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UNIQUENESS IN STRUCTURE THEOREMS FOR LCA GROUPS

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The classical Pontrjagin-van Kampen structure theorem states that any locally compact abelian (LCA) group G can be written as the direct product of a vector group R^m (where R denotes the additive group of real numbers with the usual topology, and m is a non-negative integer) and an LCA group H which contains a compact open subgroup. This important theorem, which van Kampen deduced from the work of Pontrjagin, was first stated and proved in [**5**, p. 461]. A more modern proof may be found in the monograph of Hewitt and Ross [**4**, 24.30]. As in the latter reference, the theorem is often accompanied by a uniqueness statement, namely, that if $R^m \times H$ and $R^n \times K$ are both representations of the type described above for a given LCA group G, then $m = n$. One can hardly refrain from asking whether or not we also have $H \cong K$ (here and throughout we use $\cong$ for topological isomorphism). No such assertion is to be found either in van Kampen's paper or in the book of Hewitt and Ross.

Nevertheless, the assertion that H and K are indeed topologically isomorphic does occur in the literature, but, as far as the authors are aware, only twice. The first is in Braconnier's important 1948 paper [**2**, Théorème 1, p. 4]. Braconnier, giving no proof, refers the reader to the van Kampen paper cited above, where, as we have noted, the assertion is not made. Braconnier also refers to certain sections of Pontrjagin's book [**6**], but again, no such uniqueness statement is to be found there. The second reference in the literature may be found in Corwin's 1970 paper [**3**, Theorem 1.2]. The author, making no reference to Braconnier's paper, invokes the five-lemma to sketch a proof that $H \cong K$, several of the details being left to the reader.

What Braconnier probably had in mind and what is implicit in Corwin's proof is the following line of attack in showing that $H \cong K$. Let f be a topological isomorphism from $R^m \times H$ onto $R^n \times K$. Let i be the injection from H into $R^m \times H$ and let p be the projection from $R^n \times K$ onto K. Then the composition $g = pfi$ is the desired topological isomorphism from H onto K. There is sufficient machinery developed in van Kampen's paper for Braconnier to have proved this easily, and, as pointed out above, Corwin does it by using the five-lemma.

The argument that g is the desired isomorphism between H and K is the starting point for our results. We shall formulate a rather general and quite

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elementary uniqueness theorem (Theorem 1) for direct products of topological groups (not necessarily locally compact or even abelian). This theorem will illustrate a general way to prove that maps similar in construction to g are topological isomorphisms. The theorem will provide a unified approach to various uniqueness problems for decompositions of LCA groups. In particular, it yields an immediate proof of the Braconnier-Corwin result (see Corollary 1) as well as a uniqueness statement for a more refined structure theorem due to Robertson and hitherto unpublished.

We shall employ additive notation for all groups, whether abelian or not. If A and B are topological groups, then $A \times B$ denotes the Cartesian product of A and B with the product topology.

THEOREM 1. *Let A_i and B_i be topological groups, and let C_i be a subgroup of B_i for $i = 1, 2$. Suppose that f is a topological isomorphism from $A_1 \times B_1$ onto $A_2 \times B_2$ such that*

(i) $f(A_1 \times C_1) = A_2 \times C_2$ *and*
(ii) $f(\{0\} \times C_1) = \{0\} \times C_2$.

Then $A_1 \cong A_2$ and $B_1 \cong B_2$.

Proof. We first show that $B_1 \cong B_2$. Let i be the injection from B_1 into $A_1 \times B_1$ and let p be the projection from $A_2 \times B_2$ onto B_2. Define $g\colon B_1 \to B_2$ by $g = pfi$ (all maps are written to the left of their arguments). It is evident that g is a continuous homomorphism. We will show that g is a topological isomorphism from B_1 onto B_2.

By (ii) it is clear that $g(C_1) = C_2$. Let g' be g restricted to C_1. It is easy to see that g' is a topological isomorphism from C_1 onto C_2, for in essence g' is just f restricted to C_1. We use this fact to show that g is the desired isomorphism from B_1 onto B_2. Suppose first that $g(b_1) = 0$ for some b_1 in B_1. This means that $fi(b_1) \in \ker(p) = A_2 \times \{0\} \subseteq A_2 \times C_2$, so $i(b_1) \in A_1 \times C_1$ by condition (i). Therefore $b_1 \in C_1$, so $0 = g(b_1) = g'(b_1)$, so $b_1 = 0$ and g is one-one. We complete the argument that $B_1 \cong B_2$ by constructing a continuous mapping $h\colon B_2 \to B_1$ which turns out to be the inverse of g. We do this by constructing two continuous mappings u and v such that if we set $h = u + v$ we have $gh(b_2) = b_2$ for each b_2 in B_2.

Construction of u: Pick b_2 in B_2. Find (a_1, b_1) in $A_1 \times B_1$ such that f carries (a_1, b_1) to $(0, b_2)$. Set $u(b_2) = b_1$. Then $u\colon B_2 \to B_1$ is a continuous homomorphism, as may be seen from the "sequence"

$$b_2 \longrightarrow (0, b_2) \xrightarrow{f^{-1}} (a_1, b_1) \longrightarrow b_1 = u(b_2).$$

Construction of v: Letting b_2 and (a_1, b_1) be as above, we see by condition (i) that the image under f of $(a_1, 0)$ has the form (a_2, c_2) for some $a_2 \in A_2$ and $c_2 \in C_2$. Set $v(b_2) = (g')^{-1}(c_2)$. Then $v\colon B_2 \to C_1$ is a continuous homomorphism, as may be seen from the "sequence"

$$b_2 \longrightarrow (0, b_2) \xrightarrow{f^{-1}} (a_1, b_1) \longrightarrow (a_1, 0) \xrightarrow{f} (a_2, c_2) \longrightarrow c_2 \longrightarrow v(b_2).$$

We now set $h = u + v$. It is clear that h is a continuous mapping from B_2 to B_1 (although it is not immediately clear that h is a homomorphism, since B_1 need not be abelian). Now let b_2 be an element of B_2 and retain all of the notation used in the construction of u and v. Then $gu(b_2) = g(b_1) = pfi(b_1) = pf(0, b_1) = pf(a_1, b_1) - pf(a_1, 0) = p(0, b_2) - p(a_2, c_2) = b_2 - c_2$, while $gv(b_2) = g(g')^{-1}(c_2) = c_2$. Hence $gh(b_2) = (b_2 - c_2) + c_2 = b_2$. Therefore g has h for continuous inverse (and hence h is an isomorphism), so that g is a topological isomorphism from B_1 onto B_2.

It remains to show that $A_1 \cong A_2$. This could be done by an independent and simple argument, but it is perhaps quicker to use what we have already proved. Suppose that G_i and H_i are topological groups and K_i is a subgroup of G_i for $i = 1, 2$. Suppose further that there is a topological isomorphism φ from $G_1 \times H_1$ onto $G_2 \times H_2$ such that (i′) $\varphi(K_1 \times H_1) = K_2 \times H_2$ and (ii′) $\varphi(K_1 \times \{0\}) = K_2 \times \{0\}$. An evident modification of our earlier argument shows that $G_1 \cong G_2$. Let us then take $G_i = A_i$, $H_i = C_i$ and $K_i = \{0\}$ for $i = 1, 2$ and take for φ the restriction of f to $A_1 \times C_1$. It is then evident that conditions (i′) and (ii′) are satisfied, so $A_1 \cong A_2$, which completes the proof of the theorem.

We can now give an immediate proof of the Braconnier-Corwin statement.

COROLLARY 1. *Suppose $R^m \times H \cong R^n \times K$, where m and n are non-negative integers and H and K are LCA groups each containing a compact open subgroup. Then $m = n$ and $H \cong K$.*

Proof. In Theorem 1 take $A_1 = R^m$, $A_2 = R^n$, $B_1 = H$, $B_2 = K$ and let C_1 and C_2 be the identity components of H and K respectively. If f is a topological isomorphism from $A_1 \times B_1$ onto $A_2 \times B_2$, then condition (i) holds since identity components must correspond, while condition (ii) holds since the compact elements [**4**, 9.9] ofthe identity components must correspond. Hence $R^m \cong R^n$ and $H \cong K$. Finally, it is easy to see that a continuous isomorphism between the topological groups R^m and R^n must also be a linear mapping between R^m and R^n considered as vector spaces. Hence $m = n$, and the proof is complete.

It should perhaps be pointed out here that Theorem 1 and its corollary provide a very easy proof of the uniqueness of decomposition of a compactly generated LCA group, a problem which is treated at some length in [**4**, 9.12 and 9.13]. For if G is compactly generated and $G \cong R^m \times Z^p \times H \cong R^n \times Z^q \times K$, where m, n, p, and q are non-negative integers, Z is the additive group of integers, and H and K are compact, then $m = n$ and $Z^p \times H \cong Z^q \times K$ by Corollary 1. Taking $A_1 = Z^p$, $A_2 = Z^q$, $B_1 = C_1 = H$ and $B_2 = C_2 = K$, we conclude from Theorem 1 that $Z^p \cong Z^q$ (and hence $p = q$ by a rank argument) and $H \cong K$. In sum, $m = n$, $p = q$ and $H \cong K$, which is to say that the decomposition of G is unique.

We now state and prove a structure theorem of Robertson [**8**] for LCA groups; this theorem constitutes a refinement of the Pontrjagin-van Kampen theorem. Our proof of the existence of the decomposition is a simplification of Robertson's proof, which remains unpublished; the corresponding uniqueness statement is new and follows from Theorem 1. In our opinion Robertson's theorem is the best structure theorem available for general LCA groups, and the accompanying uniqueness statement is useful in various contexts.

A word about notation. If G is an LCA group, $B(G)$ denotes the subgroup of compact (or bounded) elements of G [**4**, 7.10], while $D(G)$ denotes the maximal divisible subgroup of G [**4**, A.6]. If λ is a cardinal number, then $Q^{\lambda *}$ denotes the weak direct product of λ copies of the additive group of rational numbers, taken discrete. The dual of this group is the (full) direct product of λ copies of the dual group $\hat{Q}$ of Q with the product topology. This group will be denoted $\hat{Q}^{\lambda}$. The Pontrjagin Duality Theorem and its elementary consequences will be assumed throughout (see [**4**, §§ 23 and 24]).

LEMMA. *Let G be an LCA group having compact identity component. Then $G \cong Q^{\lambda *} \times H$, where λ is a cardinal number and H is an LCA group satisfying $D(H) \subseteq B(H)$. The representation is unique in the sense that if $Q^{\mu *} \times K$ is another such representation, then $\lambda = \mu$ and $H \cong K$.*

Proof. Let $\overline{D(G)}$ be the closure in G of $D(G)$. It is evident that each nontrivial character of $\overline{D(G)}$ must have infinite order, so the dual group $(\overline{D(G)})\hat{}$ is torsion-free. Now since G has compact identity component, G must have a compact open subgroup. Hence the same is true of $\overline{D(G)}$ and $(\overline{D(G)})\hat{}$. It follows from this that the identity component U of $(\overline{D(G)})\hat{}$ is compact. We now combine [**4**, 25.30(c), 25.8 and 25.4] to conclude that $(\overline{D(G)})\hat{}$ is topologically isomorphic with the internal direct sum of U and V, where $U \cong \hat{Q}^{\lambda}$ and V is a totally disconnected closed subgroup. By [**4**, 24.17] the elements of V are compact. Hence $\overline{D(G)}$ is itself the internal direct sum of two of its closed subgroups S and T, where $S \cong \hat{U}$, $T \cong \hat{V}$ and $T \subseteq B(G)$. Since $S \cong Q^{\lambda *}$ we have $S \cap B(G) = \{0\}$. Since S is divisible we conclude from [**4**, A.8] that there is a subgroup H of G containing $B(G)$ such that algebraically G is the internal direct sum of S and H. Now $B(G)$ is open in G by [**4**, 9.26(a)], so H is open in G. It follows from [**4**, 6.11] that the decomposition is topological, that is, $G \cong S \times H$. We also have $D(H) \subseteq B(H)$. For if $h \in D(H)$ then certainly $h \in \overline{D(G)}$, which is the direct sum of S and T, so there exist $s \in S$ and $t \in T$ such that $h = s + t$. Hence $s = h - t \in H$, since $T \subseteq B(G) \subseteq H$. But $S \cap H = \{0\}$, so $h = t \in B(G)$, which means that $h \in B(H)$. Therefore $G \cong Q^{\lambda *} \times H$ with $D(H) \subseteq B(H)$.

For the uniqueness part we use Theorem 1. Set $A_1 = Q^{\lambda *}$, $A_2 = Q^{\mu *}$, $B_1 = H$, $B_2 = K$, $C_1 = D(H)$ and $C_2 = D(K)$. It is evident that conditions (i) and

(ii) must hold for any topological isomorphism from $A_1 \times B_1$ onto $A_2 \times B_2$. Therefore $Q^{\lambda *} \cong Q^{\mu *}$, whence $\lambda = \mu$ by a rank argument [**4**, A.14] and $H \cong K$.

Remark 1. If G has compact identity component, then the statements (a) G has no subgroups topologically isomorphic with Q and (b) $D(G) \subseteq B(G)$ are equivalent. For it is clear that (b) implies (a). The other implication is most quickly proved by the lemma. If (a) holds for G then in the representation $G \cong Q^{\lambda *} \times H$ we must have $\lambda = 0$, so $G \cong H$ and $D(G) \subseteq B(G)$.

Definition. Call an LCA group G *residual* if and only if $D(G) \subseteq B(G)$ and $D(\hat{G}) \subseteq B(\hat{G})$.

It is evident that G is residual if and only if $\hat{G}$ is residual. It follows further from the remark that if G has compact identity component, then G is residual if and only if neither G nor $\hat{G}$ contains an isomorphic copy of the discrete group Q. The name "residual" is motivated by the following result, which, except for the uniqueness part, is due originally to Robertson [**8**].

THEOREM 2. *Any LCA group G can be written in the form $G \cong R^n \times Q^{\lambda *} \times \hat{Q}^{\mu} \times E$, where n is a non-negative integer, λ and μ are cardinal numbers and E is residual. The representation is unique in the sense that if $R^m \times Q^{\rho *} \times \hat{Q}^{\sigma} \times F$ is another such representation, then $m = n$, $\rho = \lambda$, $\sigma = \mu$, and $F \cong E$.*

Proof. Write $G \cong R^n \times H$, where n is a non-negative integer and H has compact open subgroup and hence compact identity component. By the lemma we can write $H \cong Q^{\lambda *} \times L$, where $D(L) \subseteq B(L)$. Now $\hat{L}$ also has compact identity component, so $\hat{L} \cong Q^{\mu *} \times M$, where $D(M) \subseteq B(M)$. Set $E = \hat{M}$. Then $G \cong R^n \times Q^{\lambda *} \times \hat{Q}^{\mu} \times E$, and one verifies directly that E is residual.

As to the uniqueness, suppose that G has two representations as in the statement of the theorem. By Corollary 1 we get $n = m$ and $Q^{\lambda *} \times \hat{Q}^{\mu} \times E \cong Q^{\rho *} \times \hat{Q}^{\sigma} \times F$. By the uniqueness part of the lemma we conclude that $\lambda = \rho$ and $\hat{Q}^{\mu} \times E \cong \hat{Q}^{\sigma} \times F$. By taking duals we have $Q^{\mu *} \times \hat{E} \cong Q^{\sigma *} \times \hat{F}$, so again by the lemma we have $\mu = \sigma$ and $\hat{E} \cong \hat{F}$, whence $E \cong F$, which completes the proof.

Remark 2. E may contain closed subgroups of the form $\hat{Q}$. For instance, let D be a discrete free abelian group having quotient Q. Set $G = \hat{D}$. Then G has a closed subgroup of the form $\hat{Q}$, but $G = E$ in the decomposition of G.

As Theorem 2 shows, each LCA group G has four "invariants" associated with it. Three of these are cardinal numbers, while the fourth is the "residual part" of G. Thus the study of LCA groups is reduced to that of residual LCA groups. For example, G is self-dual if and only if $\lambda = \mu$ and $E \cong \hat{E}$ in the decomposition of G. Admittedly this is not much help in determining the self-dual LCA groups. Nevertheless, the decomposition, besides being of some intrinsic interest, is useful in a number of situations. We cite as an example

Mackey's structure theorem for divisible torsion-free LCA groups (see [**4**, 25.33]). We present here a short proof derived from Theorem 2.

COROLLARY 2. *A divisible torsion-free LCA group G can be written uniquely in the form $G = R^n \times Q^{\lambda *} \times \hat{Q}^{\mu} \times E$, where n is a non-negative integer, λ and μ are cardinal numbers and E is a minimal divisible extension of a product of p-adic integer groups for various primes p.*

Proof. For the existence part, it suffices to show that the residual part E of G has the form mentioned. Now since E is torsion-free, $\hat{E}$ has a divisible dense subgroup (see [**7**, Theorem 5.2], or [**1**, Theorem 1] for an altogether different proof), so $\hat{E} = \overline{D(\hat{E})} \subseteq B(\hat{E})$. Thus the elements of $\hat{E}$ are compact, so E is totally disconnected [**4**, 24.17]. In particular, E has a compact open subgroup K. It then follows from [**4**, 25.8] that K is a product of p-adic integer groups. Since E is divisible, we have $E = B(E)$, so E/K is a torsion group. Hence E is a minimal divisible extension of K (see [**4**, 25.32 and A.17]), which completes the existence part of the proof.

As to the uniqueness, suppose that G has a representation of the type described. Then it is easy to see from the form of E that both E and $\hat{E}$ are totally disconnected, so the elements of E and of $\hat{E}$ are compact. In particular, $D(E) \subseteq B(E)$ and $D(\hat{E}) \subseteq B(\hat{E})$, so E is residual. It now follows from the uniqueness part of Theorem 2 that λ, μ and E are uniquely determined. We may carry the argument a bit further and show that the particular product of p-adic integer groups of which E is the minimal divisible extension is also uniquely determined, but this requires a separate argument (see [**4**, 25.33(b)]).

We close with a few remarks about cancellation of LCA groups. We say that an LCA group G is *cancellable* if and only if from $G \times H \cong G \times K$ it follows that $H \cong K$ for each pair of LCA groups H and K. It is quite difficult to say much about cancellable LCA groups generally, and to show that a given LCA group is cancellable can present quite a challenge (even the problem of describing the cancellable groups in the class of discrete abelian groups is far from solution). We show here, as a by-product of Corollary 1 and Theorem 2, that R and Q are cancellable.

As for R, suppose that $R \times H \cong R \times K$. Write $H \cong R^n \times H'$ and $K \cong R^m \times K'$, where m and n are non-negative integers and H' and K' contain compact open subgroups. Then $R^{n+1} \times H' \cong R^{m+1} \times K'$, so $m = n$ and $H' \cong K'$ by Corollary 1. Therefore $H \cong K$ and R is cancellable. The proof that Q is cancellable follows the same pattern but uses Theorem 2 instead of Corollary 1. Of course, an LCA group G is cancellable if and only if its dual $\hat{G}$ is cancellable, so $\hat{Q}$ is cancellable as well. In the absence of structure theorems involving groups other than R and Q (with corresponding uniqueness statements) it appears to be generally quite difficult to decide whether a given LCA group, even one having a simple nature, is cancellable. For example, we do not know whether the p-adic integers or p-adic numbers are cancellable groups.

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