EXPANSION OF AN E-FUNCTION IN A SERIES OF PRODUCTS OF E-FUNCTIONS

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1. Introductory. The formula to be established is

$$\Gamma(\delta-\alpha)\Gamma(\delta-\beta)\Gamma(\alpha+\beta-\delta)\,[\Gamma(\alpha)\Gamma(\beta)]^{-1}E(\alpha,\,\beta,\,\gamma:\delta:z)$$
$$=\sum_{r=0}^{\infty}\frac{z^{-2r}}{r!\,\Gamma(\gamma+r)}\,E(\gamma+r,\,\alpha+\beta-\delta+r::z)E\begin{pmatrix}\gamma+r,\ \delta-\alpha+r,\ \delta-\beta+r:z\\ \delta+r\end{pmatrix},\quad \ldots\ldots\ldots(1)$$

where $|\operatorname{amp} z|<\pi$, $z\neq 0$, $R(\alpha+\beta)>R(\delta)>R(\alpha)>0$, $R(\delta-\beta)>0$.

In proving (1) use will be m ade of the two following formulae.

If $p\geqq q+1$, $|\operatorname{amp} z|<\pi$, $z\neq 0$, $R(\alpha_n)>0$, where $n=1,2,\ldots,p-1$, $R(\rho_n-\alpha_n)>0$, where $n=1,2,\ldots,q$ [**1**, p. 352],

$$E(p\,;\ \alpha_n:q\,;\ \rho_s:z)$$
$$=\Gamma(\alpha_p)\left[\prod_{n=1}^{q}\Gamma(\rho_n-\alpha_n)\right]^{-1}\prod_{n=1}^{q}\int_0^1\lambda_n^{\alpha_n-1}(1-\lambda_n)^{\rho_n-\alpha_n-1}\,d\lambda_n$$
$$\times\prod_{n=q+1}^{p-2}\int_0^{\infty}e^{-\lambda_n}\lambda_n^{\alpha_n-1}\,d\lambda_n\int_0^{\infty}e^{-\lambda_{p-1}}\,\lambda_{p-1}^{\alpha_{p-1}-1}(1+\lambda_1\lambda_2\ldots\lambda_{p-1}/z)^{-\alpha_p}\,d\lambda_{p-1}.\quad\ldots\ldots(2)$$

If $R(\delta-\alpha-\beta)>0$, $|\operatorname{amp} z|<\pi$, $z\neq 0$ [**2**, p. 171],

$$\int_0^{\infty}e^{-\lambda}\lambda^{\delta-\alpha-\beta-1}(\lambda+z)^{-\gamma}E(\alpha,\,\beta,\,\gamma:\delta:\lambda+z)\,d\lambda=\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\delta-\alpha-\beta)}{\Gamma(\delta-\alpha)\Gamma(\delta-\beta)}\,z^{-\gamma}E(\delta-\alpha,\,\delta-\beta,\,\gamma:\delta:z).$$
$$\ldots\ldots(3)$$

2. Proof of the expansion. On substitution for each E-function on the right of (1) from (2), the right-hand side becomes

$$\sum_{r=0}^{\infty}\frac{\Gamma(\gamma+r)z^{-2r}}{r!\,\Gamma(\alpha)}\int_0^{\infty}e^{-\lambda}\lambda^{\alpha+\beta-\delta+r-1}(1+\lambda/z)^{-\gamma-r}\,d\lambda$$
$$\times\int_0^1\xi^{\delta-\alpha+r-1}(1-\xi)^{\alpha-1}\,d\xi\int_0^{\infty}e^{-\mu}\mu^{\delta-\beta+r-1}(1+\xi\mu/z)^{-\gamma-r}\,d\mu.$$

Here interchange the order of integration and summation [**3**, p. 500], and get

$$\frac{\Gamma(\gamma)}{\Gamma(\alpha)}\int_0^{\infty}e^{-\lambda}\lambda^{\alpha+\beta-\delta-1}(1+\lambda/z)^{-\gamma}\,d\lambda\int_0^1\xi^{\delta-\alpha-1}(1-\xi)^{\alpha-1}\,d\xi$$
$$\times\int_0^{\infty}e^{-\mu}\mu^{\delta-\beta-1}(1+\xi\mu/z)^{-\gamma}\left\{1-\frac{\lambda\mu\xi.}{(\lambda+z)\,(\xi\mu+z)}\right\}^{-\gamma}d\mu.$$

Hence the right-hand side is equal to

$$\frac{\Gamma(\gamma)}{\Gamma(\alpha)}z^{\gamma}\int_0^{\infty}e^{-\lambda}\lambda^{\alpha+\beta-\delta-1}(z+\lambda)^{-\gamma}\,d\lambda\int_0^1\xi^{\delta-\alpha-1}(1-\xi)^{\alpha-1}\,d\xi\int_0^{\infty}e^{-\mu}\mu^{\delta-\beta-1}\{1+\xi\mu/(z+\lambda)\}^{-\gamma}\,d\mu$$
$$=z^{\gamma}\int_0^{\infty}e^{-\lambda}\lambda^{\alpha+\beta-\delta-1}(z+\lambda)^{-\gamma}E(\delta-\alpha,\ \delta-\beta,\,\gamma:\delta:z+\lambda)\,d\lambda,$$

by (2). From (3) the result follows.

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Note. In order to ensure that the series summed under the integral signs should be convergent, it is necessary, when $\frac{1}{2}\pi \leqq |\text{amp } z| < \pi$, to revolve the λ and μ axes through acute angles such that $|\text{amp } z - \text{amp } \lambda| \leqq \frac{1}{2}\pi$ and $|\text{amp } z - \text{amp } \mu| \leqq \frac{1}{2}\pi$.

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