SOME TAUBERIAN THEOREMS FOR THE LOGARITHMIC METHOD OF SUMMABILITY

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1. Introduction. The series $\sum_{\nu=0}^{\infty} a_\nu$ is said to be summable (L) to s if the sequence $\{s_n\}$, where $s_n = a_0 + a_1 + \ldots + a_n$, is L-convergent to s, i.e., if

$$\lim_{x\to 1-} \frac{-1}{\log(1-x)} \sum_{\nu=0}^{\infty} s_\nu \frac{x^{\nu+1}}{\nu+1} = s.$$

If the sequence $\{s_n\}$ is l-convergent to s, i.e., if the sequence $\{t_n\}$, where

$$(1) \qquad t_n = \frac{1}{\alpha_n} \sum_{\nu=0}^{n} \frac{s_\nu}{\nu+1}, \qquad \alpha_n = \sum_{\nu=0}^{n} \frac{1}{\nu+1} \sim \log n,$$

converges to s, we say that $\sum_{\nu=0}^{\infty} a_\nu$ is summable (l) to s. It follows from Theorem 57 of (**3**) that summability (l) implies summability (L). Summability (L) has been discussed by Ishiguro (**5**), Borwein (**2**), and myself (**6**). Mohanty and Nanda (see **7**; **8**) and Hsiang (**4**) have used the (L) method to sum Fourier series. We shall write

$$\sigma_n = \alpha_n(s_n - t_n) = \begin{cases} 0 & (n = 0), \\ \sum_{\nu=1}^{n} a_\nu \alpha_{\nu-1} & (n \geqq 1). \end{cases}$$

The following theorems are results of further investigation.

THEOREM 1. *Let $\{a_n\}$ be a sequence such that*

$$(2) \qquad \limsup_{n\to\infty} |a_n n \log n| = H < \infty$$

and let

$$f(x) = -(\log(1-x))^{-1} \sum_{\nu=0}^{\infty} s_\nu \frac{x^{\nu+1}}{\nu+1},$$

where $s_\nu = a_0 + a_1 + \ldots + a_\nu$. Then

$$(3) \qquad \limsup_{x\to 1-} |s_{n(x)} - f(x)| \leqq H,$$

where $n(x)$ is an integer-valued function satisfying

$$(4) \qquad p > n(x)(1-x) > q > 0 \qquad (0 < x < 1)$$

and

$$(5) \qquad \limsup_{n\to\infty} |s_n - f(x_n)| \leqq H,$$

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where $\{x_n\}$ *is any sequence satisfying*

$$p > n(1 - x_n) > q > 0. \tag{6}$$

Moreover, there is a real sequence $\{a_n\}$ *satisfying* (2) *such that the equalities in* (3) *and* (5) *hold.*

THEOREM 2. *Let* $\sum_{\nu=0}^{\infty} a_\nu$ *be summable* (L) *to* s. *Then a necessary and sufficient condition that the series should converge to* s *is that*

$$\sigma_n = o(\log n),$$

i.e.,

$$s_n - t_n = o(1). \tag{7}$$

THEOREM 3. *Suppose that* $\sum_{\nu=0}^{\infty} a_\nu$ *is summable* (L) *to* s. *Then a necessary and sufficient condition that the series should be summable* (l) *to* s *is that*

$$\sigma_n = o(\log n) \ (l),$$

i.e.,

$$v_n = \sum_{\nu=0}^{n} \frac{\sigma_\nu}{\nu + 1} = o(\log^2 n). \tag{8}$$

THEOREM 4. *If* $\sum_{\nu=0}^{\infty} a_\nu$ *is summable* (L), *and*

$$s_0 + s_1 + \ldots + s_n = O(n), \tag{9}$$

then the series is summable (l) *to the same sum.*

Since Abel summability implies summability (L), we have the following corollary.

COROLLARY. *If* $\sum_{\nu=0}^{\infty} a_\nu$ *is summable* (A) *and bounded* (C, 1), *then the series is summable* (l) *to the same sum.*

THEOREM 5. *If* $\sum_{\nu=0}^{\infty} a_\nu$ *is summable* (L) *to* s *and*

$$\sigma_n = O_L(\log n), \tag{10}$$

then the series is summable (l) *to* s.

For the definition of O_L see (**3**, p. 149).

THEOREM 6. *If* $\sum_{\nu=0}^{\infty} a_\nu$ *is summable* (L) *and* $\liminf (t_n - t_m) \geqq 0$ *when* $n > m \to \infty$, $\log n/\log m \to 1$, *then the series is summable* (l) *to the same sum.*

2. Lemmas.

We require the following lemmas, of which the first is Theorem 2 of (**5**).

LEMMA 1. *If* $\sum_{\nu=0}^{\infty} a_\nu$ *is summable* (L) *to* s *and* $s_n = O_L(1)$, *then the series is summable* (l) *to* s.

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LEMMA 2. *If* $\sum_{\nu=0}^{\infty} a_\nu$ *is summable* (L) *to* s *and* $\liminf(s_n - s_m) \geqq 0$ *when* $n > m \to \infty$, $\log n/\log m \to 1$, *then the series converges to* s.

The proof of this lemma is given in (**6**).

LEMMA 3. *Let* $\sum_{\nu=0}^{\infty} a_\nu$ *be summable* (L) *to* s, *then the sequence* $\{t_n\}$ *of* (l) *means is* L-*convergent to* s.

Proof. Write

$$g(u) = \sum_{\nu=0}^{\infty} \frac{s_\nu}{\nu+1} u^\nu. \tag{11}$$

Thus summability (L) to s is the same as

$$g(u) \sim s \log(1-u)^{-1} \qquad (u \to 1-). \tag{12}$$

Write also

$$\phi(x) = \sum_{\nu=0}^{\infty} \frac{t_\nu}{\nu+1} x^\nu. \tag{13}$$

It is easily verified that the sequence $\{1/(\nu+1)\alpha_\nu\}$ is totally monotone. Hence (see **3**, Theorem 207), there is a function $\chi(u)$, bounded and non-decreasing in [0, 1] such that

$$\int_0^1 t^\nu \, d\chi(t) = \frac{1}{(\nu+1)\alpha_\nu}.$$

By an obvious change of variable we have, for $x > 0$,

$$\int_0^x u^\nu \, d\chi\left(\frac{u}{x}\right) = \frac{x^\nu}{(\nu+1)\alpha_\nu}.$$

Hence, for $0 < x < 1$,

$$\int_0^x \frac{u^\nu}{1-u} \, d\chi\left(\frac{u}{x}\right) = \int_0^x \left\{\sum_{n=\nu}^{\infty} u^n\right\} d\chi\left(\frac{u}{x}\right)$$
$$= \sum_{n=\nu}^{\infty} \int_0^x u^n \, d\chi\left(\frac{u}{x}\right)$$
$$= \sum_{n=\nu}^{\infty} \frac{x^n}{(n+1)\alpha_n}.$$

Now assume that (11) converges for $|u| < 1$. Then it converges absolutely for $|u| < 1$. Thus for any fixed x with $0 < x < 1$ we have, the inversions being justified by absolute convergence, that

$$\int_0^x \frac{g(u)}{1-u} \, d\chi\left(\frac{u}{x}\right) = \int_0^x \left\{\sum_{\nu=0}^{\infty} \frac{s_\nu u^\nu}{\nu+1}\right\} \frac{d\chi(u/x)}{1-u} = \tag{14}$$
$$\sum_{\nu=0}^{\infty} \frac{s_\nu}{\nu+1} \int_0^x \frac{u^\nu}{1-u} \, d\chi\left(\frac{u}{x}\right) = \sum_{\nu=0}^{\infty} \frac{s_\nu}{\nu+1} \sum_{n=\nu}^{\infty} \frac{x^n}{(n+1)\alpha_n} =$$
$$\sum_{n=0}^{\infty} \frac{x^n}{(n+1)\alpha_n} \sum_{\nu=0}^{n} \frac{s_\nu}{\nu+1} = \sum_{n=0}^{\infty} \frac{x^n}{n+1} t_n = \phi(x).$$

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Now assume further that (12) holds. The result will follow if we show that

$$\phi(x) \sim s \log(1 - x)^{-1}.$$

By the analogue from Stieltjes integrals of Theorem 6 of (**3**) (applied to (14), regarded as a transformation from $g(u)/\log(1 - u)^{-1}$ to $\phi(x)/\log(1 - x)^{-1}$), the result will follow if we show that, as $x \to 1-$,

(i) $$\int_0^x \frac{\log(1-u)^{-1}}{1-u}\, d\chi\left(\frac{u}{x}\right) \sim \log(1-x)^{-1},$$

(ii) $$\int_0^x \left|\frac{\log(1-u)^{-1}}{1-u}\right| d\chi\left(\frac{u}{x}\right) = O(\log(1-x)^{-1}),$$

(iii) $$\int_0^v \left|\frac{\log(1-u)^{-1}}{1-u}\right| d\chi\left(\frac{u}{x}\right) = o(\log(1-x)^{-1})$$

for any fixed v with $0 < v < 1$.

Now consider the case in which $s_n = 1$ (all n). Then $g(u) = \log(1 - u)^{-1}$. Also, by (1), $t_n = 1$ (all n) so that $\phi(x) = \log(1 - x)^{-1}$. Applying (13) to this particular case, we therefore have that

$$\int_0^x \frac{\log(1-u)^{-1}}{1-u}\, d\chi\left(\frac{u}{x}\right) = \log(1-x)^{-1} \qquad (0 < x < 1)$$

which gives (i). Further, since χ is non-decreasing, we may omit the modulus signs in (ii) and (iii) so that (ii) is a trivial consequence of (i). Also, for fixed v, the expression on the left of (iii) is bounded for $v < x < 1$ (even though the bound will, of course, depend on v). It is thus a fortiori $o(\log(1 - x)^{-1})$, and the proof is completed.

LEMMA 4. *If $\sum_{\nu=0}^{\infty} a_\nu$ is summable* (L) *to s, then any sequence of regular Hausdorff means of $\{s_n\}$ is* L*-convergent to s.*

This is Theorem 5 of (**2**).

LEMMA 5. *Suppose that $\{s_n\}$ is any bounded (real or complex) sequence. Let $\{c_n(x)\}$ be a sequence of functions defined for $0 < x < 1$ and satisfying*

(15) $$\lim_{x\to 1-} c_n(x) = 0 \quad \textit{for } n = 0, 1, \ldots,$$

(16) $$\limsup_{x\to 1-} \sum_{\nu=0}^{\infty} |c_\nu(x)| = M < +\infty.$$

Then we have that

(17) $$\limsup_{x\to 1-} \left|\sum_{\nu=0}^{\infty} c_\nu(x) s_n\right| \leqq M \limsup_{n\to\infty} |s_n|.$$

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Moreover, M is the best constant in the following sense: there exists a bounded sequence $\{s_n\}$, $0 < \lim_{n\to\infty} \sup |s_n| < +\infty$, such that the members of inequality (17) *are equal.*

This lemma is due to R. P. Agnew (**1**).

3. Proof of Theorem 1. We first prove (3). We have that

$$s_{n(x)} - f(x) = \sum_{\nu=1}^{n(x)} a_\nu \left\{1 - \frac{\beta_\nu(x)}{\beta(x)}\right\} - \sum_{\nu=n(x)+1}^{\infty} a_\nu \frac{\beta_\nu(x)}{\beta(x)},$$

where

$$\beta_{n(x)}(x) = \sum_{\nu=n(x)}^{\infty} \frac{x^{\nu+1}}{\nu+1}, \qquad \beta(x) = \beta_0(x) = -\log(1-x).$$

Let

$$c_0(x) = 0,$$

$$c_1(x) = 1 - \frac{\beta_1(x)}{\beta(x)},$$

$$c_\nu(x) = \begin{cases} \dfrac{1}{\nu \log \nu}\left(1 - \dfrac{\beta_\nu(x)}{\beta(x)}\right) & (2 \leqq \nu \leqq n(x)), \\ -\dfrac{1}{\nu \log \nu}\dfrac{\beta_\nu(x)}{\beta(x)} & (\nu > n(x)). \end{cases}$$

Since

$$\lim_{x\to 1-} c_\nu(x) = 0$$

for every fixed ν, condition (15) is satisfied.

Now consider the sum

$$\sum_{\nu=0}^{\infty} |c_\nu(x)| = \frac{1}{\beta(x)}\left\{x + \sum_{\nu=2}^{n(x)} \frac{1}{\nu \log \nu}\left(x + \frac{x^2}{2} + \ldots + \frac{x^\nu}{\nu}\right)\right\} + \frac{1}{\beta(x)} \sum_{\nu=n(x)+1}^{\infty} \frac{\beta_\nu(x)}{\nu \log \nu}$$

$$= A_{n(x)} + B_{n(x)},$$

where

$$B_{n(x)} = \frac{1}{\beta(x)} \sum_{\nu=n(x)+1}^{\infty} \frac{\beta_\nu(x)}{\nu \log \nu} = O\left(\frac{1}{(1-x)\beta(x)} \sum_{\nu=n(x)+1}^{\infty} \frac{x^\nu}{\nu^2 \log \nu}\right) = O\left(\frac{n(x)}{\log n(x)} \sum_{\nu=n(x)+1}^{\infty} \frac{1}{\nu^2 \log \nu}\right) = o(1).$$

When $0 < x < 1$ and $x \to 1-$,

$$G(x) = \frac{1}{\beta(x)}\left\{x + \sum_{\nu=2}^{n(x)} \frac{1}{\nu \log \nu}\left(1 + \frac{1}{2} + \ldots + \frac{1}{\nu}\right)\right\} \sim \frac{\log n(x)}{\beta(x)} \to 1$$

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by (4), and

$$0 < G(x) - A_{n(x)} = \frac{1}{\beta(x)} \sum_{\nu=2}^{n(x)} \frac{1}{\nu \log \nu} \left(1 - x + \frac{1-x^2}{2} + \ldots + \frac{1-x^\nu}{\nu}\right) \leqq$$

$$\frac{1-x}{\beta(x)} \sum_{\nu=2}^{n(x)} \frac{1}{\log \nu} \leqq \frac{(1-x)n(x)}{\beta(x)} \to 0.$$

Hence, $A_{n(x)} \to 1$ and (16) is satisfied.

(5) can be proved in much the same way.

4. Proof of Theorem 2. Suppose that the series $\sum_{\nu=0}^{\infty} a_\nu$ is convergent. Then, since (l) summability is regular, it is summable (l) to s. Hence (7) is necessary.

Suppose now that (7) holds and that $s_n \to s$ (L). We have that

$$s_n = a_0 + \sum_{\nu=1}^{n} \frac{\sigma_\nu - \sigma_{\nu-1}}{\alpha_{\nu-1}} = a_0 + \sum_{\nu=1}^{n-1} \sigma_\nu \left(\frac{1}{\alpha_{\nu-1}} - \frac{1}{\alpha_\nu}\right) + \frac{\sigma_n}{\alpha_{n-1}}.$$

Let

$$u_0 = a_0, \qquad u_\nu = \sigma_\nu \left(\frac{1}{\alpha_{\nu-1}} - \frac{1}{\alpha_\nu}\right) \quad (\nu \geqq 1),$$

$$r_n = u_0 + u_1 + \ldots + u_n,$$

and

$$h(x) = \frac{-1}{\log(1-x)} \sum_{\nu=0}^{\infty} r_\nu \frac{x^{\nu+1}}{\nu+1}.$$

Then, by (7), $s_n - r_{n-1} \to 0$ as $n \to \infty$. Since

$$u_n = o\left(\frac{1}{n \log n}\right),$$

it follows from the second part of Theorem 1, with $x_n = 1 - 1/n$, that $r_{n-1} - h(x_n) \to 0$ and hence $s_n - f(x_n) \to 0$. Thus, $s_n \to s$ as $n \to \infty$. (7) is therefore sufficient.

5. Proof of Theorem 3. We first prove that the condition is necessary. Suppose that the series is summable (l) to s. Then, by the regularity of summability (l), the sequence $\{t_n'\}$, where t_n' is the nth (l) mean of the sequence $\{t_n\}$, tends to s. Hence

$$t_n - t_n' = \frac{1}{\alpha_n} \sum_{\nu=1}^{n} \frac{s_\nu - t_\nu}{\nu+1} = o(1), \tag{18}$$

i.e.,

$$w_n = \sum_{\nu=1}^{n} \frac{s_\nu - t_\nu}{\nu+1} = o(\log n).$$

Now

$$v_n = \sum_{\nu=1}^{n} \frac{\alpha_\nu}{\nu+1} (s_\nu - t_\nu) = \sum_{\nu=1}^{n-1} w_\nu(\alpha_\nu - \alpha_{\nu+1}) + w_n\alpha_n = o(\log^2 n).$$

Hence (8) is necessary.

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We shall now prove that the condition is sufficient. Suppose that (8) holds and that $\sum_{n=0}^{\infty} a_n$ is summable (L) to s. Then, from (18),

$$(19) \qquad t_n - t_n' = \frac{1}{\alpha_n} \sum_{\nu=1}^{n} \frac{\sigma_\nu}{(\nu+1)\alpha_\nu} = \frac{1}{\alpha_n} \sum_{\nu=1}^{n-1} v_\nu \left(\frac{1}{\alpha_\nu} - \frac{1}{\alpha_{\nu+1}}\right) + \frac{v_n}{\alpha_n^2} = o(1).$$

By Lemma 3, $t_n \to s$ (L), Hence, by Theorem 2 and (19), $t_n \to s$. Thus, (8) is sufficient.

6. Proof of Theorem 4. Let

$$c_n = \frac{s_0 + s_1 + \ldots + s_n}{n+1}.$$

Then, since $\sum_{\nu=0}^{\infty} a_\nu$ is summable (L) to s, the sequence $\{c_n\}$, by Lemma 4, is summable (L) to the same sum. It follows from (9) that $c_n = O(1)$, and hence $\{c_n\}$ is, by Lemma 1, summable (l) to s, that is,

$$T_n = \frac{1}{\alpha_n} \sum_{\nu=0}^{n} \frac{c_\nu}{\nu+1}$$

tends to s as $n \to \infty$. Write

$$T_n' = \frac{1}{\alpha_n} \sum_{\nu=0}^{n} \frac{c_\nu}{\nu+2}.$$

Then, since c_ν is bounded,

$$T_n - T_n' = \frac{1}{\alpha_n} \sum_{\nu=0}^{n} \frac{c_\nu}{(\nu+1)(\nu+2)} = O\left(\frac{1}{\alpha_n}\right) = o(1).$$

Thus, $T_n' \to s$ as $n \to \infty$. Now

$$T_n' = \frac{1}{\alpha_n} \sum_{\nu=0}^{n} \frac{s_0 + s_1 + \ldots + s_\nu}{(\nu+1)(\nu+2)} = t_n - \frac{(n+1)c_n}{(n+2)\alpha_n}.$$

Hence, $t_n \to s$ as $n \to \infty$. This proves Theorem 4.

7. Proof of Theorems 5 and 6. We first prove Theorem 5. From Lemma 3, the sequence $\{s_n - t_n\}$ is L-convergent to 0. Also, from (10), $s_n - t_n = O_L(1)$. Therefore the sequence $\{s_n - t_n\}$ is summable (l) to 0 by Lemma 1, that is,

$$t_n - \frac{1}{\alpha_n} \sum_{\nu=0}^{n} \frac{t_\nu}{\nu+1} = o(1).$$

By lemma 3 and Theorem 2, with s_n replaced by t_n, $t_n \to s$ as $n \to \infty$. This proves Theorem 5. Theorem 6 follows from Lemmas 2 and 3.

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