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LOWER BOUNDS ON THE NUMBER OF POINTS IN THE LOWER SPECTRUM OF ELLIPTIC OPERATORS

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Let G denote an unbounded domain of Euclidean m-space E^m with regular boundary, and let L be a self-adjoint operator generated in $L^2(G)$ by a second order elliptic expression. We denote by $\mathrm{S}(L)$ the spectrum of L, by μ the least point of the essential spectrum $\mathrm{S}_e(L)$ and by $\mathrm{N}(L)$ the number of *bound states* of L; that is, the number of points in $(-\infty, \mu) \cap \mathrm{S}(L)$. There are many results in the literature dealing with the localization, significance and properties of μ, of $\mathrm{S}_e(L)$ and of $(-\infty, \mu) \cap \mathrm{S}(L)$, with most of the emphasis on the cases where $G = E^m$ or G is the exterior of a closed surface in E^m. We refer the reader to the books by Glazman [**12**], Schechter [**19**], Reed and Simon [**18**], and Faris [**9**], where extensive references are also found. We observe that it is often possible to give upper estimates for $\mathrm{N}(L)$ in terms of coefficient norms (see, for example, the recent paper by Cwikel [**7**], and the references therein). We further note that by observing the behaviour of G and of L at infinity, it is often easy to estimate μ from below, and, by the spectral theorem, to estimate from above the first n points $\lambda_1, \ldots, \lambda_n$ in $(-\infty, \mu) \cap \mathrm{S}(L)$ for some $n \geqq 1$. Assume that such an estimate has been obtained for λ_1 (the simplest case). It seems reasonable to expect that if μ/λ_1 is "large" then so ought to be $\mathrm{N}(L)$. It is the purpose of this paper to obtain calculable lower bounds on $\mathrm{N}(L)$ which will show that, under conditions to be specified below, this is the case. In general, no lower estimate for $\mathrm{N}(L)$ can remain valid if in the problem we allow finite regular, but otherwise arbitrary perturbations. However, since our bounds will depend only on estimates for μ and $\lambda_1, \ldots, \lambda_n$, and the coefficients, then they will continue to remain valid after any perturbation which does not affect these estimates, regardless of the effect such a perturbation may have on the other parts of the spectrum of L. Because of this generality, our estimates will usually be worse than those obtained by considering only a calculable specific case from the start.

Our main tool will be an extension of a fundamental result of Payne, Polya, and Weinberger [**15**] for Laplace's equation in the bounded subdomains of E^2. We shall also make use of variational arguments and of oscillation theory.

Finally, we observe that μ can be connected to the oscillation properties of L and that, under suitable hypotheses (specified in [**2**], [**17**], [**16**]), $\mathrm{N}(L)$ is infinite if and only if $Lu - \mu u = 0$ is oscillatory. Consequently, we shall always assume in the sequel that such situations where $\mathrm{N}(L)$ is known to be infinite are excluded from consideration.

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Let $x = (x_1, \ldots, x_m)$ denote a point of E^m, $m \geqq 3$, and let D_i denote differentiation with respect to x_i for $i = 1, \ldots, m$. Let Q denote any domain in E^m and write $\| \ \|_p$ for the $L^p(Q)$ norm, $(\ , \ )(Q)$ for the $L^2(Q)$ inner product. We define, for $u \in C_0^\infty(G)$, the elliptic expression:

$$\mathscr{L}u = -\sum_{i,j=1}^{m} D_i(a_{ij}D_ju) + qu,$$

where we shall always at least assume that: (1) the coefficients of $\mathscr{L}$ are real; (2) $a_{ij} \in C^1(\bar{G})$, $a_{ij} = a_{ji}$; (3) if $\alpha(x)$, $\beta(x)$ denote the smallest and largest eigenvalues of (a_{ij}) then there exist constants N, M such that $0 < N \leqq \alpha(x) \leqq \beta(x) < M$ for all x in G; (4) $q \in C(\bar{G})$. These assumptions are made so that the only singularity in the problem comes from the unboundedness of G, and therefore the standard oscillation theory results may be directly applied. A considerable weakening of these assumptions would imply the need to introduce a more general oscillation theory which would allow for oscillations at finite points of $\bar{G}$, as well as at infinity. We do not pursue this approach here, but we remark that several of the results (for example Theorem 1) may clearly be established by the same proof under weaker conditions then those stated above. We may assume that $\mathscr{L}$ is bounded below, and we let L denote its Friedrichs extension. This is the only extension considered in the sequel. If Q is any subdomain of G, then we denote by $L(Q)$ the extension of $\mathscr{L}$ restricted to $C_0^\infty(Q)$. If Q is obvious from the context we write L for $L(Q)$, $\| \ \|_p$ for $\| \ \|_p(Q)$, etc. Finally, we shall use ϵ to denote a generic positive constant, whose value may vary within the same proof, and we shall set $q_+ = \max(q, 0)$, $q_- = \max(-q, 0)$.

The first theorem is an extension of a result of [**15**], (where only the case of a bounded domain, $m = 2$, $a_{ij} = \delta_{ij}$, $q = 0$ was considered) and of Thompson [**22**] (where the results of [**15**] were extended to the case $m \geqq 2$). We emphasize that, as in [**15**], no other restrictions are placed on G.

THEOREM 1. *Assume that $\lambda_i \in \mathrm{S}(L)$ for $i = 1, \ldots, n$ and that $\mathscr{K}\|q_-\|_{m/2} < N$ where $\mathscr{K} = (m-1)^2(m-2)^{-2}m^{-1}$. Define for $\epsilon \geqq 0$ the interval*

$$I(\epsilon) = \left(-\infty, \lambda_n + \frac{4M}{nm}\{\textstyle\sum_1^n \lambda_i\}(N - \mathscr{K}\|q_-\|_{m/2})^{-1} + \epsilon\right].$$

Then $I(0) \cap \mathrm{S}(L)$ has at least $n + 1$ points.

Proof. We first note that by the Gagliardo-Niremberg Theorem [**10**, p. 24] for any $\phi \in C_0^\infty(G)$ we have:

$$(1) \quad (q_-\phi, \phi) \leqq \mathscr{K}N^{-1}\|q_-\|_{m/2}(\phi, -\textstyle\sum_{p,l} D_p(a_{p,l}D_l\phi))$$

Assume next that Q is a bounded regular subdomain of G, $\bar{Q} \subset G$. Then $L(Q)$ has a discrete spectrum and we let $\{u_i\}_1^n$ denote the first n normalized eigenfunctions corresponding to the first eigenvalues $\lambda_1, \ldots, \lambda_n$. Following [**15**] we set

$$\varphi_i = fu_i - \sum_{k=1}^{n} (fu_i, u_k)u_k \quad \text{for } i = 1, \ldots, n,$$

where f is any regular function. Clearly $\phi_i \in \overline{\text{Dom}(L)}$ in the Friedrichs norm, and by the spectral theorem we obtain:

$$(\phi_i, \phi_i)\lambda_{n+1} \leqq (\phi_i, L\phi_i) = (fu_i, L(fu_i)) - \textstyle\sum_k (fu_i, u_k)(u_k, L(fu_i)) =$$
$$(u_i^2, \textstyle\sum_{p,l} a_{p,l} D_p f D_l f) + \lambda_i (fu_i, fu_i) - \textstyle\sum_k (fu_i, u_k)(u_k, L(fu_i)).$$

Consequently,

$$(2) \qquad (\phi_i, \phi_i)(\lambda_{n+1} - \lambda_n) \leqq (u_i^2, \textstyle\sum_{p,l} a_{p,l} D_p f D_l f) - \textstyle\sum_k (fu_i, u_k)$$
$$\times (u_k, -\textstyle\sum D_p(a_{p,l} D_l f) u_i) + 2\textstyle\sum_k (fu_i, u_k)(u_k, \textstyle\sum_{p,l} a_{p,l} D_p f D_l u_i).$$

Summing over i we find that the sum of the last two terms on the right hand side of (2) is zero by symmetry. Consequently,

$$(3) \qquad \lambda_{n+1} - \lambda_n \leqq \textstyle\sum_1^n (u_i^2, \textstyle\sum_{p,l} a_{p,l} D_p f D_l f) / \textstyle\sum_1^n (\phi_i, \phi_i)$$

Clearly there exists a ξ such that

$$\textstyle\sum_1^n (D_\xi u_i, D_\xi u_i) \leqq m^{-1} \textstyle\sum_1^n (u_i, -\Delta u_i).$$

Without loss of generality, let $\xi = 1$ and set $f = x_1$. As in [**15**] we observe the inequality

$$mn^2 \leqq 4\textstyle\sum_1^n (\phi_i, \phi_i) \textstyle\sum_1^n (u_i, -\Delta u_i).$$

It follows from (3) that

$$(4) \qquad \lambda_{n+1} - \lambda_n \leqq 4Mm^{-1}n^{-1} \textstyle\sum_1^n (u_i, -\Delta u_i)$$
$$\leqq 4M(Nnm)^{-1} \textstyle\sum_1^n (u_i, -\textstyle\sum_{p,l} D_p(a_{pl} D_l u_i)).$$

By (1) and a limit argument we find

$$(u_i, -\textstyle\sum_{p,l} D_p(a_{pl} D_l u_i)) \leqq (q_- u_i, u_i) + \lambda_i$$
$$\leqq \mathscr{K} N^{-1} \|q_-\|_{m/2} (u_i, -\textstyle\sum_{p,l} D_p(a_{pl} D_l u_i)) + \lambda_i.$$

Substituting into (4) gives

$$(5) \qquad \lambda_{n+1} - \lambda_n \leqq 4M(\textstyle\sum_1^n \lambda_i) n^{-1} m^{-1} (N - \mathscr{K} \|q_-\|_{m/2})^{-1}.$$

Next, we may assume that $\lambda_1, \ldots, \lambda_n$ are the first n points of $S(L)$ in G and that $S_e(L) \cap I(0) = \emptyset$. Let $u_1, \ldots, u_n$ denote the associated normalized eigenvectors and let $\epsilon \geqq 0$ be given. Choose functions φ_i, such that $\varphi_i \in C_0^\infty$, $(\varphi_i - u_i, L(\varphi_i - u_i)) < \epsilon$ for $i = 1, \ldots, n$. By orthonormalizing (in $L^2(G)$) the set $\{\varphi_i\}$ we construct the set $\{v_i\}$ such that $(v_i, Lv_i) < \lambda_i + \epsilon$, and $(v_i, Lv_j) < \epsilon$ for $i \neq j$. Let Q denote any bounded regular subdomain such that $\bigcup_1^n \text{supp}(v_i) \subset Q \subset \bar{Q} \subset G$. We apply the spectral theorem to the subspace generated by the v_i and conclude that $L(Q)$ has eigenvalues e_i such that

$e_i \leqq \lambda_i + \epsilon$ for $i = 1, \ldots, n$. By (5) and the min-max principle we conclude that L has at least $n + 1$ eigenvalues in $I(\epsilon)$. Since ϵ is arbitrary, the result follows.

We observe that the calculations leading to (3) are somewhat different from the analogous calculations given in [**15**], although the underlying ideas are the same.

COROLLARY 1. *Assume that the spectrum of $L(G)$ is discrete and let $\lambda_1, \ldots, \lambda_n$ denote the first n eigenvalues. Then*

$$\lambda_{n+1} \leqq \lambda_n + 4Mn^{-1}m^{-1}\left(\textstyle\sum_1^n \lambda_i\right)(N - \mathscr{K}||q_-||_{m/2})^{-1}. \tag{6}$$

Furthermore, if $q_- = 0$ then (6) *is also valid for $m = 2$.*

We remark on the differences between (6) and the analogous estimate given in [**15**]. It is clear that q_- must appear in the estimate, since if q_- is sufficiently large then λ_1 could vanish and λ_2/λ_1 could not be bounded. Similarily if $q_- = 0$, but $a_{ij} \neq \delta_{ij}$, then calculations for a related problem in a paper by Gentry and Banks [**11**] indicate that if MN^{-1} is chosen large then λ_2/λ_1 can be made large by suitable choices of (a_{ij}).

By the min-max principle we have the following useful corollary.

COROLLARY 2. *Let $\lambda_1', \ldots, \lambda_n'$ denote upper estimates on the first n eigenvalues of $L(Q)$ with $Q \subset G$, and let δ denote a lower estimate on μ. Then* (6) *holds with $\lambda_1', \ldots, \lambda_n'$ in place of $\lambda_1, \ldots, \lambda_n$. In particular,*

$$N(L) \geqq J\{\ln(\delta/\lambda_1')(\ln(4Mm^{-1})N - ||q_-||_{m/2}\mathscr{K})^{-1} + 1))^{-1}\}, \tag{7}$$

where $J(x)$ denotes the integer part of x if x is nonnegative and is zero otherwise.

Since it is obvious that the addition of a constant multiple of the identity to L does not change $N(L)$, we have the following improvement on Corollary 2 (with similar improvements on the previous results):

COROLLARY 3. *Let the above conditions hold. Then*

$$N(L) \geqq \sup_{\alpha \in E} J(\ln((\mu + \alpha)/(\lambda_1 + \alpha))\{\ln\{4Mm^{-1}(N - \mathscr{K} \\ ||(\alpha + q)_-||_{m/2})^{-1} + 1\}\}^{-1}). \tag{8}$$

Corollary 3 is useful for cases where $N - ||q_-||_{m/2}\mathscr{K} \leqq 0$ but, for some α, $N - ||(\alpha + q)_-||_{m/2}\mathscr{K} > 0$. For simplicity of notation, we shall however assume in the sequel that the optimum αI has already been added to L and we will use formula (7) as opposed to the more cumbersome (8).

We next recall some of the terminology of oscillation theory, [**2**], [**13**], [**16**], [**21**], [**20**]. The operator L is termed *oscillatory in G* if and only if given any sphere R there exists a bounded domain N (called a *nodal domain*) in $G \cap \{x| \, |x| > R\}$ such that $S(L(N)) \cap (-\infty, 0] \neq \emptyset$. We define $\alpha = \sup\{k| \quad L - k$ does not oscillate$\}$ to be the oscillation constant of L in G. In view of our assumptions on the coefficients of L it is possible, by means of

[16] and [1, p. 129], to extend the decomposition theorem of Glazman and conclude that $\alpha = \mu$. These observations enable us to use the known oscillation and nonoscillation criteria to estimate $N(L)$. We simplify somewhat the presentation by restricting our considerations to the operator L_1 generated by

$$\mathscr{L}_1 u = -\Delta u + qu.$$

The analogous results for L will be obvious from the presentation and Corollaries 2 and 3.

THEOREM 2. *Assume that G contains the set $\{x \mid |x| \geqq R_0\}$ for some R_0, and that there exists a $\delta > 0$ such that for all $\epsilon > 0$*

(a) *for some $\epsilon_0 \in [0, 1)$, $g \in L^{m/2}(G)$, and all x near ∞*

$$q(x) - \delta + \epsilon \geqq -(m-2)^2 \epsilon_0 4^{-1} |x|^{-2} + g(x);$$

(if $g(x) = 0$ we may set $\epsilon_0 = 1$);

(b) $\lim_{R \to \infty} \{\int_{|R_0 < |x| < R|} |x|^{2-m} (q - \delta - \epsilon) dG + \mu(U)(m-2)^{-2} 4^{-1} \ln R\} = -\infty$,

where $\mu(U)$ is the measure of the surface of the unit sphere. Define E by the expression

$$E = \inf_i \inf_{R \geqq R_0} (\lambda_i(R) + \int_{|R_0 < |x| < R|} q(x) \omega^2(x, R) dG)$$

where $\lambda_i(R)$ denotes the i^{th} eigenvalue of $-\Delta$ in $\{R_0 < |x| < R\}$ and $\omega(x, R)$ is the associated normalized eigenfunction. Then $\delta = \mu$ and furthermore,

$$N(L_1) \geqq J\{\ln(\delta/E)(\ln\{4m^{-1}(1 - \|q_-\|_{m/2} \mathscr{K})^{-1} + 1\})^{-1}\}. \tag{9}$$

Proof. Condition (a) is an extension of Kneser's classical result which implies that $L_1 - \delta + \epsilon$ is nonoscillatory [3] while condition (b) implies that $L_1 - \delta - \epsilon$ oscillates [14]. Consequently, $\delta = \mu$ for L_1. Let $\epsilon > 0$ be given. By the min-max principle it follows that for some $R_1 > R$ we have

$$S(L_1(\{R_0 < |x| < R_1\})) \cap (-\infty, E + \epsilon) \neq \emptyset.$$

Consequently, $S(L_1) \cap (-\infty, E + \epsilon) \neq \emptyset$. Letting $\epsilon \to 0$ shows that $S(L_1) \cap (-\infty, E] \neq \emptyset$. Corollary 2 then gives the desired result.

COROLLARY 4. *Theorem 2 remains valid if condition (a) is replaced by*

$$\lim_{r \to \infty} r \int_r^\infty h_-(t) dt > -1/4$$

where

$$h(t) = \min_{|x|=t}(q(x)) + (m-1)(m-3)/4t^2 - \delta + \epsilon.$$

This corollary shows the nonoscillation of $L_1 - \delta + \epsilon$ at ∞ by replacing the Kneser criterion by an extension of Hille's Theorem [20]. By restricting, for simplicity, our attention to the first eigenvalue of $-\Delta$, we obtain:

THEOREM 3. *Assume that G contains the infinite rectangle R given by*

$$R = \{x | x_1 > 0, \quad 0 < x_i < \alpha, \quad i = 2, \ldots, m\},$$

and that for some r sufficiently large $(G - R) \cap \{x| \quad |x| > r\} = \emptyset$. *Assume further that for all* $\epsilon > 0$

(a) *for all* $|x|$ *sufficiently large*

$$q(x) - \delta + \epsilon + (m - 1)\pi^2\alpha^{-2} \geqq -4^{-1}x_1^{-2};$$

(b) *the ordinary differential equation*

$$(10) \quad -y''(t) + y(t)(C(t) + \pi^2\alpha^{-2}(m - 1)) = 0$$

is oscillatory, where

$$C(t) = 2^{m-1}\alpha^{1-m}\int_{x\in R, x_1 = t} (q(x) - \delta - \epsilon) \prod_2^m \sin^2(\pi\alpha^{-1}x_i)dx_2, \ldots, dx_m.$$

Define E by

$$E = \inf_{t>0}\{(m - 1)\pi^2\alpha^{-2} + \pi^2 t^{-2} + 2^m\alpha^{1-m}t^{-1}\int_{R\cap\{x|0<x_1<t\}} q(x) \times \prod_2^m \sin^2(\pi x_i\alpha^{-1}) \times \sin^2(\pi x_1 t^{-1})dx_1, \ldots, dx_m\}.$$

Then $\delta = \mu$, *and* (9) *holds.*

Proof. Following the procedure of [**5**] we observe that condition (a) implies that $L_1 - \delta + \epsilon$ does not oscillate. Indeed, this is a consequence of making the explicit choice $\omega = x_1^{1/2}\prod_2^m \sin(\pi\alpha^{-1}x_i)$ in the well known formula

$$(11) \quad \int_G \varphi(-\Delta\varphi) \geqq \int_G \varphi^2(-\Delta\omega)\omega^{-1}$$

which is valid for any $\varphi \in C_0^\infty(G)$ and any $\omega \in C^2(G)$, $\omega > 0$ in G. Similarily, we observe that if (10) oscillates then for any θ there is a $\zeta > \theta$ and a solution ψ of (10) with $\psi(\zeta) = \psi(\theta) = 0$. We set $\psi = 0$ outside (θ, ζ) and define $\varphi(x) = \psi(x_1)\prod_2^m \sin(\pi\alpha^{-1}x_i)$. A direct calculation shows that condition (b) implies that $(\varphi, (L_1 - \delta - \epsilon)\varphi) \leqq 0$ and, consequently, that $L_1 - \delta - \epsilon$ oscillates. Conditions (a) and (b) thus imply that $\delta = \mu$. Finally we observe that E is an upper bound on the first point of the spectrum of L_1, obtained by choosing $\varphi(x) = \sin(\pi x_1 t^{-1}) \prod_2^m \sin(\pi x_i\alpha^{-1})$ in the functional $(L_1\varphi, \varphi)/(\varphi, \varphi)$. The validity of (9) then follows from Corollary 2.

THEOREM 4. *Assume that G contains the rectangles*

$$R_k = \{x| \quad \alpha_j^k < x_j < \alpha_j^k + t_k, \quad j = 1, \ldots, m\},$$

where $k = 1, \ldots, \infty$, *and for any* r, $R_k \subset \{x| \quad |x| > r\}$ *for k sufficiently large. Define*

$$A(t, \eta, \epsilon, k) = m\pi^2 t^{-2} + (2/t)^m\int (q - \eta - \epsilon)\prod_1^m \sin^2(\pi(x_j - \alpha_j^k)t^{-1})dG,$$

the integral being taken over $R_k \cap \{x| \quad 0 < x_j - \alpha_j^k < t\}$, *and let* $\delta > 0$ *be such that for all* $\epsilon > 0$

(a) *there exists an integer* R *and a function* ω, *positive in* $G \cap \{x \mid |x| > R\}$ *such that* $q - \delta + \epsilon \geqq +\Delta\omega/\omega$ *for* $|x| > R$;

(b) $\liminf_{k\to\infty} A(t_k, \delta, \epsilon, k) < 0$.

Define $E = \inf_k \inf_{0<t\leqq t_k}(A(t, 0, 0, k))$. *Then* $\delta = \mu$, *and* (9) *holds.*

Proof. We first note that condition (a) is sufficient for the nonoscillation of $L_1 - \delta + \epsilon$ as a consequence of inequality (11). Next, we observe that by the spectral theorem, see [**21**], condition (b) implies that there exist arbitrarily large values of k such that $\mathrm{S}(L_1 - \delta - \epsilon(R_k)) \cap (-\infty, 0] \neq \emptyset$. It follows that $\delta = \mu$. Finally, we observe that once again E gives an upper bound on the first point of $\mathrm{S}(L_1)$, this time by choosing $\varphi = \prod_{j=1}^m \sin(\pi(x_j - \alpha_j^k)t^{-1})$ in $(\varphi, L_1\varphi)/(\varphi, \varphi)$. The conclusion then follows from Corollary 2.

COROLLARY 5. *Let condition* (b) *be deleted from the respective statements of Theorems* 2, 3, 4. *Then* $\mu \geqq \delta$ *and* (9) *still holds.*

Observe that positive radial functions can always be used in condition (a) of Theorem 4, but other types of functions may sometimes be used so as to take advantage of the shape of G near infinity. This was done in Theorem 3, where it was postulated that G was rectangular near infinity. Note also that any of the well known oscillation criteria may now be used in Theorem 3. Clearly the rectangle of Theorem 3 can be substituted by a cone if obvious changes are made in the statement of the theorem.

We consider the following simple example to illustrate the results obtained. Assume that $m = 3$ and that $G = E^3$. Let q be regular, nonpositive, with $q = p - 1$ and $p \geqq 0$. Further, let q approach zero at infinity. Assume that for some positive integer S we have:

$$2\int_{|x|<R} p(x)\omega^2(x, R)dG \leqq \exp(-S\ln(7/3)),$$

where ω denotes the first normalized eigenfunction of the Laplacian in $\{x \mid |x| < R\}$, and $R^2 \geqq (9\pi^2 2^{-1}\exp(S\ln(7/3))$. In this case a simple variation in Theorem 2 applied to $L_1 + I$ leads to the estimate $\mathrm{N}(L_1) \geqq S$.

By the methods developed above it may be possible in some cases to obtain better numerical results by using the estimates of Brands [**6**] and DeVries [**8**] for the calculation of the lower eigenvalues.

Finally, we observe that none of the above results took advantage of q_+. Better estimates should be obtainable by suitably using properties of q_+, but how this is to be done is not clear.

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