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ELIMINATION OF IMAGINARIES IN ORDERED ABELIAN GROUPS WITH BOUNDED REGULAR RANK

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Abstract. In this paper we study elimination of imaginaries in some classes of pure ordered abelian groups. For the class of ordered abelian groups with bounded regular rank (equivalently with finite spines) we obtain weak elimination of imaginaries once we add sorts for the quotient groups Γ/Δ for each definable convex subgroup Δ, and sorts for the quotient groups $\Gamma/(\Delta + \ell\Gamma)$ where Δ is a definable convex subgroup and $\ell \in \mathbb{N}_{\geq 2}$. We refer to these sorts as the *quotient sorts*. For the dp-minimal case we obtain a complete elimination of imaginaries if we also add constants to distinguish the cosets of $\ell\Gamma$ in Γ, where $\ell \in \mathbb{N}_{\geq 2}$.

§1. Introduction. The model theory of ordered abelian groups has been studied since the sixties, and was initiated by Robinson and Zakon in [15] who studied the completions of *regular* ordered abelian groups (see Definition 2.10). Later, the study of the elementary properties of ordered abelian groups was continued by Belegradek in [2] for the class of *poly-regular* ordered abelian groups (see Definition 2.13). Significant achievements on (relative) quantifier elimination, model completion, and definability of convex subgroups were achieved by Schmitt in [16] for the general class of ordered abelian groups. More recently, Cluckers and Halupczok obtained a (relative) quantifier elimination for ordered abelian groups in [3] in a language that is more aligned with Shelah's imaginary expansion than the one introduced by Schmitt.

The model theoretic classification of certain classes of ordered abelian groups is an active research area. Results include: the well-known result of Gurevich–Schmitt that no ordered abelian group has the independence property in [6]; the dp-minimal case characterized by Jahnke, Simon, and Walsberg in [11]; the strongly dependent case independently obtained by Dolich–Goodrick, Farré, and Halevi–Hasson in [4, 5, 7], respectively, and the distal case in [1] due to Aschenbrenner, Chernikov, Gehret, and Ziegler.

The next natural step regarding the model theory of ordered abelian groups was understanding a reasonable language where one will have elimination of imaginaries. The answer to this problem, interesting in its own sake, has a significant impact in clarifying the problem of elimination of imaginaries for henselian valued fields. At the heart of the model theory of henselian valued fields is the well known *Ax-Kochen/Ershov* theorem, that broadly states that the first-order theory of a henselian finitely ramified valued field is completely determined by the first-order theory of its

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residue field and its value group. In a pure henselian valued field, the value group is a pure ordered abelian group and it is interpretable in the structure.

Following the Ax-Kochen principle one can first attempt to solve the problem of elimination of imaginaries for henselian valued fields by following two orthogonal directions:

1. The first one is to make the value group as tame as possible (e.g., to assume that it is definably complete) and to understand the obstacles that the residue field naturally contributes to the problem. This research path was successfully finalized by Hils and Rideau-Kikuchi in [9].
2. Alternatively, one can make the residue field very tame (e.g., algebraically closed) and study the issues that the complexity of the value group brings to the problem. The work in [19] clarifies the picture for the equicharacteristic zero case, and this paper is the first milestone towards the solution.

We use an abstract criterion isolated by Hrushovski in [10] to show the following two results:

THEOREM. *Let* Γ *be an ordered abelian group of bounded regular rank* (*equivalently with finite spines*). *Then* Γ *admits weak-elimination of imaginaries once the* quotient sorts *are added*.

THEOREM. *Let* Γ *be a dp-minimal ordered abelian group. Then* Γ *admits elimination of imaginaries once the* quotient sorts *are added, and we add constants to distinguish the cosets of* $\ell\Gamma$ *in* Γ, *where* $\ell \in \mathbb{N}_{\geq 2}$.

This paper is organized as follows:

1. Section 2: We present the state of model theory of ordered abelian groups and introduce the class of ordered abelian groups with bounded regular rank.
2. Section 3: We characterize the definable end-segments in an ordered abelian group with bounded regular rank and show that they can be coded in the quotient sorts.
3. Section 4: We introduce Hrushovski's theorem to achieve a weak elimination of imaginaries result for the class of ordered abelian groups with bounded regular rank. This criterion requires us to check two conditions: the density of definable types, proved in Proposition 4.4; and the coding of definable types, proved in Proposition 4.2.
4. Section 5: We briefly summarize the main results for pure ordered abelian groups.

The case of direct sums of the integers with the lexicographic order has been done independently by Liccardo in [13], as part of her Ph.D. thesis under D'Aquino. Hils and Mennuni in [8] have independently obtained the result for the regular case.

§2. Preliminaries.

2.1. Elimination of imaginaries. Let T be a first-order theory and let $\mathfrak{M}$ be its monster model. Let $D \subseteq \mathfrak{M}^k$ be some definable set and E some definable equivalence relation over D. The equivalence class $e = a/E$ is said to be an *imaginary element*. Imaginaries in model theory were introduced by Shelah in [18]. Later in [14], Makkai proposed to construct the many sorted structure $\mathfrak{M}^{eq}$, where we add a sort S_E for

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each definable equivalence relation E and a map π_E sending each element to its class. Since then, the model theoretic community has presented and studied imaginary elements in this way and refers to the multi-sorted structure $\mathfrak{M}^{eq}$ as the *imaginary expansion of* $\mathfrak{M}$. We call the sorts S_E *imaginary sorts* while we refer to $\mathfrak{M}$ as the *home sort*.

Any formula $\phi(\mathbf{x}, \mathbf{y})$ induces an equivalence relation in $\mathfrak{M}^{|\mathbf{y}|}$ defined as

$$E_\phi(\mathbf{y}_1, \mathbf{y}_2) \text{ if and only if } \forall \mathbf{x}\big(\phi(\mathbf{x}, \mathbf{y}_1) \leftrightarrow \phi(\mathbf{x}, \mathbf{y}_2)\big).$$

Let $\mathbf{b} \in \mathfrak{M}^{|\mathbf{y}|}$ and $X := \phi(\mathbf{x}, \mathbf{b})$. We call the class $\mathbf{b}/E_\phi$ the *code* of X and denote it as $\ulcorner X \urcorner$. We denote by dcl^{eq} and acl^{eq} the definable closure and the algebraic closure in the expansion $\mathfrak{M}^{eq}$.

DEFINITION 2.1.
1. We say that T has *elimination of imaginaries* if for any imaginary element e there is a tuple a in the home sort such that $e \in \mathrm{dcl}^{eq}(a)$ and $a \in \mathrm{dcl}^{eq}(e)$.
2. We say that T has *weak elimination of imaginaries* if for any imaginary element e there is a tuple a in the home sort such that $e \in \mathrm{dcl}^{eq}(a)$ and $a \in \mathrm{acl}^{eq}(e)$.
3. We say that T *codes finite sets* if for every model $M \models T$ and every finite subset S of M, the code $\ulcorner S \urcorner$ is interdefinable with a tuple of elements in M.

The following is a folklore fact.

FACT 2.2. *Let T be a complete multi-sorted theory. If T has weak elimination of imaginaries and codes finite sets then T eliminates imaginaries.*

2.2. Ordered abelian groups of bounded regular rank. In this section we summarize several results about the classification of ordered abelian groups and their model theoretic behavior. We start by recalling the following folklore fact.

FACT 2.3. *Let $(\Gamma, <, +, 0)$ be a non-trivial ordered abelian group. Then the topology induced by the order in Γ is discrete if and only if Γ has a minimal positive element. In this case we say that Γ is* discrete, *otherwise we say that it is* dense.

The following notions were isolated in the sixties by Robinson and Zakon in [15] to understand some complete extensions of the theory of ordered abelian groups.

DEFINITION 2.4. Let Γ be an ordered abelian group and $n \in \mathbb{N}_{\geq 2}$.
1. Let $\gamma \in \Gamma$. We say that γ is *n-divisible* if there is some $\beta \in \Gamma$ such that $\gamma = n\beta$.
2. We say that Γ is *n-divisible* if every element $\gamma \in \Gamma$ is n-divisible.
3. Γ is said to be *n-regular* if any interval with at least n points contains an n-divisible element.

EXAMPLE 2.5. We include some examples to illustrate the previous definitions.
1. Consider the ordered abelian group $(\mathbb{Z}, +, \leq, 0)$, the elements $2, 4, 6$ are 2-divisible while 1 is not.
2. The groups $(\mathbb{Q}, +, \leq, 0)$ and $(\mathbb{Z}, +, \leq, 0)$ are n-regular for each natural number $n \in \mathbb{N}_{\geq 2}$. The group $(\mathbb{Z} \oplus \mathbb{Z}, \leq_{lex}, +, 0)$, where $\leq_{lex}$ is the lexicographic order, is not 2-regular, because the interval $\big((1, -1), (1, 4)\big) = \{(1, 0), (1, 1), (1, 2), (1, 3)\}$ does not contain a 2-divisible element.

The following definitions were introduced by Schmitt in [16, 17].

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DEFINITION 2.6. We fix an ordered abelian group Γ and $n \in \mathbb{N}_{\geq 2}$. Let $\gamma \in \Gamma$. We define:

- $A(\gamma) =$ the largest convex subgroup of Γ not containing γ.
- $B(\gamma) =$ the smallest convex subgroup of Γ containing γ.
- $C(\gamma) = B(\gamma)/A(\gamma)$.
- $A_n(\gamma) =$ the smallest convex subgroup C of Γ such that $B(\gamma)/C$ is n-regular.
- $B_n(\gamma) =$ the largest convex subgroup C of Γ such that $C/A_n(\gamma)$ is n-regular.

In [16, Chapter 2], Schmitt shows that the groups $A_n(\gamma)$ and $B_n(\gamma)$ are definable in the language of ordered abelian groups $\mathcal{L}_{OAG} = \{+, -, \leq, 0\}$ by a first-order formula using only the parameter γ.

We recall that the set of convex subgroups of an ordered abelian group is totally ordered by inclusion.

DEFINITION 2.7. Let Γ be an ordered abelian group and $n \in \mathbb{N}_{\geq 2}$, we define the *n-regular rank* to be the order type of

$$\big(\{A_n(\gamma) \mid \gamma \in \Gamma \backslash \{0\}\}, \subseteq\big).$$

The n-regular rank of an ordered abelian group Γ is a linear order, and when it is finite we can identify it with its cardinal. In [5], Farré emphasizes that we can characterize it without mentioning the subgroups $A_n(\gamma)$. The following is [5, Remark 2.2].

FACT 2.8. *Let Γ be an ordered abelian group and $n \in \mathbb{N}_{\geq 2}$, then*:

1. *Γ has n-regular rank equal to 0 if and only if $\Gamma = \{0\}$.*
2. *Γ has n-regular rank equal to 1 if and only if Γ is n-regular and not trivial.*
3. *Γ has n-regular rank equal to m if there are $\Delta_0, \dots, \Delta_m$ convex subgroups of Γ, such that*:
   - *$\{0\} = \Delta_0 < \Delta_1 < \cdots < \Delta_m = \Gamma$.*
   - *For each $0 \leq i < m$, the quotient group Δ_{i+1}/Δ_i is n-regular.*
   - *The quotient group Δ_{i+1}/Δ_i is not n-divisible for $0 < i < m$.*

   In this case we define $RJ_n(\Gamma) = \{\Delta_0, \dots, \Delta_{m-1}\}$. The elements of this set are called the n-regular jumps.

EXAMPLE 2.9. Let $G = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{n\text{–times}}$ with the lexicographic order $\leq_{lex}$. The 3-regular rank of G is equal to n. This is witnessed by the sequence:

$$\{\mathbf{0}\} \leq \underbrace{\{0\} \oplus \cdots \oplus \{0\}}_{n-1 \text{ times}} \oplus \mathbb{Z} \leq \cdots \leq \{0\} \oplus \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{n-1\text{–times}} \leq \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}.$$

2.2.1. Regular groups and poly-regular groups.

DEFINITION 2.10. An ordered abelian group Γ is said to be *regular* if it is n-regular for all $n \in \mathbb{N}$.

EXAMPLE 2.11. $(\mathbb{Z}, +, \leq, 0)$ and $(\mathbb{Q}, +, \leq, 0)$ are standard examples of regular groups. By [2, Theorem 1.2] any archimedean group is regular.

Robinson and Zakon in their seminal paper [15] completely characterized the possible completions of the theory of regular groups, obtained by extending the

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first-order theory of ordered abelian groups with axioms asserting that for each $n \in \mathbb{N}$ if an interval contains at least n-elements then it contains an n-divisible element. The following is [15, Theorem 4.7].

THEOREM 2.12. *The possible completions of the theory of regular groups are*:

1. *the theory of discrete regular groups, and*
2. *the completions of the theory of dense regular groups T_χ where*

$$\chi =: \text{Primes} \to \mathbb{N} \cup \{\infty\}$$

is a function specifying the index $\chi(p) = [\Gamma : p\Gamma]$.

Robinson and Zakon proved as well that each of these completions is the theory of some archimedean group. In particular, any discrete regular group is elementarily equivalent to $(\mathbb{Z}, \leq, +, 0)$. This theory is known as Presburger arithmetic, introduced in 1929 by M. Presburger, who proved that it admits quantifier elimination in the well-known *Presburger Language*

$$\mathcal{L}_{\text{Pres}} = \{0, 1, +, -, <, (\equiv_m)_{m \in \mathbb{N}_{\geq 2}}\}.$$

Given an ordered abelian group Γ we naturally see it as an $\mathcal{L}_{\text{Pres}}$-structure. The symbols $\{0, +, -, <\}$ take their obvious interpretation. If Γ is discrete, the constant symbol 1 is interpreted as the least positive element of Γ, and by 0 otherwise. For each $m \in \mathbb{N}_{\geq 2}$ the binary relation symbol $\equiv_m$ is interpreted as the equivalence modulo m, i.e., for any $g, h \in \Gamma$ $g \equiv_m h$ if and only if $g - h \in m\Gamma$.

The theory of a dense ordered abelian group admits quantifier elimination in the Presburger language if and only if it is regular. This is a result of Weispfenning in [20, Theorem 2.9].

DEFINITION 2.13. Let Γ be an ordered abelian group. We say that it is *poly-regular* if it is elementarily equivalent to a subgroup of the lexicographically ordered group $\mathbb{R}^n$.

In [2] Belegradek studied poly-regular groups and proved that an ordered abelian group is poly-regular if and only if it has finitely many proper definable convex subgroups, and all the proper convex subgroups are definable over the empty set. In [20, Theorem 2.9] Weispfenning obtained quantifier elimination for the class of poly-regular groups in the language of ordered abelian groups extended with predicates to distinguish the subgroups $\Delta + \ell\Gamma$ where Δ is a convex subgroup and $\ell \in \mathbb{N}_{\geq 2}$.

2.2.2. Ordered abelian groups with bounded regular rank.

DEFINITION 2.14. Let Γ be an ordered abelian group. We say that it has *bounded regular rank* if it has finite n-regular rank for each $n \in \mathbb{N}_{\geq 2}$. For notation, we will use $RJ(\Gamma) = \bigcup_{n \in \mathbb{N}_{\geq 2}} RJ_n(\Gamma)$.

The class of ordered abelian groups of bounded regular rank extends the class of poly-regular groups and regular groups. The terminology of *bounded regular rank* becomes clear with the following Proposition (item 3).

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PROPOSITION 2.15. *Let* Γ *be an ordered abelian group. The following are all equivalent*:

1. Γ *has finite p-regular rank for each prime number p*.
2. Γ *has finite n-regular rank for each natural number* $n \geq 2$.
3. *There is some cardinal* κ *such that for any* $H \equiv \Gamma$, $|RJ(H)| \leq \kappa$.
4. *For any* $H \equiv \Gamma$, *any definable convex subgroup of H has a definition without parameters.*
5. *There is some cardinal* κ *such that for any* $H \equiv \Gamma$, *H has at most* κ *definable convex subgroups.*

Moreover, in this case $RJ(\Gamma)$ *is the collection of all proper definable convex subgroups of* Γ *and all are definable without parameters. In particular, there are only countably many definable convex subgroups.*

PROOF. This is [5, Proposition 2.3]. ⊣

2.2.3. Quantifier elimination and the quotient sorts. In [3] Cluckers and Halupczok introduced a language $\mathcal{L}_{qe}$ to obtain quantifier elimination for ordered abelian groups relative to the *auxiliary sorts* S_n, T_n, and T_n^+, whose precise description can be found in [3, Definition 1.5]. This language is similar in spirit to the one introduced by Schmitt in [16], but has lately been preferred by the community as it is more in line with the many-sorted language of Shelah's imaginary expansion $\mathfrak{M}^{eq}$. Schmitt does not distinguish between the sorts S_n, T_n, and T_n^+. Instead for each $n \in \mathbb{N}$ he works with a single sort $Sp_n(\Gamma)$ called the *n-spine* of Γ, whose description can be found in [6, Section 2]. In [3, Section 1.5] it is explained how the auxiliary sorts of Cluckers and Halupczok are related to the n-spines $Sp_n(\Gamma)$ of Schmitt. In [5, Section 2], it is shown that an ordered abelian group Γ has bounded regular rank if and only if all the n-spines are finite, and $Sp_n(\Gamma) = RJ_n(\Gamma)$. In this case, we define the regular rank of Γ as the cardinal $|RJ(\Gamma)|$, which is either finite or $\aleph_0$. Instead of saying that Γ is an ordered abelian group with finite spines, we prefer to use the classical terminology of bounded regular rank, as it emphasizes the relevance of the n-regular jumps and the role of the divisibilities to describe the definable convex subgroups.

DEFINITION 2.16 (The language $\mathcal{L}$). Let Γ be an ordered abelian group with bounded regular rank. We view Γ as a multi-sorted structure in the language $\mathcal{L}$, where:

1. We add a sort for the ordered abelian group Γ, and we equip it with a copy of the language $\mathcal{L}_{\text{Pres}}$ extended with predicates to distinguish each of the convex subgroups $\Delta \in RJ(\Gamma)$. We refer to this sort as the *main sort*.
2. For each $\Delta \in RJ(\Gamma)$ we add a sort for the ordered abelian group Γ/Δ, equipped with a copy of the Presburger language
   $\mathcal{L}_{\text{Pres}}^{\Delta} = \{0^{\Delta}, 1^{\Delta}, +^{\Delta}, -^{\Delta}, <^{\Delta}, (\equiv_m^{\Delta})_{m \in \mathbb{N}_{\geq 2}}\}$.
   We add as well a map $\rho_{\Delta} : \Gamma \to \Gamma/\Delta$, interpreted as the natural quotient map.

In [5, Theorem 2.4] Farré obtained a quantifier elimination statement for the class of ordered abelian groups with bounded regular rank in the language $\mathcal{L}$ extended with a set of constants in the home sort. However, we present a slightly different language where we add the constants for the minimal element in Γ/Δ (if it exists)

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instead of adding a representative in the home sort whose projection is the minimal class in Γ/Δ. For this purpose we highlight that the following statement is a direct consequence of [1, Proposition 3.14].

THEOREM 2.17. *Let Γ be an ordered abelian group with bounded regular rank. Then Γ admits quantifier elimination in the language $\mathcal{L}$.*

NOTATION 2.18. *We will be mainly interested in the description of the definable sets in the* main sort. *For this purpose we will slightly abuse the language, to simplify the notation. For each $k \in \mathbb{Z}$ and $\Delta \in RJ(\Gamma)$ we define $k^\Delta = k \cdot 1^\Delta$, where 1^Δ is the minimal element in Γ/Δ if it exists. We will sometimes indicate k^Δ simply as $k + \Delta$. We introduce the following notation*:

1. *We write $\tau(\mathbf{x}) + \Delta < \beta + k + \Delta$ for the formula $\rho_\Delta(\tau(\mathbf{x})) <^\Delta \rho_\Delta(\beta) + k^\Delta$.*
2. *We write $\tau(\mathbf{x}) \equiv_\Delta \beta + k$ for the formula $\rho_\Delta(\tau(\mathbf{x})) = \rho_\Delta(\beta) + k^\Delta$.*
3. *We write $\tau(\mathbf{x}) \equiv_{\Delta+m\Gamma} \beta + k$ for the formula $\rho_\Delta(\tau(\mathbf{x})) \equiv_m^\Delta \rho_\Delta(\beta) + k^\Delta$. The latter is interpreted as $\rho_\Delta(\tau(\mathbf{x})) - (\rho_\Delta(\beta) + k^\Delta) \in m(\Gamma/\Delta)$.*

Here $\tau(\mathbf{x})$ is a term in the language of ordered abelian groups in m variables, $\mathbf{x} = (x_1, \dots, x_m)$ and $\beta \in \Gamma$.

DEFINITION 2.19.
1. A set $S \subset \Gamma$ is said to be an *end-segment* (respectively, an initial segment) if for any $x \in S$ and $y \in \Gamma$, $x < y$ (respectively, $y < x$) we have that $y \in S$.
2. Let $n \in \mathbb{N}_{\geq 2}$, $\Delta \in RJ(\Gamma)$, $\beta \in \Gamma \cup \{-\infty, +\infty\}$, and $\square \in \{\geq, >\}$.
   $\{\eta \in \Gamma \mid n\eta + \Delta \square \beta + \Delta\}$ is an end-segment of Γ. We call the end-segments of this form *divisibility end-segments*. We define *divisibility initial segments* analogously.
3. A *mid-segment* is a nonempty set C of the form $C = U \cap L$ where U is a divisibility end-segment and L is a divisibility initial segment.
4. A *basic positive congruence formula* is a formula of the form $zx \equiv_{\Delta+l\Gamma} \beta + k$ where $\beta \in \Gamma$, $z, k \in \mathbb{Z}$, $l \in \mathbb{N}_{\geq 2}$, and $\Delta \in RJ(\Gamma)$. Likewise, a *basic negative formula* is a formula of the form $zx \not\equiv_{\Delta+l\Gamma} \beta + k$. A *basic congruence formula* is either a basic positive congruence formula or a basic negative formula.
5. A *finite congruence restriction* is a finite conjunction of basic congruence formulas.
6. A *nice set* is a set of the form $C \cap X$, where C is a mid-segment and X is defined by a finite congruence restriction.

The following is a direct consequence of quantifier elimination.

COROLLARY 2.20. *Let Γ be an ordered abelian group with bounded regular rank. Let $X \subseteq \Gamma$ be a definable set. Then X is a finite union of nice sets.*

We will consider an extension $\mathcal{L}_Q$ of the language $\mathcal{L}$, where for each natural number $n \in \mathbb{N}_{\geq 2}$ and $\Delta \in RJ(\Gamma)$ we add a sort for the quotient group $\Gamma/(\Delta + n\Gamma)$ and a map $\pi_\Delta^n : \Gamma \to \Gamma/(\Delta + n\Gamma)$. We refer to the sorts in the language $\mathcal{L}_Q$ as *quotient sorts*.

The following fact will be very useful to show weak elimination of imaginaries for ordered abelian groups with bounded regular rank.

FACT 2.21. *Let Γ be an ordered abelian group of finite n-regular rank witnessed by the sequence $\{0\} = \Delta_0 < \Delta_1 < \cdots < \Delta_l = \Gamma$ and fix some definable convex subgroup*

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H. Then Γ/H is also a group of finite n-regular rank. Moreover, if Γ is an ordered abelian group of bounded regular rank, then $H \in RJ(\Gamma)$ and each coset of Δ_i/H in Γ/H is interdefinable with an element of Γ/Δ_i.

PROOF. Let Γ be an ordered abelian group and H a convex subgroup. Assume that Γ has finite n-regular rank, witnessed by the sequence $\{0\} = \Delta_0 < \Delta_1 < \cdots < \Delta_l = \Gamma$ and let r be the smallest index such that $\Delta_r \subseteq H \subseteq \Delta_{r+1}$. We aim to show that $\Delta_{r+1}/H < \cdots < \Delta_l/H = \Gamma/H$ witnesses that Γ/H has finite n-regular rank. For each $r \leq i < l$, by the isomorphism theorem $(\Delta_{i+1}/H)/(\Delta_i/H) \cong \Delta_{i+1}/\Delta_i$. As Δ_{i+1}/Δ_i is n-regular and not n-divisible, so is $(\Delta_{i+1}/H)/(\Delta_i/H)$.

The second part of the statement follows immediately by the isomorphism theorem and Proposition 2.15. ⊣

2.3. A survey of model theoretic results on ordered abelian groups. In 1984 the classification of the model theoretic complexity of ordered abelian groups was initiated by Gurevich and Schmitt in [6], who proved that no ordered abelian group has the independence property. During the last years finer classifications have been achieved, and we present the state of the field in this subsection.

DEFINITION 2.22. Let Γ be an ordered abelian group and let p be a prime number. We say that p is a *singular prime* if $[\Gamma : p\Gamma] = \infty$. If Γ does not have singular primes we call it *non-singular*.

The following result corresponds to [11, Proposition 5.1].

PROPOSITION 2.23. *Let Γ be an ordered abelian group. The following conditions are equivalent*:

1. *Γ is non-singular.*
2. *Γ is dp-minimal.*

The following is [1, Theorem 3.13].

PROPOSITION 2.24. *Let Γ be an ordered abelian group with bounded regular rank (i.e., each $Sp_n(\Gamma)$ is finite). The following statements are equivalent*:

1. *Γ is distal.*
2. *Γ is dp-minimal.*

The following statement was independently achieved in [4, 5, 7].

PROPOSITION 2.25. *Let Γ be an ordered abelian group. The following conditions are equivalent*:

1. *Γ is strongly dependent.*
2. *Γ has finite dp-rank.*
3. *Γ has bounded regular rank and finitely many singular primes.*

Moreover, let $\mathcal{P} = \{p \in \mathbb{N} \mid p \text{ is a singular prime}\}$. Then

$$dp - \operatorname{rank}(\Gamma) \leq 1 + \sum_{p \in \mathcal{P}} |RJ_p(G)|.$$

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§3. Definable end-segments. In this subsection we characterize the definable end-segments (or initial segments) in an ordered abelian group with bounded regular rank (equivalently with finite spines). We also show that they can be coded in the quotient sorts.

DEFINITION 3.1. Let Γ be an ordered abelian group with bounded regular rank:

1. Given $S \subseteq \Gamma$ an end-segment (or an initial segment) we denote by Δ_S the stabilizer of S, i.e., $\Delta_S := \{\beta \in \Gamma \mid \beta + S = S\}$.
2. Let $S \subseteq \Gamma$ be an end-segment and $\Delta \in RJ(\Gamma)$. We consider the projection map $\rho_\Delta : \Gamma \to \Gamma/\Delta$, and we write S_Δ to indicate $\rho_\Delta(S)$. The set
$$S_\Delta = \{\gamma \in \Gamma/\Delta \mid \exists y \in S\ \rho_\Delta(y) = \gamma\}$$
is a definable end-segment of Γ/Δ, as it is the projection of an end-segment.
3. Let $\Delta \in RJ(\Gamma)$ and $S \subseteq \Gamma$ be an end-segment. We say that S is Δ-*decomposable* if it is a union of Δ-cosets.
4. Let X and Y be definable sets. We say that Y is *coinitial* (or *cofinal*) in X if for any $y \in X$ there is some element $z \in X \cap Y$ such that $z \leq y$ (respectively, $z \geq y$).

FACT 3.2. *Let $S \subseteq \Gamma$ be a definable end-segment. Then Δ_S is a definable convex subgroup of Γ, and therefore $\Delta_S \in RJ(\Gamma)$. Furthermore, $\Delta_S = \bigcup_{\Delta \in \mathcal{C}} \Delta$, where*
$$\mathcal{C} = \{\Delta \in RJ(\Gamma) \mid S \text{ is } \Delta\text{-decomposable}\}.$$

PROOF. We first show that $\Delta_S \subseteq \bigcup_{\Delta \in \mathcal{C}} \Delta$. Note Δ_S is a definable convex subgroup, so $\Delta_S \in RJ(\Gamma)$. We aim to show that S is Δ_S-decomposable, so it is sufficient to show that for any $\gamma \in S$, $\gamma + \Delta_S \subseteq S$. Fix some $\gamma \in \Gamma$. If $\delta \in \Delta_S$ then $\gamma + \delta \in S$, so $\gamma + \Delta_S \subseteq S$.

We now prove that $\bigcup_{\Delta \in \mathcal{C}} \Delta \subseteq \Delta_S$. Let $\Delta \in \mathcal{C}$ and fix some $\delta \in \Delta$. We want to show that $\delta + S \subseteq S$ and $S \subseteq \delta + S$. Because S is Δ-decomposable, $\gamma + \Delta \subseteq S$ for any $\gamma \in S$. In particular, $\gamma + \delta \in S$. As γ is an arbitrary element in S, we conclude that $\delta + S \subseteq S$. It is only left to show that $S \subseteq \delta + S$. Let $\gamma \in S$, then $\gamma - \delta \in S$ because $\gamma + \Delta \subseteq S$. As $\gamma = \delta + (\gamma - \delta) \in \delta + S$, we have $S \subseteq \delta + S$, as required. ⊣

PROPOSITION 3.3. *Let Γ be an ordered abelian group of bounded regular rank. Any definable end-segment is a divisibility end-segment.*

PROOF. Let $S \subseteq \Gamma$ be a definable end-segment such that $S \neq \Gamma$. By Fact 3.2, Δ_S is a definable convex subgroup of Γ and S is Δ_S-decomposable. To simplify the notation we will denote $\hat{\Gamma} = \Gamma/\Delta_S$ and $\hat{S} = S_{\Delta_S} = \rho_{\Delta_S}(S)$. It is sufficient to prove that $\hat{S}$ is a divisibility end-segment in $\hat{\Gamma}$.

CLAIM 3.3.1. *Note that for any $k \in \mathbb{N}$ exactly one of the following occurs*:

- *$\hat{\Gamma}$ is k-regular.*
- *There is a nontrivial k-regular convex subgroup Λ_k of $\hat{\Gamma}$ and a coset $\eta + \Lambda_k$ such that $\hat{S} \cap (\eta + \Lambda_k) \neq \emptyset$ and $\hat{S}^c \cap (\eta + \Lambda_k) \neq \emptyset$.*

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PROOF. Let $\{0\} = \Delta_0 < \Delta_1 < \cdots < \Delta_l = \Gamma$ be the sequence of convex subgroups witnessing that Γ has k-finite regular rank equal to l. Let $r \leq l$ be the smallest index such that $\Delta_S \subsetneq \Delta_r$. If $r = l$ then $\hat{\Gamma}$ is k-regular. Otherwise the quotient group $\Lambda_k = \Delta_r/\Delta_S$ satisfies the required conditions. Indeed, as Δ_r/Δ_{r-1} is k-regular so is Δ_r/Δ_S. Additionally, S is not Δ_r-decomposable. If it were, then we would have $\Delta_r \subseteq \Delta_S$ which contradicts $\Delta_S \subsetneq \Delta_r$. Then there is some coset $\eta + \Delta_r$ such that $S \cap (\eta + \Delta_r) \neq \emptyset$ and $S^c \cap (\eta + \Delta_r) \neq \emptyset$ because otherwise S would be Δ_r-decomposable. Thus $\hat{S} \cap (\hat{\eta} + \Lambda_k) \neq \emptyset$ and $\hat{S}^c \cap (\hat{\eta} + \Lambda_k) \neq \emptyset$, where $\hat{\eta} = \eta + \Delta_S$. ⊣

We may assume that $\hat{S}$ does not have a minimum because otherwise the statement follows immediately. By Corollary 2.20 applied to $\hat{\Gamma}$, $\hat{S}$ is a finite union of nice sets $C_i \cap X_i$, where $C_i = U_i \cap L_i$. As $\hat{S}$ is a definable end-segment, it is sufficient to understand the co-initial description of $\hat{S}$. Without loss of generality we may assume that $U_i \subseteq U_1$ for all i. Let $\hat{\Delta} \in RJ(\hat{\Gamma})$. Then $\hat{\Delta} = \Delta/\Delta_S$ for some $\Delta_S \subsetneq \Delta \in RJ(\Gamma)$. Thus there is a coset $\eta + \hat{\Delta}$ such that $\hat{S} \cap (\eta + \hat{\Delta}) \neq \emptyset$ and $\hat{S}^c \cap (\eta + \hat{\Delta}) \neq \emptyset$ because S is not Δ-decomposable.

Hence, each of the congruence formulas involving the groups $\hat{\Delta} + k\hat{\Gamma}$ does not change its truth value over $U_1 \cap (\eta + \hat{\Delta})$. Therefore it does not change its truth value co-initially in U_1.

Consider a conjunction of congruence restrictions of the form:

$$C(x) := \left(\bigwedge_{i \leq s} x \equiv_{k_i \hat{\Gamma}} c_i \right) \wedge \left(\bigwedge_{j \leq l} \neg (x \equiv_{r_j \hat{\Gamma}} d_j) \right).$$

Let M be the least common multiple of all the k_i's and r_j's involved in the definition of $C(x)$. By the previous Claim 3.3.1, Γ is M-regular or we can find an M-regular group Λ_M and a coset that intersects $\hat{S}$ and its complement. We first assume the existence of a non-trivial convex subgroup Λ_M and a coset $\eta + \Lambda_M$ such that $\hat{S} \cap (\eta + \Lambda_M) \neq \emptyset$ and $\hat{S}^c \cap (\eta + \Lambda_M) \neq \emptyset$. Let $Y = C(x) \cap (U_1 \cap (\eta + \Lambda_M))$.

CLAIM 3.3.2. *If $Y \neq \emptyset$, then $C(x)$ is co-initial in U_1.*

PROOF. Let $x_0 \in Y$ and $U' = \big(U_1 \cap (\eta + \Lambda_M)\big) - x_0$. U' is a definable end-segment of Λ_M without a minimum. Fix an element $\delta \in U'$. As U' does not have a minimum and Λ_M is M-regular, we can find an element $\gamma \in \Lambda_M$ such that $M\gamma \in U'$ and $M\gamma < \delta$. Then $z = M\gamma + x_0 \in Y$ and $z < x_0 + \delta$. Thus $C(x)$ is co-initial in U_1. ⊣

Likewise, if $\hat{\Gamma}$ is M-regular we can conclude that $C(x)$ is co-initial in U_1. Consequently, the congruence restrictions are irrelevant in the definition of the end-segment S. It must be the case then that $S = U_1$, as desired. ⊣

Even though it may seem like any divisibility cut defined by a formula of the form $nx \square \beta$ where $n \in \mathbb{N}_{\geq 2}$, $\square \in \{\geq, >\}$ and $\beta \in \Gamma$ could be coded by β, this statement is false and requires a slightly more delicate treatment. We introduce the following example to motivate the reader to not dismiss the technical work in Proposition 3.3.

EXAMPLE 3.4. Consider the ordered abelian group $(\mathbb{Z} \oplus \mathbb{Z}, \leq_{lex}, +, 0)$ where $\leq_{lex}$ is the lexicographic order. Let $S = \{z \in \mathbb{Z}^2 \mid 2z \geq (1, 1)\}$. Note that for any $\beta \in \mathbb{Z}$, S is also defined by the formula $2z \geq (1, \beta)$.

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LEMMA 3.5. *Let Γ be an ordered abelian group of bounded regular rank. Let $\{0\} = \Delta_0 \leq \Delta_1 \leq \cdots \leq \Delta_l = \Gamma$ be the sequence of convex subgroups witnessing that Γ has finite n-regular rank. Then any divisibility end-segment S defined by a formula $nx \square \beta$ where $n \in \mathbb{N}_{\geq 1}$, $\square \in \{\geq, >\}$ and $\beta \in \Gamma$ is coded by a tuple of elements in the sorts $\Gamma \cup \{\Gamma/\Delta_i \mid i \leq l\}$.*

PROOF. We argue by induction in the n-regular rank of Γ that S can be coded in the sorts $\Gamma \cup \{\Gamma/\Delta_i \mid i \leq l\}$. For the base case, we suppose that Γ is n-regular. We first assume that Γ is dense, and we aim to prove that β and $\ulcorner S \urcorner$ are interdefinable. It is clear that $\ulcorner S \urcorner \in dcl^{eq}(\beta)$. For the converse let σ be any automorphism of the monster model $\mathfrak{M}$ and suppose that $\sigma(\beta) \neq \beta$. Without loss of generality, $\beta < \sigma(\beta)$. By density we can find n-elements in the interval $(\beta, \sigma(\beta))$. By n-regularity and density there is an element δ such that $\beta < n\delta < \sigma(\beta)$. Thus $\sigma(S) \subsetneq S$.

We now assume that Γ is discrete and let 1 be its minimal element. There is a unique natural number $0 \leq i \leq n-1$ such that $\beta + i$ is n-divisible, because $\{\beta, \beta + 1, \ldots, \beta + (n-1)\}$ is an interval with at least n-elements. Let i_0 be the index such that $\beta + i_0$ is n-divisible. Then $x \in S$ if and only if $nx \geq \beta + i_0$. Thus $\frac{\beta + i_0}{n}$ is the minimal element of S and thereby it is a code for S.

We proceed to show the inductive step, and we consider the sequence $\{0\} = \Delta_0 < \Delta_1 < \cdots < \Delta_{l+1} = \Gamma$ witnessing that Γ has n-regular rank equal to $l+1$. Let $\rho_{\Delta_1} : \Gamma \to \Gamma/\Delta_1$ be the canonical projection map, and note that Γ/Δ_1 is an ordered abelian group of n-regular rank l. First we suppose that $\rho_{\Delta_1}(\beta)$ is not n-divisible. Then S is interdefinable with $S_{\Delta_1} = \{\eta \in \Gamma/\Delta_1 \mid n\eta > \rho_{\Delta_1}(\beta)\}$. By the induction hypothesis, such end-segment can be coded in the sorts $\Gamma/\Delta_1 \cup \{(\Gamma/\Delta_1)/(\Delta_i/\Delta_1) \mid 2 \leq i \leq l\}$. As each of the sorts $(\Gamma/\Delta_1)/(\Delta_i/\Delta_1)$ can be canonically identified with Γ/Δ_i, the conclusion of the statement follows.

We consider the case where $\rho_{\Delta_1}(\beta)$ is n-divisible, i.e., there is some $\eta \in \Gamma$ such that $n\rho_{\Delta_1}(\eta) = \rho_{\Delta_1}(\beta)$. Note that $\rho_{\Delta_1}(\eta) = \min(S_{\Delta_1})$. If Δ_1 is discrete, then S has a minimum and this minimal element is a code for S. Thus without loss of generality Δ_1 is dense. We aim to show that β and $\ulcorner S \urcorner$ are interdefinable. In fact, let σ be an automorphism of the monster model $\mathfrak{M}$ fixing $\ulcorner S \urcorner$. We want to show that it fixes also β. We argue by contradiction, and we assume that $\beta < \sigma(\beta)$. As $\rho_{\Delta_1}(\beta) \in dcl^{eq}(\ulcorner S \urcorner)$, we have $\sigma(\beta) - \beta \in \Delta_1$. Fix some element $\eta \in \Gamma$ such that $n\eta + \Delta_1 = \beta + \Delta_1$. We can find elements $\delta_1 < \delta_2 \in \Delta_1$ such that $\beta = n\eta + \delta_1$ and $\sigma(\beta) = n\eta + \delta_2$. By n-regularity and density of Δ_1 we can find an element $\gamma \in \Delta_1$ such that $\delta_1 < n\gamma < \delta_2$, so we have $\beta < n(\gamma + \eta) < \sigma(\beta)$ and hence $S \subsetneq \sigma(S)$, as desired. ⊣

PROPOSITION 3.6. *Let Γ be an ordered abelian group of bounded regular rank and let $S \subseteq \Gamma$ be a definable end-segment. Then $\ulcorner S \urcorner$ is interdefinable with a tuple of elements in the sorts $\Gamma \cup \{\Gamma/\Delta \mid \Delta \in RJ(\Gamma)\}$. Consequently, any initial segment is also coded in the sorts $\Gamma \cup \{\Gamma/\Delta \mid \Delta \in RJ(\Gamma)\}$.*

PROOF. By Proposition 3.3 it is sufficient to code divisibility end-segments. We may assume that $S = \{\eta \in \Gamma \mid n\eta + \Delta \geq \beta + \Delta\}$. Therefore S is interdefinable with $S_\Delta = \{z \in \Gamma/\Delta \mid nz \geq \rho_\Delta(\beta)\}$; this is a definable end-segment of Γ/Δ. The statement follows immediately from Lemma 3.5 combined with Fact 2.21.

The second part of the statement follows by noticing that any initial segment is the complement of an end-segment. ⊣

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§4. An abstract criterion to eliminate imaginaries. The following is [10, Lemma 1.17].

THEOREM 4.1. *Let T be a first-order theory with home sort K. Let $\mathcal{G}$ be some collection of sorts. If the following conditions all hold, then T has weak elimination of imaginaries in the sorts $\mathcal{G}$.*

1. *Density of definable types: for every non-empty definable set $X \subseteq K$ there is an $acl^{eq}(\ulcorner X \urcorner)$-definable type in X.*
2. *Coding definable types: every definable type in K^n has a code in $\mathcal{G}$ (possibly infinite). That is, if p is any (global) definable type in K^n, then the set $\ulcorner p \urcorner$ of codes of the definitions of p is interdefinable with some (possibly infinite) tuple from $\mathcal{G}$.*

PROOF. A very detailed proof can be found in [12, Theorem 6.3]. The first part of the proof shows weak elimination of imaginaries as it is shown that for any imaginary element e we can find a tuple $a \in \mathcal{G}$ such that $e \in dcl^{eq}(a)$ and $a \in acl^{eq}(e)$. ⊣

We will use this criterion to prove that any pure ordered abelian group with bounded regular rank admits weak elimination of imaginaries once the quotient sorts are added.

4.1. Coding of definable types. In this subsection we show that any definable type $p(\mathbf{x})$ can be coded in the quotient sorts.

PROPOSITION 4.2. *Let Γ be an ordered abelian group and let $p(\mathbf{x}) \in S_n(\Gamma)$ be a definable type. Then $p(\mathbf{x})$ can be coded in the quotient sorts.*

PROOF. Let $p(\mathbf{x})$ be a definable type in n variables over Γ. By quantifier elimination (Theorem 2.17), $p(\mathbf{x})$ is completely determined by formulas of the following forms:

- *First kind:*

$$\phi_1(\mathbf{x}, \beta) := \sum_{i \leq n} z_i x_i + \Delta < \beta + k + \Delta$$

or

$$\psi_1(\mathbf{x}, \beta) := \sum_{i \leq n} z_i x_i + \Delta > \beta + k + \Delta,$$

where $\beta \in \Gamma$, $\Delta \in RJ(\Gamma)$, and $k, z_i \in \mathbb{Z}$.

- *Second kind:*

$$\phi_2(\mathbf{x}, \beta) := \sum_{i \leq n} z_i x_i \equiv_{\Delta + l\Gamma} \beta + k,$$

where $\beta \in \Gamma$, $\Delta \in RJ(\Gamma)$, $k, z_i \in \mathbb{Z}$, and $l \in \mathbb{N}_{\geq 2}$.

- *Third kind:*

$$\phi_3(\mathbf{x}, \beta) := \sum_{i \leq n} z_i x_i \equiv_{\Delta} \beta + k,$$

where $\beta \in \Gamma$, $\Delta \in RJ(\Gamma)$, and $z_i \in \mathbb{Z}$.

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The set $\{\beta \in \Gamma \mid \phi_1(\mathbf{x}, \beta) \in p(\mathbf{x})\}$ is an end-segment of Γ, so it can be coded in the quotient sorts by Propositions 3.3 and 3.6. Likewise, the set $\{\beta \in \Gamma \mid \psi_1(\mathbf{x}, \beta) \in p(\mathbf{x})\}$ is an initial segment of Γ, and it admits a code in the quotient sorts.

Let $X = \{\beta \in \Gamma \mid \phi_2(\mathbf{x}, \beta) \in p(\mathbf{x})\}$, then X is either empty or we can take $\beta_0 \in X$ and $\ulcorner X \urcorner$ is interdefinable with $\pi^l_\Delta(\beta_0) \in \Gamma/(\Delta + l\Gamma)$.

Lastly, $Z = \{\beta \in \Gamma \mid \phi_3(\mathbf{x}, \beta) \in p(\mathbf{x})\}$ is either empty or for any element $\beta_0 \in Z$, we have that $\ulcorner Z \urcorner$ is interdefinable with $\rho_\Delta(\beta_0) \in \Gamma/\Delta$. ⊣

4.2. Density of definable types. In this subsection we prove the first condition required in Hrushovski's criterion: the density of definable types in algebraically closed sets.

The following will be a useful fact to obtain our result.

FACT 4.3. *Let $X \subseteq \Gamma$ be a definable set without a minimum element. Then there is a $\ulcorner X \urcorner$ definable end-segment S such that X is co-initial in S.*

PROOF. Let $I = \{\beta \in \Gamma \mid (-\infty, \beta] \cap X = \emptyset\}$. I is a $\ulcorner X \urcorner$-definable initial segment of Γ. Let $S = \Gamma \backslash I$, it is sufficient to verify that X is co-initial in S. Let $\beta \in S$, then $(-\infty, \beta] \cap X \neq \emptyset$. Because X does not have a minimum, we can find an element $x \in X$ such that $x < \beta$, as required. ⊣

PROPOSITION 4.4. *Let Γ be an ordered abelian group of bounded regular rank and $X \subseteq \Gamma$ a definable set. There is a global type $p(x) \vdash x \in X$ such that $p(x)$ is definable over* $\mathrm{acl}^{eq}(\ulcorner X \urcorner)$.

PROOF. Let $X \subseteq \Gamma$ be a 1-definable set. If X has a minimum element a, the statement follows immediately by taking the type of this element. Thus we may assume that X does not have a minimum, by Fact 4.3 there is a $\ulcorner X \urcorner$-definable end-segment S such that X is co-initial in S. In particular, the type

$$\Sigma^{gen}_S(x) = \{x \in S \cap X\} \cup \{x \notin B \mid B \subsetneq S \text{ and } B \text{ is a definable end-segment}\}$$

is a consistent partial type which is $\ulcorner S \urcorner$-definable.

Let $\pi : \mathbb{N} \to \mathbb{N} \times \mathbb{N}_{\geq 1}$ be some fixed bijection. We now build by induction an increasing sequence of partial consistent types $(\Sigma_i(x) \mid i < \omega)$ in the following way:

- **Stage** 0**:** Set $\Sigma_0(x) = \Sigma^{gen}_S(x)$.
- **Stage** $i + 1$**:** Let $\pi(i) = (k, l)$, at this stage we want to decide the congruence modulo the subgroup $\Delta_k + l\Gamma$. To keep the notation simple we assume that $l \geq 2$ and we use the projection map $\pi^l_{\Delta_k} := \Gamma \to \Gamma/(\Delta_k + l\Gamma)$. If $l = 1$ we argue in the same manner to fix the coset of Δ_k and instead we use the projection map $\rho_{\Delta_k} : \Gamma \to \Gamma/\Delta_k$.
  We proceed by cases:
  a) Set $\Sigma_{i+1}(x) = \Sigma_i(x) \cup \{\pi^l_{\Delta_k}(x) \neq \pi^l_{\delta_k}(\beta) \mid \beta \in \Gamma\}$ if it is consistent.
  b) Otherwise, let $A_i = \{\eta_1, \ldots, \eta_{r_i}\} \subseteq \Gamma/(\Delta_k + l\Gamma)$ be the finite set of cosets such that $\Sigma_i(x) \cup \{\pi^l_{\Delta_k}(x) = \eta_j\}$ is consistent. Take an element $\hat{\eta} \in A_i$ and set $\Sigma_{i+1}(x) = \Sigma_i(x) \cup \{\pi^l_{\Delta_k}(x) = \hat{\eta}\}$.

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Let $\mathfrak{M}$ be the monster model and

$$\mathcal{J} = \{i \in \mathbb{N} \mid \Sigma_i(x) \cup \{\pi^l_{\Delta_k}(x) \neq \pi^l_{\delta_k}(\beta) \mid \beta \in \Gamma\} \text{ is inconsistent}\}.$$

CLAIM 4.4.1. *For any $\sigma \in Aut(\mathfrak{M}/\operatorname{acl}^{eq}(\ulcorner X \urcorner))$ the following conditions hold*:
1. *For any $i \in \mathbb{N}$ $\sigma(\Sigma_i(x)) = \Sigma_i(x)$.*
2. *For any $i \in \mathcal{J}$ $\sigma(A_i) = A_i$.*
In particular, as σ is arbitrary, then $A_i \subseteq \operatorname{acl}^{eq}(\ulcorner X \urcorner)$.

PROOF. We argue by induction on i to show that for any $\sigma \in Aut(\mathfrak{M}/\operatorname{acl}^{eq}(\ulcorner X \urcorner))$ we have that $\sigma(\Sigma_i(x)) = \Sigma_i(x)$ and if $i \in \mathcal{J}$ then $\sigma(A_i) = A_i$. For the base case, fix some $\sigma \in Aut(\mathfrak{M}/\operatorname{acl}^{eq}(\ulcorner X \urcorner))$. Then $\sigma(\Sigma_0(x)) = \Sigma_0(x)$ because $\Sigma^{gen}_S(x)$ is $\ulcorner S \urcorner$-definable and $\ulcorner S \urcorner \in \operatorname{dcl}^{eq}(\ulcorner X \urcorner)$. Suppose the statement holds for i and fix some $\sigma \in Aut(\mathfrak{M}/\operatorname{acl}^{eq}(\ulcorner X \urcorner))$. If $\Sigma_{i+1}(x) = \Sigma_i(x) \cup \{\pi^l_{\Delta_k}(x) \neq \pi^l_{\Delta_k}(\beta) \mid \beta \in \Gamma\}$, then

$$\begin{aligned}\sigma(\Sigma_{i+1}(x)) &= \sigma(\Sigma_i(x)) \cup \{\pi^l_{\Delta_k}(x) \neq \pi^l_{\Delta_k}(\sigma(\beta)) \mid \beta \in \Gamma\} \\ &= \Sigma_i(x) \cup \{\pi^l_{\Delta_k}(x) \neq \pi^l_{\Delta_k}(\sigma(\beta)) \mid \beta \in \Gamma\} = \Sigma_{i+1}(x).\end{aligned}$$

Let's assume that $\Sigma_{i+1}(x) = \Sigma_i(x) \cup \{\pi^l_{\Delta_k}(x) = \eta\}$ for some $\eta \in A_i$. We first argue that $\sigma(A_i) = A_i$. By definition of A_i:

$$\mu \in A_i \text{ if and only if } \Sigma_i(x) \cup \{\pi^l_{\Delta_k}(x) = \mu\} \text{ is consistent.}$$

Let $\mu \in A_i$, then $\Sigma_i(x) \cup \{\pi^l_{\Delta_k}(x) = \mu\}$ is consistent. As σ is an automorphism, then

$$\sigma(\Sigma_i(x)) \cup \{\pi^l_{\Delta_k}(x) = \sigma(\mu)\} \text{ is consistent.}$$

By the induction hypothesis, $\sigma(\Sigma_i(x)) = \Sigma_i(x)$. Hence,

$$\Sigma_i(x) \cup \{\pi^l_{\Delta_k}(x) = \sigma(\mu)\} \text{ is consistent.}$$

Consequently, $\sigma(\mu) \in A_i$ and we conclude that $\sigma(A_i) \subseteq A_i$. We argue in a similar manner with σ^{-1} to show that $A_i \subseteq \sigma(A_i)$.
As for any $\sigma \in Aut(\mathfrak{M}/\operatorname{acl}^{eq}(\ulcorner X \urcorner))$, $\sigma(A_i) = A_i$ and A_i is a finite set, then $A_i \subseteq \operatorname{acl}^{eq}(\ulcorner X \urcorner)$. In particular, $\eta \in \operatorname{acl}^{eq}(\ulcorner X \urcorner)$ where $\Sigma_{i+1}(x) = \Sigma_i(x) \cup \{\pi^l_{\Delta_k}(x) = \eta\}$. Then for any $\sigma \in Aut(\mathfrak{M}/\operatorname{acl}^{eq}(\ulcorner X \urcorner))$ we have that $\sigma(\Sigma_{i+1}(x)) = \Sigma_{i+1}(x)$, as required. ⊣

Let $\Sigma_\infty(x) = \bigcup_{i \in \mathbb{N}} \Sigma_i(x)$, this is a partial consistent type and $\Sigma_\infty(x) \vdash x \in X$. By quantifier elimination $\Sigma_\infty(x)$ determines a complete type $p(x)$. Then $p(x) \vdash x \in X$, and $p(x)$ is $acl^{eq}(\ulcorner X \urcorner)$-definable because $p(x)$ is completely determined by the data in $\Sigma_\infty(x)$, which is definable over $\operatorname{acl}^{eq}(\ulcorner X \urcorner)$ by Claim 4.4.1. ⊣

§5. Main Results.

THEOREM 5.1. *Let Γ be an ordered abelian group of bounded regular rank (equivalently with finite spines). Then Γ admits weak-elimination of imaginaries in the language $\mathcal{L}_Q$, once the quotient sorts are added.*

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PROOF. By 4.1 it is sufficient to check that we have density of definable types and that we can code definable types in the quotient sorts. The first condition is Proposition 4.4 and the second one is Proposition 4.2. ⊣

5.1. The dp-minimal case. In this section we show that a better statement can be achieved for the dp-minimal case.

DEFINITION 5.2. Let Γ be an ordered abelian group and H some definable subgroup. A subset $\mathcal{C} \subseteq \Gamma$ is said to be a *complete set of representatives modulo* H if:

1. Given any $\gamma \in \Gamma$ there is some $\beta \in \mathcal{C}$ such that $\gamma - \beta \in H$.
2. For any $\beta \neq \eta \in \mathcal{C}$ we have that $\beta + H \neq \eta + H$.

FACT 5.3. *Let Γ be an ordered abelian group, Δ a convex subgroup, and $k \in \mathbb{N}$. Let $\mathcal{C}$ be a complete set of representatives of Γ modulo $k\Gamma$, then some subset $\mathcal{C}_0 \subseteq \mathcal{C}$ is a complete set of representatives modulo $\Delta + k\Gamma$.*

PROOF. Let $\mathcal{C} \subseteq \Gamma$ be a complete set of representatives of Γ modulo $k\Gamma$ and let $\pi^k_\Delta : \Gamma \to \Gamma/(\Delta + k\Gamma)$ be the projection map. $\pi^k_\Delta(\mathcal{C}) = \Gamma/(\Delta + k\Gamma)$, because for any $\gamma \in \Gamma$, there is some $\beta \in \mathcal{C}$ such that $\gamma - \beta \in k\Gamma$, in particular, $\gamma - \beta \in \Delta + k\Gamma$. For each coset $\eta \in \Gamma/(\Delta + k\Gamma)$ choose an element $c_\eta \in \mathcal{C}$ such that $\pi^k_\Delta(c_\eta) = \eta$. The set $\mathcal{C}_0 = \{c_\eta \mid \eta \in \Gamma/(\Delta + k\Gamma)\}$ is a complete set of representatives. ⊣

By Proposition 2.23, an ordered abelian group is dp-minimal if and only if it does not have singular primes, i.e., for any p prime number $[\Gamma : p\Gamma] < \infty$. We consider the language $\mathcal{L}_{dp}$ extending $\mathcal{L}_Q$, where for each $k \in \mathbb{N}_{\geq 2}$ we add constants for the elements of the finite groups $\Gamma/k\Gamma$.

COROLLARY 5.4. *Let Γ be a dp-minimal ordered group. Then Γ admits elimination of imaginaries in the language $\mathcal{L}_{dp}$, where the quotient sorts are added.*

PROOF. By Theorem 5.1 and Fact 2.2 it is sufficient to show that we can also code finite sets. Let Δ definable convex subgroup and $k \in \mathbb{N}$, the group $\Gamma/(\Delta + k\Gamma)$ is also finite. We first argue that $\Gamma/(\Delta + k\Gamma) \subseteq \mathrm{dcl}(\emptyset)$. Consider the $\emptyset$-definable function

$$\begin{aligned} f :& \Gamma/k\Gamma \to \Gamma/(\Delta + k\Gamma) \\ & \gamma + k\Gamma \to \gamma + (\Delta + k\Gamma). \end{aligned}$$

By Fact 5.3 f is surjective.

Hence, it is enough to prove that finite sets of tuples in $\mathcal{S} = \{\Gamma/\Delta \mid \Delta \in RJ(\Gamma)\}$ can be coded in the quotient sorts. As each of the sorts Γ/Δ is linearly ordered, there is a definable order induced over the finite products of quotients of Γ/Δ, and thereby any finite set of tuples in $\mathcal{S}$ is already coded in $\mathcal{S}$. ⊣

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