(0, 0) \neq 0$. Assume $Z(0, 0) = 0$. Let $s = \Re Z(x, y)$ and $t = y$. Then in (s, t) coordinates,

$$Z = s + i\phi(s, t)$$

and we may take

$$L = \frac{\partial}{\partial t} - \left(\frac{i\phi_t}{1 + i\phi_s}\right)\frac{\partial}{\partial s}.$$

The finite type assumption then implies that

$$\phi(s,t)=\alpha(s)+t^k\beta(s,t), \quad \beta(0)\neq 0,$$

for some smooth β. □

Proposition III.5.3. *Suppose L and Σ are as in Proposition III.5.2. If the uniform type k is even, then there are coordinates in which $Z(x,y)=x+iy^k$ is a first integral.*

Proof. By Proposition III.5.2, we have a first integral

$$Z(s,t)=s+i\phi(s,t), \quad \text{where}$$

$$\phi(s,t)=\alpha(s)+t^k\beta(s,t), \quad \beta(0)\neq 0.$$

We may assume $\beta>0$ near the origin. For ϵ small, let $\Omega=Z(D_\epsilon(0))$ where $D_\epsilon(0)$ denotes the disk centered at 0 of radius ϵ. Let Ω' be a smooth subdomain of Ω such that $0\in\partial\Omega'$ and the boundary part of Ω' near 0 is $\{(s,\alpha(s))\}$. By the Riemann mapping theorem, there exists a holomorphic function H which is a diffeomorphism up to $\partial\Omega'$ such that

$$H(\Omega')\subset\{(x,y):y>0\},$$

and $H(s+i\alpha(s))\in\mathbb{R}$. Let $W(s,t)=H\circ Z(s,t)$. Then $LW=0$ and $\mathrm{d}W\neq 0$ in a neighborhood of the origin. From the form of ϕ and the fact that $\Im H\circ Z(s,0)=0$, we have

$$\Im H\circ Z(s,t)=t^k\tilde{\beta}(s,t),$$

where $\tilde{\beta}>0$ near the origin. Let $x=\Re H\circ Z(s,t)$ and $y=t\tilde{\beta}(s,t)^{\frac{1}{k}}$. It can easily be checked that these are coordinates near 0 and in these coordinates,

$$W(x,y)=x+iy^k$$

is a first integral. □

Definition III.5.4. *A locally integrable structure $(\mathcal{M},\mathcal{V})$ is called hypocomplex if every solution u is locally of the form $H\circ Z$ where H is holomorphic and $Z=(Z_1,\ldots,Z_m)$ is a complete set of first integrals.*

Proposition III.5.5. *Suppose L and Σ are as in Proposition III.5.2. If k is odd, then there are coordinates (x,y) in which $Z(x,y)=x+iy^k$ is a first integral of L if and only if for any first integral W of L, there is a biholomorphism near 0 mapping $W(\Sigma)$ into the real axis.*

PROOF. Since k is odd, if we take the first integral $Z(s,t) = s + i\phi(s,t)$ with $\phi(s,t) = \alpha(s) + t^k\beta(s,t)$, $\beta(0) \neq 0$ as in Proposition III.5.2, we see that L is hypocomplex. Therefore, to prove the necessity, we only need to do it for this first integral. Suppose then (x,y) are coordinates in which $x + iy^k$ is a solution. Let $F = U + iV$ denote this diffeomorphism and we may assume $F(0) = 0$. Then F maps the characteristic set of L to that of

$$\frac{\partial}{\partial y} - iky^{k-1}\frac{\partial}{\partial x},$$

and so $V(s,0) = 0$ for s near 0. Moreover, by the hypocomplexity of L, there is a holomorphic function H defined near the origin such that

$$U(s,t) + iV(s,t)^k = H(s + i\phi(s,t)).$$

Since $U + iV^k$ and Z are homeomorphisms, H is a biholomorphism (near 0). We also have

$$H(s + i\alpha(s)) = U(s,0) \in \mathbb{R},$$

showing that $H(Z(\Sigma)) \subseteq \mathbb{R}$. Note also that from the equations

$$\Im H(s,\alpha(s)) = 0 \quad \text{and} \quad \mathrm{d}\Im H(0) \neq 0,$$

we conclude that $\alpha(s)$ is real-analytic. In the latter statement, we have assumed as we may that $\alpha'(0) = 0$ and used the consequent fact that $H'(0)$ is real. Conversely, suppose H is a biholomorphism near 0 such that $H \circ Z(\Sigma) \subseteq \mathbb{R}$ where we take $Z(s,t)$ as before. Thus $H(s + i\alpha(s)) \in \mathbb{R}$. Define $F(s,t) = W^{-1} \circ H \circ Z(s,t)$, where $W(x,y) = x + iy^k$. F is a homeomorphism and away from $t = 0$, it is a diffeomorphism. Since $\Re F(s,t) = \Re H \circ Z(s,t)$, $\Re F(s,t)$ is smooth. Next note that since $\Im H \circ Z(s,t)$ vanishes to order k at $t = 0$ and $(\Im F)^k = \Im H \circ Z$, there is a nonvanishing smooth function $g(s,t)$ near the origin such that

$$\Im F(s,t) = g(s,t)t.$$

The latter, together with the fact that $H'(0) \in \mathbb{R}$ (we assume $\alpha'(0) = 0$), implies that F is a diffeomorphism near the origin. Clearly, using $(\Re F, \Im F)$ as new coordinates, we get $x + iy^k$ as a first integral for L. $\square$

REMARK III.5.6. In Proposition III.5.5, when $Z(s,t) = s + i\phi(s,t)$ with $\phi(s,t) = \alpha(s) + it^k\beta(s,t)$, the proof shows that the two equivalent conditions are equivalent to the real-analyticity of $\alpha(s)$.

Thus, we have:

COROLLARY III.5.7. *Suppose L and Σ are as in Proposition III.5.2 and L is real-analytic. Then there are real-analytic coordinates (x, y) in which the function $x+iy^k$ is a first integral of L. In other words, when L is real-analytic and Σ is a manifold of dimension 1, up to a real-analytic local diffeomorphism (and up to a nonvanishing multiple), there is only one real-analytic vector field of uniform type k.*

The preceding corollary was also proved in [**Me1**]. Proposition III.5.5 can be generalized as follows:

PROPOSITION III.5.8. *Suppose L_1 and L_2 are two vector fields of the same uniform odd type on their respective characteristic sets Σ_1 and Σ_2. Then there exists a local diffeomorphism mapping the structure generated by L_1 to the one generated by L_2 if and only if for any first integrals Z_1 and Z_2 of L_1 and L_2 respectively, there exists a local biholomorphism mapping $Z_1(\Sigma_1)$ onto $Z_2(\Sigma_2)$.*

The proof is similar to that of Proposition III.5.5.

In Proposition III.5.2 and the subsequent discussion, we assumed that Σ is a one-dimensional manifold. However, in general, as the following examples show, Σ may not be one-dimensional.

EXAMPLE III.5.9. Let $\mathcal{V}_1$ be the structure in the plane defined by

$$Z_1 = x + i(x^2 y + y^3).$$

Then the characteristic set $\Sigma_1 = \{(0,0)\}$ and the type there is 3.

EXAMPLE III.5.10. Let $\mathcal{V}_2$ be the structure defined by

$$Z_2 = x + i(x^4 y + y^3).$$

Again the characteristic set $\Sigma_2 = \{(0,0)\}$ and the type is 3.

We remark that in any neighborhood of the origin, the structures $\mathcal{V}_1$ and $\mathcal{V}_2$ are not equivalent. More generally, we have:

PROPOSITION III.5.11. *Suppose L is elliptic except at the origin and is of finite type there. Then the type is odd. In particular, L is hypocomplex.*

PROOF. Write $L = \dfrac{\partial}{\partial y} + ib(x, y)\dfrac{\partial}{\partial x}$, with b real-valued. Then $b(p) \equiv 0$ if and only if $p = 0$. Hence b cannot change sign in any neighborhood of the origin.

It follows that the type at the origin is odd. Since b does not change sign, L is locally solvable (Theorem IV.1.6) and hence locally integrable. □

We are now ready to state and prove the key lemma from [**Cor2**] concerning approximate solutions:

LEMMA III.5.12. *Let L be a locally integrable, planar vector field of uniform finite type on its characteristic set Σ which we assume is a one-dimensional manifold. Assume that u is a nontrivial approximate solution on a side of Σ and that u is continuous up to the boundary piece Σ. Then the set*

$$\{p \in \Sigma : u(p) = 0\}$$

has zero measure with respect to arclength on Σ.

In view of the preceding propositions, Lemma III.5.12 will be a consequence of:

LEMMA III.5.13. *Let L be locally integrable near the origin with a first integral $Z(x, t) = x + i\Phi(x, t)$. Suppose that*

$$\Phi_t(x, t) = a(x, t)t^k,$$

where k is a positive integer and a is never zero for $|x|$, $|t| \le \delta$, $(\delta > 0)$. Let u be a nontrivial function satisfying on $|x| < \delta$, $0 < t < \delta$,

$$|Lu(x, t)| \le M|u(x, t)|$$

and continuous up to $t = 0$. Then the set

$$\{x : |x| < \delta, u(x, 0) = 0\}$$

has zero Lebesgue measure.

PROOF. We may assume that $a(x, t) > 0$ for every (x, t). The map $(x, t) \mapsto (x, \Phi(x, t))$ is a diffeomorphism from the region $|x| < \delta$, $0 < t < \delta$ onto the open set in the plane:

$$\Omega = \{z = x + iy : |x| < \delta, \Phi(x, 0) < y < \Phi(x, \delta)\}.$$

Denote by $z \to (x, \Psi(x, y))$ the inverse of this diffeomorphism and set

$$v(x, y) = u(x, \Psi(x, y)), \quad x + iy \in \Omega. \tag{III.8}$$

By the chain rule, we have

$$\left|\left(\frac{\partial v}{\partial x} + i\frac{\partial v}{\partial y}\right)(x, y)\right| \le K\Phi_t(x, \Psi(x, y))^{-1}|v(x, y)|. \tag{III.9}$$

Now we have for $t \geq 0$

$$\Phi(x,t) - \Phi(x,0) = t^{k+1} A(x,t)$$

where $A > 0$ for $|x| \leq \delta$, $0 \leq t \leq \delta$. Hence

$$y - \Phi(x,0) = \Psi(x,y)^{k+1} A(x, \Psi(x,y)), \quad x + iy \in \Omega.$$

Since

$$\Phi_t(x,t) \geq \epsilon t^k, \quad \epsilon > 0,$$

we get

$$\Phi_t(x, \Psi(x,y)) \geq \epsilon \Psi(x,y)^k = \epsilon \left(\frac{y - \Phi(x,0)}{A(x, \Psi(x,y))} \right)^{k/(k+1)}.$$

Consequently, (III.9) implies for $x + iy \in \Omega$:

$$\left| \left(\frac{\partial v}{\partial x} + i \frac{\partial v}{\partial y} \right)(x,y) \right| \leq K'[y - \Phi(x,0)]^{-k/(k+1)} |v(x,y)|. \tag{III.9$'$}$$

Observe that since $(x,t) \mapsto (x, \Phi(x,t))$ is also a homeomorphism from $|x| < \delta$, $0 \leq t < \delta$ onto

$$\Omega' = \{z = x + iy : |x| < \delta, \Phi(x,0) \leq y < \Phi(x,\delta)\},$$

the function v is in fact continuous on Ω'.

Fix $0 < \delta' < \delta$ arbitrary. It suffices to show that the Lebesgue measure of the set

$$\{x : |x| < \delta', v(x, \Phi(x,0)) = 0\}$$

is zero. Consider now a simply connected open subset U of Ω that is bounded by a smooth Jordan curve γ for which there is a decomposition $\gamma = \gamma_1 \cup \gamma_2$ with

$$\gamma_1 = \{x + i\Phi(x,0) : |x| \leq \delta'\}, \quad \gamma_2 \subset \Omega'.$$

By the Riemann mapping theorem there is a biholomorphism $\zeta = G(z)$ from U onto the unit disk $|\zeta| < 1$. Since G is necessarily a smooth diffeomorphism from $\overline{U}$ onto $|\zeta| \leq 1$, $v'(\zeta) = v(G^{-1}(\zeta))$ will be continuous on $|\zeta| \leq 1$ and will satisfy (III.9$'$):

$$\left| \frac{\partial v'}{\partial \bar{\zeta}}(\zeta) \right| \leq K(1 - |\zeta|)^{-\frac{k}{k+1}} |v'(\zeta)|, \quad |\zeta| < 1.$$

The lemma now follows from Lemma III.5.14. □

LEMMA III.5.14. *Let D be the unit disk in the complex z-plane and let $v \in C(\overline{D})$ be not identically zero and satisfy*

$$\left|\frac{\partial v}{\partial \bar{z}}(z)\right| \leq K(1-|z|)^{-\alpha}|v(z)|, \quad z \in D, \tag{III.10}$$

for some $0 \leq \alpha < 1$. Then the set

$$\{\omega \in T : v(\omega) = 0\}$$

has zero Lebesgue measure (here T denotes the boundary of D).

PROOF. The main step in the proof is to show the following property:

There is a solution $S \in \bigcap_{\sigma<1} C^\sigma(D)$ of the equation

$$\frac{\partial S}{\partial \bar{z}} = \frac{1}{v}\frac{\partial v}{\partial \bar{z}} \quad \textit{in } D \textit{ satisfying} \tag{$*$}$$

$$\sup_{r<1} \int_0^{2\pi} |S(re^{i\theta})| d\theta < \infty.$$

Let us show right away that $(*)$ implies the conclusion of Lemma III.5.14. Write $v = \exp\{S\}h$ with $h \in \mathcal{O}(D)$. There is $p \in \mathbb{Z}_+$ such that v/z^p is continuous in $\overline{D}$ and does not vanish at the origin. Moreover, (III.10) is satisfied when v/z^p is substituted for v. Summing up, this argument shows that there is no loss of generality in assuming from the outset that $v(0) \neq 0$. Applying Jensen's inequality to the holomorphic function h gives, if $r < 1$,

$$\log|v(0)| \leq \Re S(0) - \frac{1}{2\pi}\int_0^{2\pi} \Re S(re^{i\theta}) d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log|v(re^{i\theta})| d\theta$$

and consequently $(*)$ implies

$$\log|v(0)| \leq C + \frac{1}{2\pi}\int_0^{2\pi} \log|v(re^{i\theta})| d\theta \tag{III.11}$$

where $C > 0$ is independent of r. A standard application of Fatou's lemma in (III.11) shows that $\log^-|v(e^{i\theta})| \in L^1(T)$, whence the sought conclusion.

We now proceed to the proof of $(*)$. To simplify the notation, we set $F = v_{\bar{z}}/v$. We observe that there is $p > 1$ such that $F \in L^p(D)$ (indeed it suffices to take $1 < p < 1/\alpha$). We set

$$S(z) = \frac{1}{\pi}\int\int_D \frac{F(z')}{z - z'} dx' dy'.$$

Then

$$\frac{\partial S}{\partial \bar{z}} = F;$$

moreover, since $p > 1$, it also follows (*cf.* [**V**], theorem 1.35) that

$$\frac{\partial S}{\partial z} = \Pi(F),$$

where Π denotes the singular integral operator

$$\Pi(g)(z) = -\frac{1}{\pi}\int\int_D \frac{g(z')}{(z-z')^2}\mathrm{d}x'\mathrm{d}y'.$$

Since Π is a bounded linear operator in $L^p(D)$ if $1 < p < \infty$ (*cf.* [**V**], page 64) we obtain $S \in L_1^p(D)$.

Since $F \in L^\infty(\{|z| < R\})$ for $R < 1$, any solution of the equation $\partial u/\partial \bar{z} = F$ belongs to $\bigcap_{\sigma<1} C^\sigma(D)$. Hence $(*)$ will follow if we can establish the following property:

$$\sup_{1/2\le r<1}\int_0^{2\pi} |S(re^{i\theta})|\mathrm{d}\theta < \infty. \tag{III.12}$$

We observe that $\partial/\partial r = \mathrm{e}^{i\theta}\partial/\partial z + \mathrm{e}^{-i\theta}\partial/\partial\bar{z}$, $\partial/\partial\theta = ir(\mathrm{e}^{i\theta}\partial/\partial z - \mathrm{e}^{-i\theta}\partial/\partial\bar{z})$ from which we derive that $(r,\theta)\mapsto S(r\mathrm{e}^{i\theta})$ belongs to the Sobolev space $L_1^1(]1/4, 1[\times]0, 2\pi[)$. Thus $r\mapsto S(r\mathrm{e}^{i\theta})$ is absolutely continuous for almost all θ. By first integrating on $[1/2, r]$ and afterwards on $[0, 2\pi]$ we conclude that

$$\int_0^{2\pi} |S(r\mathrm{e}^{i\theta})|\mathrm{d}\theta \le \int_0^{2\pi}\left|S\left(\frac{1}{2}\mathrm{e}^{i\theta}\right)\right|\mathrm{d}\theta + \int_0^{2\pi}\int_{1/2}^1\left|\frac{\partial S}{\partial r}(r'\mathrm{e}^{i\theta})\right|\mathrm{d}r'\mathrm{d}\theta,$$

for every $r\in[1/2, 1]$, from which (III.12) follows. This completes the proof of Lemma III.5.14. □

III.6 Real-analytic structures in the plane

We will continue using the notation of the previous section and assume in addition that Φ is real-analytic. If $\Phi_t(0, 0) \ne 0$ then L is elliptic near the origin, and the results we will discuss are well known in this case. We next discuss the case when $\Phi_t(0, 0) = 0$ but Φ_t is not identically zero. We factor out $\Phi_t(x, t) = x^l\Psi(x, t)$, where Ψ is real-analytic and $\Psi(0,\cdot)$ does not vanish identically. Applying the Weierstrass preparation theorem to Ψ allows us to describe the zero set Σ_0 of the function Φ_t as the zero set of $(x, t)\mapsto x^l p(x, t)$, where p is a distinguished polynomial in the t-variable with no multiple factors. Hence we can state:

There is a disjoint decomposition

$$\Sigma_0 = F_0 \cup V_1^+ \cup\cdots\cup V_\alpha^+ \cup V_1^- \cup\cdots\cup V_\beta^- \tag{$**$}$$

in a small neighborhood of the origin $|x| < \epsilon$, $|t| < \epsilon$, where F_0 is either $\{(0,0)\}$ or is equal to the segment $\{0\} \times (-\epsilon, \epsilon)$ (according to either $l = 0$ or $l > 0$), and each V_j^+ (resp. V_k^-) is defined by an analytic graph $\{(x, \gamma_j(x)) : 0 < x < \delta\}$ (resp. $\{(x, \sigma_k(x)) : -\delta < x < 0\}$), where $\gamma_1 < \gamma_2 < \cdots < \gamma_\alpha$ (resp. $\sigma_1 < \sigma_2 < \cdots < \sigma_\beta$) and

$$\lim_{x\to 0^-} \sigma_k(x) = \lim_{x\to 0^+} \gamma_j(x) = 0, \quad \forall j, k.$$

As a consequence we observe that in a neighborhood of each point $(x_0, t_0) \in \Sigma_0 \backslash F_0$ we can write $\Phi_t(x, t) = (t - g(x))^k a(x, t)$, where $k \geq 1$, a and g are real-analytic and a never vanishes.

In what follows, for any set S and a number k, $\mathcal{H}^k(S)$ will denote the k-dimensional Hausdorff measure of S.

We can now prove:

Proposition III.6.1. *Suppose that $\Phi_t(0,0) = 0$, $\Phi_t \not\equiv 0$. Let u be a nontrivial C^1 function defined for $|x| < \epsilon$, $|t| < \epsilon$ and satisfying:*

$$|Lu(x, t)| \leq M|u(x, t)|$$

and denote its zero set by S. Then:

(1) *If u does not vanish identically on $\{0 < x < \epsilon, |t| < \epsilon\}$, then $S \cap \{x > 0\}$ has a trivial one-dimensional Hausdorff measure (likewise for $x < 0$).*
(2) *If $F_0 = \{0\} \times (-\epsilon, \epsilon)$, then $S \cap F_0 \neq \emptyset \Rightarrow F_0 \subset S$.*
(3) *If $F_0 = \{(0,0)\}$ and if u does not vanish identically then S has a trivial one-dimensional Hausdorff measure.*

Proof. Assume first that $F_0 = \{0\} \times (-\epsilon, \epsilon)$. Then $L = \partial u / \partial t$ over F_0 (since $Z_t(0, t) = i\Phi_t(0, t) = 0$), which gives

$$\left| \frac{\partial u}{\partial t}(0, t) \right| \leq M|u(0, t)|.$$

By Gronwall's inequality, it follows that if $u(0, t_0) = 0$ for some t_0, then $u(0, t) = 0$ for all t.

Now we consider the general case. Fix a point $(x_0, t_0) \in \Sigma_0 \backslash F_0$ and write $\Phi_t(x, t) = (t - g(x))^k a(x, t)$ in a neighborhood of (x_0, t_0) as before. After the change of variables $x' = x$, $t' = t - g(x)$, the analysis near (x_0, t_0) reduces to the situation treated in Lemma III.5.13. In particular, we obtain that u cannot vanish identically in any component of the set $W^+ = \{(x, t) : 0 < x < \epsilon, |t| < \epsilon, \Phi_t(x, t) \neq 0\}$ and also that the one-dimensional Hausdorff measure of $S \cap (\Sigma_0 \backslash F_0)$ is trivial. Since the vector field L defines a complex structure

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on W^+, it follows that the one-dimensional Hausdorff measure of $S \cap W^+$ is also trivial. The proof of Proposition III.6.1 follows from these arguments. □

COROLLARY III.6.2. *Suppose u is a C^1-approximate solution defined for $|x| < \epsilon$, $|t| < \epsilon$ and vanishing for $t = 0$. Then u vanishes identically.*

PROOF. Consider the new C^1-approximate solution $\tilde{u}$ defined as u for $t > 0$ and zero for $t \leq 0$. If Φ_t does not vanish identically, it follows from Proposition III.6.1 and the discussion that precedes it that $\tilde{u}$ vanishes identically. If however $\Phi_t \equiv 0$, then $L = \frac{\partial}{\partial t}$ and we reach the same conclusion by applying Gronwall's inequality. □

III.6.1 Real-analytic structures with $m = 1$

As a consequence of Corollary III.6.2 we obtain:

THEOREM III.6.3. *Uniqueness in the Cauchy problem for C^1-approximate solutions holds for real-analytic locally integrable structures with $m = 1$.*

PROOF. Since this is a local statement we can work in local coordinates $(x, t) = (x, t_1, \ldots, t_n)$ centered at the origin for which there is a real-analytic, real-valued function $\Phi(x, t)$ satisfying

$$\Phi(0, 0) = \Phi_x(0, 0) = 0 \tag{III.13}$$

such that, if

$$Z(x, t) = x + i\Phi(x, t),$$

then the bundle $\mathcal{V}$ is spanned by the linearly independent, pairwise commuting vector fields

$$L_j = \frac{\partial}{\partial t_j} - \frac{Z_{t_j}}{Z_x}\frac{\partial}{\partial x}, \quad j = 1, \ldots, n. \tag{III.14}$$

Let u be a C^1-approximate solution defined for $|x| < \delta$, $|t| < \delta$:

$$|Lu| \leq M|u|.$$

The conclusion will follow after we show that if u vanishes for $t = 0$ then u vanishes identically.

Fix t_0, $0 < |t_0| < \delta$ and define

$$Z_0(x, s) = Z(x, st_0), \quad |x| < \delta, \quad |s| < 1.$$

Consider also the vector field

$$L_0 = \frac{\partial}{\partial s} - \frac{Z_{0s}}{Z_{0x}}\frac{\partial}{\partial x}$$

as well as the C^1 function

$$u_0(x,s) = u(x, st_0).$$

We have

$$L_0 u_0(x,s) = \sum_{j=1}^{n} (L_j u)(x, st_0) t_{0j}$$

and thus

$$|L_0 u_0(x,s)| \le M' |u_0(x,s)|,$$

showing that u_0 is a C^1-approximate solution for the structure defined by L_0 in $|x| < \delta$, $|s| < 1$. Moreover, u_0 vanishes for $s = 0$. Therefore, by Corollary III.6.2 and a standard propagation argument, u_0 vanishes identically for $|x| < \delta$, $|s| < 1$. Hence $u(x, t_0) = 0$ for all $|x| < \delta$. $\square$

Let $\mathcal{V}$ be a real-analytic locally integrable structure over a connected, real-analytic manifold Ω of dimension N. When $m = 1$ (Ω has then dimension $n+1$) the orbits of the structure $\mathcal{V}$ have either dimension $n+1$ (open subsets of Ω) or dimension n.

Introduce the projection over Ω of the characteristic set of $\mathcal{V}$:

$$\Sigma = \{p \in \Omega : T'_p \cap \overline{T'_p} \ne 0\}.$$

It is easy to see that Σ is an analytic subset of Ω. Since Ω is connected we either have $\dim \Sigma \le n$ or $\Sigma = \Omega$.

Assume first that $\Sigma = \Omega$: in this case $\mathcal{V}$ defines a real structure on Ω in the sense that $\mathcal{V} = \mathbb{C} \otimes \mathcal{V}_0$, where $\mathcal{V}_0$ is an involutive vector sub-bundle of $T\Omega$ of rank n. The leaves of the foliation defined by $\mathcal{V}_0$ are precisely the n-dimensional (Nagano) leaves.

Next suppose that the dimension of the analytic set Σ is $\le n$. On $\Omega \backslash \Sigma$ the bundle $\mathcal{V}$ defines an elliptic structure and every n-dimensional leaf is contained in Σ; in particular, it follows that the union of all n-dimensional leaves is a set of $(n+1)$-dimensional measure zero. We now prove:

Theorem III.6.4. *Let $\mathcal{V}$ be a real-analytic locally integrable structure over a connected, real-analytic, $(n+1)$-dimensional manifold Ω with $m = 1$. Let u be a nontrivial C^1-approximate solution on Ω and let S denote its zero set. Then:*

(1) *If $\mathcal{M}$ is an $(n+1)$-dimensional leaf, then either $\mathcal{M} \cap S = \mathcal{M}$ or $\mathcal{H}^n(\mathcal{M} \cap S) = 0$.*
(2) *If S has nonempty intersection with some n-dimensional leaf $\mathcal{M}$, then $\mathcal{M} \subset S$.*

PROOF. Suppose that $\Sigma = \Omega$. By the preceding discussion any point $p \in \Omega$ is the center of a system of coordinates $(U; x, t_1, \dots, t_n)$ over which $\mathcal{V}$ is spanned by the vector fields $\partial/\partial t_j$, $j = 1, \dots, n$. On U we have $|\mathrm{d}_t u| \leq M|u|$ and consequently if $u(0,0) = 0$ then $u(0,t) = 0$ for all t thanks to Gronwall's inequality.

This argument also provides a proof of (2): if $\mathcal{M}$ is an n-dimensional leaf then $\mathcal{V}|_{\mathcal{M}} \subset \mathbb{C}T\mathcal{M}$ and it defines a real structure over $\mathcal{M}$ for which $u|_{\mathcal{M}}$ is also a C^1-approximate solution. Again Gronwall's inequality gives $S \cap \mathcal{M} \neq \phi \Rightarrow \mathcal{M} \subset S$.

Next we observe first that (1) is valid when $n = 1$. Indeed, let $\mathcal{M}$ be a two-dimensional leaf on which u is not identically zero and $p \in \mathcal{M}$. Then p is the center of a system of coordinates (x, t) as in Proposition III.6.1 for which there is $Z(x,t) = x + i\Phi(x,t)$, whose differential spans T' and $\Phi_t \not\equiv 0$. Either $\Phi_t(0,0) \neq 0$ or $\Phi_t(0,0) = 0$ and $F_0 = \{(0,0)\}$. In any of these cases we obtain that the one-dimensional Hausdorff measure of the zero set of the restriction of u to a small neighborhood of p is trivial.

Hence it remains to prove property (1) assuming that the full result has been proved for smaller values of n. Since any $(n+1)$-dimensional leaf is a connected open subset of Ω, we can assume that Ω itself is a leaf.

Decompose Σ into its regular and singular parts, $\Sigma = \Sigma_r \cup \Sigma_s$. The dimension of Σ_s is $\leq n-1$ and then it follows that $\Omega' := \Omega \backslash \Sigma_s$ is open, connected, and that $\mathcal{H}^n(\Sigma_s) = 0$. This observation allows us to assume from the outset that Σ is an embedded, real-analytic hypersurface of Ω. Denote by $\iota : \mathbb{C}T^*\Omega|_\Sigma \longrightarrow \mathbb{C}T^*\Sigma$ the pullback map, let $\mathcal{N} = \iota(T'|_\Sigma)$ and

$$\Sigma^* = \{p \in \Sigma : \dim \mathcal{N}_p = 1\}.$$

Since any component of Σ cannot be a leaf it follows that $\Sigma \backslash \Sigma^*$ is an analytic subset of Σ of dimension $\leq n-1$ and consequently has trivial n-dimensional Hausdorff measure. Any point $p_0 \in \Sigma^*$ is the center of a system of coordinates $(U_0; x, t_1, \dots, t_n)$ for which all properties described at the beginning of the proof of Theorem III.6.3 hold and that

$$U_0 \cap \Sigma = \{t_n = 0\}.$$

We make the following claim:

(γ) *If v is a C^1-approximate solution that vanishes on a nonempty open subset of Ω, then v vanishes identically.*

PROOF OF (γ). Let p_l be a sequence of points in Ω, $p_l \to p$ such that v vanishes identically in a neighborhood of each p_l. If $p \notin \Sigma$ then v vanishes identically near p since $\mathcal{V}$ is an elliptic structure in $\Omega\backslash\Sigma$. Suppose now that $p \in \Sigma$ and take a coordinate system $(V, y_1, \ldots, y_{n+1})$, $V = \{|y| < r\}$, centered at p such that $\Sigma \cap V = \{y_1 = 0\}$. Since $\mathcal{V}$ is an elliptic structure in $\{y \in V : y_1 \neq 0\}$ and since $p_l \in V$ for some l it follows necessarily that v vanishes identically on one of the sides $y_1 > 0$ or $y_1 < 0$. Suppose that the first case occurs and take $y^* \in \Sigma^* \cap V$. By Theorem III.6.3 it follows that v vanishes identically in a full neighborhood of y^* and consequently in the whole component $y_1 < 0$.

Since u is a C^1-approximate solution on $\Omega\backslash\Sigma$ with respect to an elliptic structure (with $m = 1$, $\nu = n - 1$ according to the notation of Chapter I), which does not vanish identically on any component of $\Omega\backslash\Sigma$, we have $\mathcal{H}^n(S \cap (\Omega\backslash\Sigma)) = 0$. Hence it suffices to show that $\mathcal{H}^n(S \cap \Sigma^*) = 0$ or, for that matter, that

$$\mathcal{H}^n(S \cap (U_0 \cap \Sigma)) = 0 \tag{III.15}$$

according to the preceding notation.

The differential of $Z|_{t_n=0}$ defines a locally integrable structure on $U_0 \cap \Sigma$ with $m = 1$. Moreover, the restriction of u to $U_0 \cap \Sigma$ is a C^1-approximate solution for this structure, which is furthermore not identically zero on any n-dimensional leaf thanks to Theorem III.6.3 and (γ). If such a structure is not real, then (III.15) holds by the induction hypothesis. Suppose now that this structure is real, which is the same as saying that $\Phi|_{t_n=0}$ depends only on x. Taking

$$U_0 \cap \Sigma = \{(x, t') : |x| < \delta, |t'| < \delta\},$$

Gronwall's inequality gives

$$S \cap (U_0 \cap \Sigma) = \{x : |x| < \delta, u(x, 0, 0) = 0\} \times \{t' : |t'| < \delta\}. \tag{III.16}$$

Since moreover Φ_t is not identically zero, there is a line segment $\mathfrak{p}$ through the origin in t-space such that Φ restricted to $(-\delta, \delta) \times \mathfrak{p}$ is not a function of x alone. This means that the differential of the restriction of Z defines a locally integrable structure on $(-\delta, \delta) \times \mathfrak{p}$ which satisfies the hypothesis of Proposition III.6.1. The restriction of u to $(-\delta, \delta) \times \mathfrak{p}$ is a C^1-approximate solution and does not vanish on any nonempty open subset of $(-\delta, \delta) \times \mathfrak{p}$, once more thanks to Theorem III.6.3 and (γ). But then we can apply Proposition III.6.1 in order to infer that the Lebesgue measure of $\{x : |x| < \delta, u(x, 0, 0) = 0\}$ is zero, which according to (III.16) gives (III.15).

The proof of Theorem III.6.4 is now complete. □

Corollary III.6.5. *Let u be a C^1-approximate solution on Ω. Then $d_0'u/u$ which can be regarded as a section of $\mathbb{C}T^*\Omega/T'$ with L^∞ coefficients, is d_1'-closed.*

Proof. We can of course assume that u is not identically equal to zero. By [**HaP**] (corollary 2.4) in conjunction with Theorem III.6.4 (1) it follows that $d_1'(d_0'u/u) = 0$ on the union of all $(n+1)$-dimensional leaves. Now let p be a point belonging to an n-dimensional leaf; we have to show that $d_1'(d_0'u/u) = 0$ in a neighborhood of p.

We can find a coordinate system $(U; x, t_1, \ldots, t_n)$ centered at p, with $U = \{(x,t) : |x| < \delta, |t| < \delta\}$ such that T' is spanned, over U, by the differential of the function $Z(x,t) = x + i\Phi(x,t)$, where Φ is real-valued, real-analytic, and satisfies (III.13). We necessarily have $\Phi(0,\cdot) = 0$, since p belongs to an n-dimensional leaf. We must analyze two cases: either (i) $\Phi \equiv 0$ or else, by taking $\delta > 0$ small, (ii) $\Phi(x,\cdot) \neq 0$ for all $x \in (-\delta, \delta)$, $x \neq 0$.

Under case (i) the complex d' over U equals the complex d_t, and our claim can easily be checked. We consider case (ii). Since $\{(x,t) \in U : x > 0\}$ and $\{(x,t) \in U : x < 0\}$ are contained in $(n+1)$-dimensional leaves, taking into account the representation of the operator d_0' given by

$$Lu = \sum_{j=1}^{n} L_j u \, \mathrm{d}t_j,$$

it suffices to show that

$$L\chi_\epsilon \wedge Lu/u \to 0 \quad \text{in } L^1(U, \Lambda^2((\mathbb{C}T^*\Omega/T')), \tag{III.17}$$

where $\chi_\epsilon \in \mathcal{C}^\infty(\mathbb{R})$ depends only on x and satisfies $\chi_\epsilon = 1$ for $|x| > \epsilon$, $\chi_\epsilon = 0$ for $|x| \leq \epsilon/2$, and $|\chi_\epsilon'| \leq C\epsilon^{-1}$. Now

$$L\chi_\epsilon = -i\chi_\epsilon'(x)\frac{\mathrm{d}_t\Phi(x,t)}{Z_x(x,t)}$$

and thus, since $\mathrm{d}_t\Phi(x,t) = O(|x|)$, the L^∞ norm of $L\chi_\epsilon$ is bounded uniformly in ϵ. From this (III.17) follows immediately, and the proof is complete. □

III.7 Further applications of Sussmann's orbits

In this chapter, the focus has been on the applications of Sussmann's orbits to a variety of questions on unique continuation. However, Sussmann's orbits have also been applied to several problems in involutive structures. In particular, it is now known that many properties of CR functions propagate along orbits. Here we will very briefly mention some of the results that involved orbits.

As mentioned in Section III.3, orbits were used by Tumanov ([**Tu1**]) and Baouendi and Rothschild ([**BR**]) to prove necessary and sufficient conditions for the holomorphic extension of all CR functions into wedges. In [**Tr**], Trepreau showed that the wedge extendability of continuous CR functions propagates along the orbits of a CR manifold in $\mathbb{C}^n$. Another proof of this result appeared in [**Jo2**]. In the same paper [**Tr**], Trepreau also described the variation of the direction of extendability along orbits by proving that the wave front set of a CR function is a union of orbits in the conormal bundle with respect to a natural CR structure there. These results were generalized by Tumanov in [**Tu2**], where he showed that CR-extendability of a CR function on a generic CR manifold $\mathcal{M}$ in $\mathbb{C}^n$ propagates along orbits. A CR function on $\mathcal{M}$ is said to be *CR-extendable* at $p \in \mathcal{M}$ if it extends to be CR on some manifold with boundary attached to $\mathcal{M}$ near p. Moreover, Tumanov described the variation of the directions of CR extendability in terms of a certain differential geometric partial connection and the corresponding parallel displacement in a quotient bundle of the normal bundle of $\mathcal{M}$. This description is dual to that of Trepreau. Merker ([**Mer1**]) gave a simplified presentation of Tumanov's connection and used it to prove that if $\mathcal{M}$ is a generic CR manifold consisting of a single orbit, then each continuous CR function on $\mathcal{M}$ is wedge-extendable at each point of $\mathcal{M}$. This result was also obtained by [**Jo2**] independently using a different proof. In Joricke's approach, the key idea is the deformation of the manifold $\mathcal{M}$ so as to produce minimal points in such a way that all points outside a truncated cone C (in suitable local coordinates on $\mathcal{M}$) are left fixed. The cone C has an axis in $\Re\mathcal{V}$, a vertex p, and the deformed CR manifold is minimal at the central point p.

The concept of Sussmann's orbit has been used to characterize the first-order linear partial differential operators which are locally solvable (see [**T5**]). Orbits were used by Hounie ([**Ho1**] and [**Ho2**]) in his work on globally solvable and globally hypoelliptic complex vector fields on manifolds.

For tube structures, Hounie and Tavares [**HT5**] have given a necessary and sufficient condition for the validity of Hartog's phenomenon for solutions in terms of the behavior of orbits. Orbits have also been relevant in the study of removable singularities, as shown in numerous works including [**HT2**], [**Jo1**], [**Mer2**], [**KR**], [**MP1**], [**MP2**], and [**MP3**]. The paper [**CR1**] of Chirka and Rea uses orbits to study the regularity of CR mappings. For earlier works exhibiting orbits as propagators of support and singularities, see [**DH**], [**HS**], and [**Z**].

Notes

As indicated in the introduction, the concept of orbits and its basic properties were presented in Sussmann's paper [**Su**]. Lemma III.1.8 and some of its consequences appeared in [**BM**]. The theorems on the strong unique continuation for L^1_{loc} solutions were proved in [**BH2**]. The propagation of support for solutions and the link to the uniqueness for the noncharacteristic Cauchy problem has been studied by many mathematicians: Strauss and Treves in [**ST**], and Cardoso and Hounie in [**CH**] studied the Cauchy problem for a single smooth vector field satisfying the solvability condition $\mathcal{P}$ of Nirenberg and Treves. Hunt, Polking and Strauss ([**HPS**]) considered the uniqueness problem for a hypersurface in a complex manifold. Hunt ([**Hu**]) proved uniqueness for the noncharacteristic Cauchy problem for locally realizable CR manifolds under some hypotheses on the Levi form. Treves proved his theorem on propagation of support along orbits by using the uniqueness theorem for the noncharacteristic Cauchy problem in locally integrable structures—a consequence of the Baouendi–Treves approximation formula.

The description of the zero set of approximate solutions in real-analytic structures where $m = 1$ and for certain planar vector fields follows Cordaro's paper ([**Cor2**]).

For additional references to the concept of orbits and their applications, we mention the books by Baouendi, Ebenfelt and Rothschild ([**BER**]), Treves ([**T5**]), and the manuscript [**MP3**] by Merker and Porten.