Forum of Mathematics, Sigma (2023), **Vol. 11**:e21 1–40
doi:10.1017/fms.2023.17

CAMBRIDGE UNIVERSITY PRESS

RESEARCH ARTICLE

Extremal Gromov-Witten invariants of the Hilbert scheme of 3 Points

Jianxun Hu [1] and Zhenbo Qin [2]

[1]School of Mathematics, Sun Yat-Sen University, Guangzhou 510275, China; E-mail: stsjxhu@mail.sysu.edu.cn.
[2]Department of Mathematics, University of Missouri, Columbia, MO 65211, USA; E-mail: qinz@missouri.edu.

Received: 8 December 2021; **Revised:** 28 December 2022; **Accepted:** 23 February 2023

2020 Mathematics Subject Classification: *Primary* – 14C05, 14N35

Abstract
We determine all the extremal Gromov-Witten invariants of the Hilbert scheme of 3 points on a smooth projective complex surface. Our result for the genus-1 case verifies a conjecture that we propose for the genus-1 extremal Gromov-Witten invariant of the Hilbert scheme of n points with n being arbitrary. The main ideas in the proofs are to use geometric arguments involving the cosection localization theory of Kiem and J. Li [17, 23], algebraic manipulations related to the Heisenberg operators of Grojnowski [13] and Nakajima [34], and the virtual localization formulas of Gromov-Witten theory [12, 20, 30].

Contents

1. Introduction

Hilbert schemes have been studied extensively since the pioneering work of Grothendieck [14]. It is well known [3, 10, 16] that the Hilbert schemes of points, parametrizing 0-dimensional closed subschemes, on algebraic surfaces are smooth and irreducible. In fact, these Hilbert schemes are crepant resolutions of the symmetric products of the corresponding surfaces, via the Hilbert-Chow morphism which maps a 0-dimensional closed subscheme to its support (counting with multiplicities). Their extremal Gromov-Witten invariants are defined via the moduli spaces of stable maps whose images are contracted by the Hilbert-Chow morphism and are motivated by Ruan's Cohomological Crepant Resolution Conjecture [39] which eventually evolves to the Crepant Resolution Conjecture of Bryan and Graber [4], Coates, Corti, Iritani and Tseng [5], and Coates and Ruan [7]. Their 1-point genus-0 extremal Gromov-Witten invariants are obtained in [27]. Okounkov and Pandharipande [36] studied the genus-0 equivariant extremal Gromov-Witten theory of the Hilbert schemes of points on the affine plane $\mathbb{C}^2$. Using cosection localization theory [17], J. Li and W-P. Li [23] determined the 2-point genus-0 extremal Gromov-Witten invariants of the Hilbert schemes of points on surfaces. The structures of the 3-point genus-0 extremal Gromov-Witten invariants of these Hilbert schemes are analyzed in [28] where Ruan's Cohomological Crepant Resolution Conjecture for the Hilbert-Chow morphism is verified. Higher genus equivariant extremal Gromov-Witten theory of the Hilbert schemes of points on $\mathbb{C}^2$ is investigated by Pandharipande and Tseng [37]. We refer to the survey book [38] for more details and to [33, 35] for related works.

In this paper, we work out explicitly all the extremal Gromov-Witten invariants of the Hilbert scheme of 3 points on a smooth projective complex surface X. Let $X^{[n]}$ be the Hilbert scheme of n points on X. For $n \geq 2$, the extremal k-point genus-g Gromov-Witten invariants of $X^{[n]}$ are of the form

$$\langle \gamma_1, \ldots, \gamma_k \rangle_{g, d\beta_n}$$

where $d \geq 0$, $\gamma_1, \ldots, \gamma_k \in H^*(X^{[n]}, \mathbb{C})$, and β_n is (the homology class of the curve)

$$\left\{ \xi + x_2 + \ldots + x_{n-1} \in X^{[n]} | \mathrm{Supp}(\xi) = \{x_1\} \right\} \cong \mathbb{P}^1$$

with $x_1, x_2, \ldots, x_{n-1}$ being distinct and fixed points in X. By degree reasons, if $g \geq 2$, all genus-g extremal Gromov-Witten invariants of $X^{[n]}$ are equal to 0.

We begin with genus-0 extremal Gromov-Witten invariants of $X^{[3]}$. By the Fundamental Class Axiom and the Divisor Axiom, these invariants are reduced to the following 1-point, 2-point and 3-point genus-0 extremal Gromov-Witten invariants:

$$\langle \tilde{\omega}_1 \rangle_{0, d\beta_3}, \quad \langle \tilde{\omega}_1, \tilde{\omega}_2 \rangle_{0, d\beta_3}, \quad \langle \omega_1, \omega_2, \omega_3 \rangle_{0, d\beta_3}$$

with $\tilde{\omega}_1, \tilde{\omega}_2 \in H^*(X^{[3]}, \mathbb{C})$ and $\omega_1, \omega_2, \omega_3 \in H^4(X^{[3]}, \mathbb{C})$. The invariants $\langle \tilde{\omega}_1 \rangle_{0, d\beta_3}$ and $\langle \tilde{\omega}_1, \tilde{\omega}_2 \rangle_{0, d\beta_3}$ have been computed in [27] and [23], respectively. When $X = \mathbb{P}^2$, $\langle \omega_1, \omega_2, \omega_3 \rangle_{0, d\beta_3}$ is partially calculated in [8]. The calculations in the 2003 paper [8] for $X = \mathbb{P}^2$ are incomplete due to the lack of understanding of the invariants $\langle \tilde{\omega}_1, \tilde{\omega}_2 \rangle_{0, d\beta_3}$ which appear later in the 2011 paper [23]. To state our result, we fix a linear basis of $H^4(X^{[3]}, \mathbb{C})$ via the Heisenberg operators of Grojnowski [13] and Nakajima [34]:

$$\begin{aligned} &\mathfrak{a}_{-1}(1_X)^2 \mathfrak{a}_{-1}(x)|0\rangle, \quad \mathfrak{a}_{-3}(1_X)|0\rangle, \quad \mathfrak{a}_{-1}(\alpha_i)\mathfrak{a}_{-2}(1_X)|0\rangle, \\ &\mathfrak{a}_{-2}(\alpha_i)\mathfrak{a}_{-1}(1_X)|0\rangle, \quad \mathfrak{a}_{-1}(\alpha_i)\mathfrak{a}_{-1}(\alpha_j)\mathfrak{a}_{-1}(1_X)|0\rangle \end{aligned} \tag{1.1}$$

where $\{\alpha_1, \ldots, \alpha_s\}$ is a linear basis of $H^2(X, \mathbb{C})$, $1 \leq i, j \leq s$ and 1_X and x stand for the fundamental classes of X and a point in X, respectively. Let $\langle \alpha, \beta \rangle = \alpha \cdot \beta$ denote the standard pairing for $\alpha, \beta \in H^*(X, \mathbb{C})$.

Theorem 1.1. *Let X be a simply connected projective surface. Let $\mathfrak{B}^4$ stand for the linear basis of $H^4(X^{[3]}, \mathbb{C})$ from (1.1). Let $d \geq 1$ and $\omega_1, \omega_2, \omega_3 \in \mathfrak{B}^4$. Then,*

$$\langle \omega_1, \omega_2, \omega_3 \rangle_{0, d\beta_3} = 0$$

if the unordered triple $(\omega_1, \omega_2, \omega_3)$ is not one of the following cases:

(i) $\Big(\mathfrak{a}_{-2}(1_X)\mathfrak{a}_{-1}(\alpha_i)|0\rangle, \mathfrak{a}_{-2}(1_X)\mathfrak{a}_{-1}(\alpha_j)|0\rangle, \mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(\alpha_k)|0\rangle\Big)$;
(ii) $\omega_1 = \omega_2 = \mathfrak{a}_{-3}(1_X)|0\rangle$, *and* $\omega_3 = \mathfrak{a}_{-2}(1_X)\mathfrak{a}_{-1}(\alpha_i)|0\rangle$ *or* $\mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(\alpha_i)|0\rangle$;
(iii) $\omega_1 = \omega_2 = \omega_3 = \mathfrak{a}_{-3}(1_X)|0\rangle$.

Moreover, $\langle \omega_1, \omega_2, \omega_3 \rangle_{0,d\beta_3} = 8\langle \alpha_i, \alpha_j \rangle \langle K_X, \alpha_k \rangle$ *in case (i), and*

$$\langle \omega_1, \omega_2, \omega_3 \rangle_{0,d\beta_3} = -2 \langle K_X, \alpha_i \rangle \, dc_{3,d}$$

in case (ii), where $c_{3,d}$ *is the universal constant from (3.16). In case (iii),*

$$\langle \omega_1, \omega_2, \omega_3 \rangle_{0,d\beta_3} = \left(-18 + 5dc_{3,d} - 2\sum_{i=1}^{d-1} ic_{3,i} + \frac{1}{3}\sum_{i=1}^{d-1} ic_{3,i} \, (d-i)c_{3,d-i} \right) K_X^2.$$

The universal constants $c_{3,d}$ appearing in Theorem 1.1 come from (3.16) which governs the 2-point genus-0 extremal Gromov-Witten invariants of $X^{[n]}$ via (3.15). The assumption that X is simply connected is intended only to shorten the statement of Theorem 1.1. The proof of Theorem 1.1 uses geometric arguments involving applications of cosection localization theory [17, 23] and algebraic manipulations involving the composition law of Gromov-Witten theory and the Heisenberg algebra of Grojnowski [13] and Nakajima [34].

As an application of Theorem 1.1, we obtain a direct proof of Ruan's Cohomological Crepant Resolution Conjecture for the Hilbert-Chow morphism $\rho_3 : X^{[3]} \to X^{(3)}$ (we remark that Ruan's Cohomological Crepant Resolution Conjecture for the Hilbert-Chow morphism $\rho_n : X^{[n]} \to X^{(n)}$ has been proved in [28] for all $n \geq 1$ via a representation theoretic approach).

Corollary 1.2. *Let X be a simply connected smooth projective surface. Then Ruan's Cohomological Crepant Resolution Conjecture for the Hilbert-Chow morphism* $\rho_3 : X^{[3]} \to X^{(3)}$ *holds (i.e., the Chen-Ruan cohomology ring of* $X^{(3)}$ *is isomorphic to the quantum corrected cohomology ring of* $X^{[3]}$*).*

We refer to the proof of Corollary 4.11 (= Corollary 1.2) for the precise statement of Ruan's Cohomological Crepant Resolution Conjecture for the Hilbert-Chow morphism $\rho_n : X^{[n]} \to X^{(n)}$.

Next, we consider the genus-1 extremal Gromov-Witten invariants of $X^{[3]}$. To put our result in perspective, note that all the genus-1 extremal Gromov-Witten invariants of $X^{[n]}$ with $n \geq 2$ can be reduced to $\langle\rangle_{1,d\beta_n}$. Let $\chi(X)$ be the Euler characteristic of X. We propose the following conjecture for the invariants $\langle\rangle_{1,d\beta_n}$.

Conjecture 1.3. *Let X be a smooth projective surface. Let* $n \geq 2$ *and* $d \geq 1$*. Then there exists a universal polynomial* $p_{n,d}(s,t)$*, independent of X, in variables s and t such that* $p_{n,d}(s^2,t) \cdot s^2$ *has degree n in s and t, and*

$$\langle\rangle_{1,d\beta_n} = p_{n,d}\big(K_X^2, \chi(X)\big) \cdot K_X^2.$$

Indeed, by [15, Theorem 1.2], Conjecture 1.3 holds for $n = 2$:

$$\langle\rangle_{1,d\beta_2} = \frac{1}{12d} \cdot K_X^2$$

(i.e., $p_{2,d}(s,t)$ is the constant polynomial $1/(12d)$). When $X = \mathbb{C}^2$, [37, (0.8)] presents a formula for $\langle\rangle_{1,d\beta_2}$ in the equivariant setting. We prove that Conjecture 1.3 holds for $n = 3$ and $d \geq 1$ as well (see Lemma 5.1):

$$\langle\rangle_{1,d\beta_3} = (a_d + b_d \cdot \chi(X)) \cdot K_X^2 \tag{1.2}$$

where a_d and b_d are universal constants depending only on d. A major part of our paper is to determine the universal constants a_d and b_d.

Theorem 1.4. *Let X be a smooth projective surface. Let* $d \geq 1$*, and let* f_d *be the constant defined in Lemma 5.10. Then,* $\langle\rangle_{1,d\beta_3}$ *is equal to*

https://doi.org/10.1017/fms.2023.17 Published online by Cambridge University Press

$$\left(f_d - \left(\frac{-d^2+d+16}{96d} + \frac{d}{48}\sum_{d_1=1}^{d-1}\frac{1}{d_1} - \frac{1}{48}\sum_{\delta\vdash d}\frac{d^2-d_1d_2}{d_1d_2\cdot|\mathrm{Aut}(\delta)|}\right) + \frac{1}{12d}\cdot\chi(X)\right)\cdot K_X^2$$

where $\delta = (d_1, d_2) \vdash d$ denotes a length-2 partition of d.

Using the definition of f_d in Lemma 5.10, one easily computes that $f_1 = 7/24$ (see also Example 5.11). However, for a general $d \geq 1$, it is unclear how to simplify the definition of f_d presented in Lemma 5.10. Note from (1.2) that to prove Theorem 1.4, it suffices to calculate a_d and b_d when X is a smooth projective toric surface. When X is a smooth projective toric surface, the torus

$$\mathbb{T} = (\mathbb{C}^*)^2$$

acts on X with finitely many fixed points x_i, $1 \leq i \leq \chi(X)$, which are the origins of the local affine charts $U_i \cong \mathbb{C}^2$, $1 \leq i \leq \chi(X)$. The induced $\mathbb{T}$-action on $X^{[3]}$ has finitely many fixed points and finitely many $\mathbb{T}$-invariant curves contracted by the Hilbert-Chow morphism. We then utilize the virtual localization formulas of Gromov-Witten theory ([12, 20] for the general setting and [8, 30] for our present setting of $X^{[3]}$). In the end, we reduce the computation of $\langle\rangle_{1,d\beta_3}$ to a certain summation $\sum_{\Gamma\in\mathcal{T}_{d,i}} - \sum_{\Gamma\in\mathcal{S}_{d,i,i}}$ over the local chart U_i, in terms of stable graphs Γ. To make our introduction here shorter, we refter to (5.13), (5.19) and (5.20) for notations and details. Next, we prove a reduction lemma (Lemma 5.4) which asserts that $\sum_{\Gamma\in\mathcal{T}_{d,i}} - \sum_{\Gamma\in\mathcal{S}_{d,i,i}}$ over the local chart $U_i \cong \mathbb{C}^2$ is of the form

$$a_d \cdot \frac{(w_i+z_i)^2}{w_iz_i} \tag{1.3}$$

where w_i and z_i are the weights for the torus action on U_i, and $a_d \in \mathbb{Q}$ is independent of i and X and depends only on d. This key reduction lemma implies that when evaluating $\sum_{\Gamma\in\mathcal{T}_{d,i}} - \sum_{\Gamma\in\mathcal{S}_{d,i,i}}$, we can ignore the stable graphs $\Gamma \in \mathcal{S}_{d,i,i}$ with more than 2 edges (see Lemma 5.6) and the stable graphs $\Gamma \in \mathcal{T}_{d,i}$ with more than 5 edges (see Lemma 5.9 for precise statements).

We remark that our reduction lemma (Lemma 5.4) may not be valid if one is only interested in calculating the analogous summation $\sum_{\Gamma\in\mathcal{T}_d}$, in the equivariant setting, for the Hilbert scheme $(\mathbb{C}^2)^{[n]}$. The reason is that in this new setting, the analogous summation $\sum_{\Gamma\in\mathcal{S}_{d,i,i}}$ does not arise, and thus $\sum_{\Gamma\in\mathcal{T}_d}$ cannot partially cancel with $\sum_{\Gamma\in\mathcal{S}_{d,i,i}}$ to simplify the computations. We refer to the related discussions on [37, p. 8] following [37, Theorem 5].

As for Conjecture 1.3 with $n > 3$, there are two possible approaches. The first one is to use the standard decomposition $\varphi = (\varphi_1, \ldots, \varphi_\ell)$ from [23] (see (3.13)) associated to a genus-1 extremal stable map $\varphi : C \to X^{[n]}$, as in the proof of Lemma 5.1 which is only for $X^{[3]}$. Intuitively, the standard decomposition splits the Hilbert scheme $X^{[n]}$ and the extremal stable map $\varphi : C \to X^{[n]}$ according to the support of $\varphi(C)$. However, complication arises when at least two of the maps $\varphi_1, \ldots, \varphi_\ell$ are not constant. The second approach to Conjecture 1.3 is to utilize the standard versus reduced method of Zinger, Vakil and Zinger, J. Li and Zinger, and Coates and Manolache (see [6, 26, 40, 41, 42] and the references therein), which transfers the computation of the standard Gromov-Witten invariants $\langle\rangle_{1,d\beta_n}$ to those of the reduced Gromov-Witten invariants. Roughly speaking, this approach splits the moduli space of genus-1 extremal stable maps into the main component (which gives rise to the reduced Gromov-Witten invariants) and the 'ghost' components (which are related to the genus-0 extremal Gromov-Witten invariants). A starting point might be to try the cases when both $n > 3$ and $d \geq 1$ are small.

Finally, the paper is organized as follows. Section 2 contains a brief introduction to Gromov-Witten theory. Section 3 presents some background materials of the Hilbert schemes of points on surfaces, including the Heisenberg algebra of Grojnowski [13] and Nakajima [34], Lehn's boundary operator [21], and their 1-point and 2-point genus-0 extremal Gromov-Witten invariants [23, 27]. Theorem 1.1 and Theorem 1.4 are proved in Section 4 and Section 5, respectively.

https://doi.org/10.1017/fms.2023.17 Published online by Cambridge University Press

2. Stable maps and Gromov-Witten invariants

Let Y be a smooth projective variety. A k-pointed *stable map* to Y consists of a complete nodal curve D with k distinct ordered smooth points $p_1, \dots, p_k$ and a morphism $\mu : D \to Y$ such that the data $(\mu, D, p_1, \dots, p_k)$ has only finitely many automorphisms. In this case, the stable map is denoted by $[\mu : (D; p_1, \dots, p_k) \to Y]$. For a fixed homology class $\beta \in H_2(Y, \mathbb{Z})$, let $\overline{\mathfrak{M}}_{g,k}(Y, \beta)$ be the coarse moduli space parameterizing all the stable maps $[\mu : (D; p_1, \dots, p_k) \to Y]$ such that $\mu_*[D] = \beta$ and the arithmetic genus of D is g. Then, we have the i-th evaluation map:

$$\mathrm{ev}_i : \overline{\mathfrak{M}}_{g,k}(Y, \beta) \to Y \tag{2.1}$$

defined by $\mathrm{ev}_i([\mu : (D; p_1, \dots, p_k) \to Y]) = \mu(p_i) \in Y$. It is known [1, 2, 11, 24, 25] that the coarse moduli space $\overline{\mathfrak{M}}_{g,k}(Y, \beta)$ is projective and has a virtual fundamental class $[\overline{\mathfrak{M}}_{g,k}(Y, \beta)]^{\mathrm{vir}} \in A_d(\overline{\mathfrak{M}}_{g,k}(Y, \beta))$ where

$$d = -(K_Y \cdot \beta) + (\dim(Y) - 3)(1 - g) + k \tag{2.2}$$

is the expected complex dimension of $\overline{\mathfrak{M}}_{g,k}(Y, \beta)$, and $A_d(\overline{\mathfrak{M}}_{g,k}(Y, \beta))$ is the Chow group of d-dimensional cycles in the moduli space $\overline{\mathfrak{M}}_{g,k}(Y, \beta)$.

The Gromov-Witten invariants are defined by using the virtual fundamental class $[\overline{\mathfrak{M}}_{g,k}(Y, \beta)]^{\mathrm{vir}}$. Recall that an element

$$\gamma \in H^*(Y, \mathbb{C}) \overset{\mathrm{def}}{=} \bigoplus_{j=0}^{2\dim_{\mathbb{C}}(Y)} H^j(Y, \mathbb{C})$$

is *homogeneous* if $\gamma \in H^j(Y, \mathbb{C})$ for some j; in this case, we take $|\gamma| = j$. Let $\gamma_1, \dots, \gamma_k \in H^*(Y, \mathbb{C})$ such that every γ_i is homogeneous and

$$\sum_{i=1}^{k} |\gamma_i| = 2d. \tag{2.3}$$

Then, we have the k-point Gromov-Witten invariant defined by:

$$\langle \gamma_1, \dots, \gamma_k \rangle_{g,\beta} = \int_{[\overline{\mathfrak{M}}_{g,k}(Y,\beta)]^{\mathrm{vir}}} (\mathrm{ev}_1 \times \cdots \times \mathrm{ev}_k)^*(\gamma_1 \otimes \ldots \otimes \gamma_k). \tag{2.4}$$

The *Fundamental Class Axiom* of the Gromov-Witten theory asserts that

$$\langle \gamma_1, \dots, \gamma_{k-1}, 1_Y \rangle_{g,\beta} = 0 \tag{2.5}$$

if either $k + 2g \geq 4$ or $\beta \neq 0$ and $k \geq 1$. The *Divisor Axiom* states that

$$\langle \gamma_1, \dots, \gamma_{k-1}, \gamma_k \rangle_{g,\beta} = \int_\beta \gamma_k \cdot \langle \gamma_1, \dots, \gamma_{k-1} \rangle_{g,\beta} \tag{2.6}$$

if $\gamma_k \in H^2(Y, \mathbb{C})$ and if either $k + 2g \geq 4$ or $\beta \neq 0$ and $k \geq 1$. A special case of the *Composition Law* (see the formulas (3.3) and (3.6) in [20]) states that

$$\begin{aligned} &\langle \gamma_1\gamma_2, \gamma_3, \gamma_4 \rangle_{0,\beta} + \langle \gamma_1, \gamma_2, \gamma_3\gamma_4 \rangle_{0,\beta} \\ &+ \sum_{\beta_1+\beta_2=\beta,\, \beta_1,\beta_2 \neq 0} \sum_a \langle \gamma_1, \gamma_2, \Delta_a \rangle_{0,\beta_1} \cdot \langle \Delta^a, \gamma_3, \gamma_4 \rangle_{0,\beta_2} \end{aligned}$$

https://doi.org/10.1017/fms.2023.17 Published online by Cambridge University Press

$$
\begin{aligned}
&= \langle \gamma_1\gamma_3, \gamma_2, \gamma_4\rangle_{0,\beta} + \langle \gamma_1, \gamma_3, \gamma_2\gamma_4\rangle_{0,\beta} \\
&\quad + \sum_{\beta_1+\beta_2=\beta,\, \beta_1,\beta_2\neq 0} \sum_a \langle \gamma_1, \gamma_3, \Delta_a\rangle_{0,\beta_1} \cdot \langle \Delta^a, \gamma_2, \gamma_4\rangle_{0,\beta_2}
\end{aligned} \tag{2.7}
$$

where $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in H^*(Y, \mathbb{C})$ are cohomology classes of even degrees, $\{\Delta_a\}_a$ denotes a homogeneous linear basis of $H^*(Y, \mathbb{C})$ and $\{\Delta^a\}_a$ is the linear basis of $H^*(Y, \mathbb{C})$ dual to $\{\Delta_a\}_a$ with respect to the standard pairing on $H^*(Y, \mathbb{C})$ (in the sense that $\langle \Delta_a, \Delta^b\rangle = \delta_{a,b}$ for all a and b).

3. Hilbert schemes of points on surfaces

In this section, we will review Hilbert schemes of points on surfaces, the Heisenberg algebra actions on the cohomology of these Hilbert schemes constructed by Grojnowski [13] and Nakajima [34], and Lehn's boundary operator [21]. Moreover, we will recall the 1-point and 2-point genus-0 extremal Gromov-Witten invariants of these Hilbert schemes from [23, 27].

3.1. Hilbert schemes of points and Heisenberg algebra actions

Let X be a smooth projective complex surface, and let $X^{[n]}$ be the Hilbert scheme of points in X. An element in $X^{[n]}$ is represented by a length-n 0-dimensional closed subscheme ξ of X. For $\xi \in X^{[n]}$, let I_ξ and $\mathcal{O}_\xi$ be the corresponding sheaf of ideals and structure sheaf, respectively. It is known [10, 16] that $X^{[n]}$ is a smooth irreducible variety of dimension $2n$. The boundary of $X^{[n]}$ is defined to be

$$
B_n = \Big\{\xi \in X^{[n]} \,|\, |\mathrm{Supp}(\xi)| < n\Big\}.
$$

For fixed distinct points $x_1, \ldots, x_{n-1} \in X$, define the curve

$$
\beta_n = \Big\{\xi + x_2 + \ldots + x_{n-1} \in X^{[n]} | \mathrm{Supp}(\xi) = \{x_1\}\Big\} \cong \mathbb{P}^1. \tag{3.1}
$$

We also regard β_n as a homology class in $H_2(X^{[n]}, \mathbb{Z})$. For a subset $Y \subset X$, define

$$
M_n(Y) = \{\xi \in X^{[n]} \,|\, \mathrm{Supp}(\xi) \text{ is a point in } Y\}.
$$

Sending an element in $X^{[n]}$ to its support (counting with multiplicities) in the n-th symmetric product $X^{(n)}$ of X, we obtain the Hilbert-Chow morphism

$$
\rho_n : X^{[n]} \to X^{(n)}
$$

which is a crepant resolution of singularities. A curve in $X^{[n]}$ is contracted by ρ_n if and only if it is homologous to $d\beta_n$ for some positive integer d.

Grojnowski [13] and Nakajima [34] geometrically constructed a Heisenberg algebra action on the cohomology of the Hilbert schemes $X^{[n]}$. Denote the Heisenberg operators by $\mathfrak{a}_m(\alpha)$ where $m \in \mathbb{Z}$ and $\alpha \in H^*(X, \mathbb{C})$. Put

$$
\mathbb{H}_X = \bigoplus_{n=0}^{+\infty} H^*(X^{[n]}, \mathbb{C}).
$$

The operators $\mathfrak{a}_m(\alpha) \in \mathrm{End}(\mathbb{H}_X)$ satisfy the following commutation relation:

$$
[\mathfrak{a}_m(\alpha), \mathfrak{a}_n(\beta)] = -m \cdot \delta_{m,-n} \cdot \langle \alpha, \beta\rangle \cdot \mathrm{Id}_{\mathbb{H}_X} \tag{3.2}
$$

where we have used $\delta_{m,-n}$ to denote 1 if $m = -n$ and 0 otherwise. The space $\mathbb{H}_X$ is an irreducible representation of the Heisenberg algebra generated by the operators $\mathfrak{a}_m(\alpha)$ with the highest weight

https://doi.org/10.1017/fms.2023.17 Published online by Cambridge University Press

vector being $|0\rangle = 1 \in H^*(X^{[0]}, \mathbb{C}) = \mathbb{C}$. In particular, $H^*(X^{[n]}, \mathbb{C})$ is the linear span of Heisenberg monomial classes

$$\mathfrak{a}_{-n_1}(\alpha_1) \cdots \mathfrak{a}_{-n_k}(\alpha_k)|0\rangle \tag{3.3}$$

where $k \geq 0, n_1, \ldots, n_k > 0$ and $\alpha_1, \ldots, \alpha_k \in H^*(X, \mathbb{C})$.

Fix closed real cycles $X_1, \ldots, X_k$ of the surface X in general position in the sense that any subset of the X_i's meet transversally in the expected dimension. Define

$$W(n_1, X_1; \ldots; n_k, X_k) \subset X^{[n]}$$

to be the closed subset consisting of all $\xi \in X^{[n]}$ which admit filtrations

$$\xi = \xi_k \supset \ldots \supset \xi_1 \supset \xi_0 = \emptyset$$

with $\ell(\xi_i) = \ell(\xi_{i-1}) + n_i$ and

$$\mathrm{Supp}(I_{\xi_{i-1}}/I_{\xi_i}) = x_i \in X_i \tag{3.4}$$

for $1 \leq i \leq k$. Let $W(n_1, X_1; \ldots; n_k, X_k)^0 \subset W(n_1, X_1; \ldots; n_k, X_k)$ be the open subset consisting of all $\xi \in W(n_1, X_1; \ldots; n_k, X_k)$ such that the points $x_1, \ldots, x_k$ in (3.4) are distinct.

Lemma 3.1 [38, Proposition 3.16]. *Let $\ell, k \geq 0$, $s_i \geq 0$ $(1 \leq i \leq \ell)$, $n_i > 0$ $(1 \leq i \leq k)$. Let $\alpha_1, \ldots, \alpha_k \in \bigoplus_{i=1}^{4} H^i(X, \mathbb{C})$ be represented by the cycles $X_1, \ldots, X_k \subset X$, respectively, such that $X_1, \ldots, X_k$ are in general position. Then,*

$$\left(\prod_{i=1}^{\ell} \frac{\mathfrak{a}_{-i}(1_X)^{s_i}}{s_i!}\right)\left(\prod_{i=1}^{k} \mathfrak{a}_{-n_i}(\alpha_i)\right)|0\rangle$$

is represented by the closure of

$$W(\underbrace{1, X; \ldots; 1, X}_{s_1 \text{ times}}; \ldots; \underbrace{\ell, X; \ldots; \ell, X}_{s_\ell \text{ times}}; n_1, X_1; \ldots; n_k, X_k)^0. \tag{3.5}$$

It follows that $1_{X^{[n]}} = 1/n! \cdot \mathfrak{a}_{-1}(1_X)^n|0\rangle \in H^0(X^{[n]}, \mathbb{C})$, where 1_X denotes the fundamental cohomology class of the surface X, and

$$\beta_n = \mathfrak{a}_{-2}(x)\mathfrak{a}_{-1}(x)^{n-2}|0\rangle, \tag{3.6}$$

$$B_n = \frac{1}{(n-2)!}\mathfrak{a}_{-1}(1_X)^{n-2}\mathfrak{a}_{-2}(1_X)|0\rangle, \tag{3.7}$$

where x denotes the fundamental cohomology class of a point $x \in X$. For simplicity, we do not distinguish a homology class and its Poincaré dual.

Let $\tau_2 : X \to X^2$ be the diagonal embedding and $\tau_{2*} : H^*(X, \mathbb{C}) \to H^*(X^2, \mathbb{C})$ be the induced map. For $\alpha \in H^*(X, \mathbb{C})$ and $m_1, m_2 \in \mathbb{Z}$, define

$$\mathfrak{a}_{m_1}\mathfrak{a}_{m_2}(\tau_{2*}\alpha) = \sum_i \mathfrak{a}_{m_1}(\alpha_{i,1})\mathfrak{a}_{m_2}(\alpha_{i,2})$$

https://doi.org/10.1017/fms.2023.17 Published online by Cambridge University Press

if $\tau_{2*}\alpha = \sum_i \alpha_{i,1} \otimes \alpha_{i,2}$ under the Künneth decomposition of $H^*(X^2, \mathbb{C})$. For $n \in \mathbb{Z}$ and $\alpha \in H^*(X, \mathbb{C})$, define the linear operator $\mathfrak{L}_n(\alpha) \in \mathrm{End}(\mathbb{H}_X)$ by

$$\mathfrak{L}_n = \begin{cases} -\dfrac{1}{2} \cdot \displaystyle\sum_{m\in\mathbb{Z}} \mathfrak{a}_m \mathfrak{a}_{n-m} \tau_{2*}, & \text{if } n \neq 0, \\ -\displaystyle\sum_{m>0} \mathfrak{a}_{-m} \mathfrak{a}_m \tau_{2*}, & \text{if } n = 0. \end{cases} \tag{3.8}$$

We have the commutation relation

$$[\mathfrak{L}_m(\alpha), \mathfrak{a}_n(\beta)] = -n \cdot \mathfrak{a}_{m+n}(\alpha\beta). \tag{3.9}$$

Lehn [21] defined the boundary operator $\mathfrak{d} \in \mathrm{End}(\mathbb{H}_X)$ by putting

$$\mathfrak{d} \cdot \gamma_n = -\frac{1}{2} B_n \cup \gamma_n \tag{3.10}$$

for $\gamma_n \in H^*(X^{[n]}, \mathbb{C})$. For a linear operator $\mathfrak{f} \in \mathrm{End}(\mathbb{H}_X)$, define its derivative $\mathfrak{f}'$ by

$$\mathfrak{f}' \;=\; [\mathfrak{d}, \mathfrak{f}].$$

A fundamental result proved in [21] states that

$$\mathfrak{a}_n'(\alpha) = n \cdot \mathfrak{L}_n(\alpha) - \frac{n(|n|-1)}{2} \cdot \mathfrak{a}_n(K_X\alpha). \tag{3.11}$$

3.2. 1*-point and* 2*-point genus-*0 *extremal Gromov-Witten invariants*

In this subsection, let X be a simply connected smooth projective surface. We start with the 1-point genus-0 extremal Gromov-Witten invariants of $X^{[n]}$.

Lemma 3.2 [27, Theorem 3.5]. *Let X be a simply connected smooth projective surface. Let $n \geq 2$, $d \geq 1$ and $\gamma \in H^*(X^{[n]}, \mathbb{C})$ be a Heisenberg monomial class (3.3). Then, $\langle\gamma\rangle_{0,d\beta_n} = 0$ unless $\gamma = \mathfrak{a}_{-2}(\alpha)\mathfrak{a}_{-1}(x)^{n-2}|0\rangle$ for some $\alpha \in H^2(X, \mathbb{C})$. Moreover, if $\gamma = \mathfrak{a}_{-2}(\alpha)\mathfrak{a}_{-1}(x)^{n-2}|0\rangle$ for some $\alpha \in H^2(X, \mathbb{C})$, then*

$$\langle\gamma\rangle_{0,d\beta_n} = \frac{2}{d^2} \cdot \langle K_X, \alpha\rangle.$$

The 2-point genus-0 extremal Gromov-Witten invariants of $X^{[n]}$ have been studied in [23] via cosection localizations [17]. By abusing notations, denote

$$[\varphi : (C; p_1, \ldots, p_k) \to X^{[n]}] \in \overline{\mathfrak{M}}_{g,k}(X^{[n]}, d\beta_n)$$

by φ. Since $\varphi_*[C] = d\beta_n$, the composition $\rho_n \circ \varphi$ is a constant map. Let

$$\mathrm{Spt} : \overline{\mathfrak{M}}_{g,k}(X^{[n]}, d\beta_n) \to X^{(n)}$$

be the induced map. If $\mathrm{Spt}(\varphi) = \sum_{i=1}^{\ell} n_i x_i \in X^{(n)}$ where $x_1, \ldots, x_\ell$ are distinct, then the morphism φ factors through the product of punctual Hilbert schemes:

$$\varphi = (\varphi_1, \ldots, \varphi_\ell) : C \to \prod_{i=1}^{\ell} M_{n_i}(x_i) \subset X^{[n]} \tag{3.12}$$

where φ_i is a morphism from C to $M_{n_i}(x_i)$. The collection

$$\varphi = (\varphi_1, \ldots, \varphi_\ell) \tag{3.13}$$

is defined to be the standard decomposition of φ, and the point x_i is called the support of φ_i. Note that the collection $\{\varphi_1, \ldots, \varphi_\ell\}$ is unique up to the ordering of the φ_i's. Fix a meromorphic section θ of $\mathcal{O}_X(K_X)$. Let D_0 and D_∞ be the vanishing and pole divisors of θ, respectively.

Lemma 3.3 [23, Proposition 3.3]. *Let $\Lambda_\theta \subset \overline{\mathfrak{M}}_{g,k}(X^{[n]}, d\beta_n)$ be the subset consisting of the stable maps $\varphi = (\varphi_1, \ldots, \varphi_\ell) \in \overline{\mathfrak{M}}_{g,k}(X^{[n]}, d\beta_n)$ such that for each i, either φ_i is a constant map or the support $x_i = \mathrm{Spt}(\varphi_i)$ lies in $D_0 \cup D_\infty$. Then the virtual fundamental class $[\overline{\mathfrak{M}}_{g,k}(X^{[n]}, d\beta_n)]^{\mathrm{vir}}$ is supported in Λ_θ.*

Let (μ^1, μ^2, μ^3) denote a triple of partitions with $|\mu^1| + |\mu^2| + |\mu^3| = n$. Let $r = \ell(\mu^1)$, $s = \ell(\mu^2)$ and $t = \ell(\mu^3)$ be the lengths. For cohomology classes $\mathbf{c}_1, \ldots, \mathbf{c}_s \in H^2(X, \mathbb{C})$, define the class $A^\mu_{\mathbf{c}} \in H^*(X^{[n]}, \mathbb{C})$ by

$$A^\mu_{\mathbf{c}} = \prod_{i=1}^{r} \mathfrak{a}_{-\mu^1_i}(x) \cdot \prod_{j=1}^{s} \mathfrak{a}_{-\mu^2_j}(\mathbf{c}_j) \cdot \prod_{k=1}^{t} \mathfrak{a}_{-\mu^3_k}(1_X)|0\rangle. \tag{3.14}$$

For a part μ^2_j of μ^2, let $A^{\mu-\mu^2_j}_{\mathbf{c}}$ be the cohomology class in $H^*(X^{[n-\mu^2_j]}, \mathbb{C})$ obtained from $A^\mu_{\mathbf{c}}$ with the factor $\mathfrak{a}_{-\mu^2_j}(\mathbf{c}_j)$ deleted. We similarly define $A^{\mu-\mu^3_i}_{\mathbf{c}}$.

The following lemma summarizes some of the main results in [23] and computes the 2-point genus-0 extremal Gromov-Witten invariants of $X^{[n]}$.

Lemma 3.4 [23]. *Let $d \geq 1$. Assume that $\langle A^\lambda_{\mathbf{e}}, A^\mu_{\mathbf{c}} \rangle_{0,d\beta_n} \neq 0$. Then,*

$$\ell(\lambda^3) = \ell(\mu^1) + \delta$$
$$\ell(\mu^3) = \ell(\lambda^1) + (1 - \delta)$$

where $\delta = 0$ or 1. If $\delta = 0$, then $\lambda^3 = \mu^1$, and there exists an integer $\ell = \mu^3_i = \lambda^2_j$ for some i and j such that the partition λ^1 is obtained from μ^3 with ℓ deleted, and the partition μ^2 is obtained from λ^2 with ℓ deleted; moreover,

$$\langle A^\lambda_{\mathbf{e}}, A^\mu_{\mathbf{c}} \rangle_{0,d\beta_n} = \sum_{\ell=\mu^3_i=\lambda^2_j} \langle A^{\lambda-\lambda^2_j}_{\mathbf{e}}, A^{\mu-\mu^3_i}_{\mathbf{c}} \rangle \cdot \langle K_X, \mathbf{e}_j \rangle \cdot c_{\ell,d} \tag{3.15}$$

where the universal constant $c_{\ell,d}$ is defined by the equation

$$\sum_{d\geq 1} d\, c_{\ell,d}\, q^d = (-1)^\ell \ell^2 \left(\frac{\ell(-q)^\ell}{(-q)^\ell - 1} - \frac{q}{1+q} \right). \tag{3.16}$$

4. Genus-0 extremal Gromov-Witten invariants of $X^{[3]}$

In this section, X is a simply connected smooth projective surface. We will study the genus-0 extremal Gromov-Witten invariants $\langle \omega_1, \ldots, \omega_k \rangle_{0,d\beta_3}$ of $X^{[3]}$. Put

$$\langle \omega_1, \ldots, \omega_k \rangle_{0,d} = \langle \omega_1, \ldots, \omega_k \rangle_{0,d\beta_3}$$

https://doi.org/10.1017/fms.2023.17 Published online by Cambridge University Press

for simplicity. In view of the Fundamental Class Axiom (2.5), the Divisor Axiom (2.6) and the dimension constraint (2.3), the genus-0 extremal Gromov-Witten invariants of $X^{[3]}$ are reduced to the invariants

$$\langle \tilde{\omega}_1 \rangle_{0,d}, \quad \langle \tilde{\omega}_1, \tilde{\omega}_2 \rangle_{0,d}, \quad \langle \omega_1, \omega_2, \omega_3 \rangle_{0,d}$$

with $\omega_1, \omega_2, \omega_3 \in H^4(X^{[3]}, \mathbb{C})$. The invariants $\langle \tilde{\omega}_1 \rangle_{0,d}$ and $\langle \tilde{\omega}_1, \tilde{\omega}_2 \rangle_{0,d}$ have been dealt with by Lemma 3.2 and Lemma 3.4, respectively. Therefore, it remains to calculate the 3-point invariants $\langle \omega_1, \omega_2, \omega_3 \rangle_{0,d}$ with $\omega_1, \omega_2, \omega_3 \in H^4(X^{[3]}, \mathbb{C})$.

To begin with, we fix a linear basis of $H^*(X^{[3]}, \mathbb{C})$ which allows us to apply the composition law (2.7). Let $\{\alpha_1, \ldots, \alpha_s\}$ be a linear basis of $H^2(X, \mathbb{C})$. By (3.3), a linear basis $\mathfrak{B}^2$ of $H^2(X^{[3]}, \mathbb{C})$ consists of the cohomology classes

$$B_3, \quad \frac{1}{2}\mathfrak{a}_{-1}(\alpha_i)\mathfrak{a}_{-1}(1_X)^2|0\rangle \tag{4.1}$$

where $1 \le i \le s$, a linear basis $\mathfrak{B}^{10}$ of $H^{10}(X^{[3]}, \mathbb{C})$ consists of

$$\beta_3, \quad \mathfrak{a}_{-1}(\alpha_i)\mathfrak{a}_{-1}(x)^2|0\rangle \tag{4.2}$$

where $1 \le i \le s$, and a linear basis $\mathfrak{B}^8$ of $H^8(X^{[3]}, \mathbb{C})$ consists of the classes

$$\begin{gathered} \mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-1}(x)^2|0\rangle, \quad \mathfrak{a}_{-3}(x)|0\rangle, \quad \mathfrak{a}_{-1}(\alpha_i)\mathfrak{a}_{-2}(x)|0\rangle, \\ \mathfrak{a}_{-2}(\alpha_i)\mathfrak{a}_{-1}(x)|0\rangle, \quad \mathfrak{a}_{-1}(\alpha_i)\mathfrak{a}_{-1}(\alpha_j)\mathfrak{a}_{-1}(x)|0\rangle \end{gathered} \tag{4.3}$$

where $1 \le i \le j \le s$. A linear basis $\mathfrak{B}^4$ of $H^4(X^{[3]}, \mathbb{C})$ consists of the classes

$$\begin{gathered} \mathfrak{a}_{-1}(1_X)^2\mathfrak{a}_{-1}(x)|0\rangle, \quad \mathfrak{a}_{-3}(1_X)|0\rangle, \quad \mathfrak{a}_{-1}(\alpha_i)\mathfrak{a}_{-2}(1_X)|0\rangle, \\ \mathfrak{a}_{-2}(\alpha_i)\mathfrak{a}_{-1}(1_X)|0\rangle, \quad \mathfrak{a}_{-1}(\alpha_i)\mathfrak{a}_{-1}(\alpha_j)\mathfrak{a}_{-1}(1_X)|0\rangle \end{gathered} \tag{4.4}$$

where $1 \le i \le j \le s$, and a linear basis $\mathfrak{B}^6$ of $H^6(X^{[3]}, \mathbb{C})$ consists of

$$\begin{gathered} \mathfrak{a}_{-2}(1_X)\mathfrak{a}_{-1}(x)|0\rangle, \quad \mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(x)|0\rangle, \quad \mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-1}(\alpha_i)\mathfrak{a}_{-1}(x)|0\rangle, \\ \mathfrak{a}_{-3}(\alpha_i)|0\rangle, \quad \mathfrak{a}_{-2}(\alpha_i)\mathfrak{a}_{-1}(\alpha_{j'})|0\rangle, \quad \mathfrak{a}_{-1}(\alpha_i)\mathfrak{a}_{-1}(\alpha_j)\mathfrak{a}_{-1}(\alpha_k)|0\rangle \end{gathered} \tag{4.5}$$

where $1 \le i, j' \le j \le k \le s$. The point class in $H^{12}(X^{[3]}, \mathbb{C})$ is $\mathfrak{a}_{-1}(x)^3|0\rangle$.

Definition 4.1. Let X be a simply connected smooth projective surface. Let $\{\alpha_1, \ldots, \alpha_s\}$ be a linear basis of $H^2(X, \mathbb{C})$. Define $\{\Delta_a\}$ to be the linear basis of the total cohomology $H^*(X^{[3]}, \mathbb{C})$ that consists of

$$\mathfrak{a}_{-1}(x)^3|0\rangle, \quad \mathfrak{B}^i \quad (i = 2, 4, 6, 8, 10), \quad 1_{X^{[3]}}. \tag{4.6}$$

Let $\omega_1, \omega_2, \omega_3 \in \mathfrak{B}^4 \subset H^4(X^{[3]}, \mathbb{C})$. The next lemma identifies all the unordered triples $(\omega_1, \omega_2, \omega_3)$ such that $\langle \omega_1, \omega_2, \omega_3 \rangle_{0,d}$ may not be 0. The idea is to use geometric argument involving the reduction Lemma 3.3. In order to apply Lemma 3.3, we fix a meromorphic section θ of the canonical line bundle $\mathcal{O}_X(K_X)$ and let D_0 and D_∞ be the vanishing and pole divisors of θ, respectively.

Lemma 4.2. *Let $d \ge 1$ and $\omega_1, \omega_2, \omega_3 \in \mathfrak{B}^4 \subset H^4(X^{[3]}, \mathbb{C})$. Then*

$$\langle \omega_1, \omega_2, \omega_3 \rangle_{0,d} = 0 \tag{4.7}$$

if the unordered triple $(\omega_1, \omega_2, \omega_3)$ *is not one of the following:*

(i) $\Big(\mathfrak{a}_{-2}(1_X)\mathfrak{a}_{-1}(\alpha_i)|0\rangle,\ \mathfrak{a}_{-2}(1_X)\mathfrak{a}_{-1}(\alpha_j)|0\rangle,\ \mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(\alpha_k)|0\rangle\Big)$*;*
(ii) $\omega_1 = \omega_2 = \mathfrak{a}_{-3}(1_X)|0\rangle$*, and* $\omega_3 = \mathfrak{a}_{-2}(1_X)\mathfrak{a}_{-1}(\alpha_i)|0\rangle$ *or* $\mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(\alpha_i)|0\rangle$*;*
(iii) $\omega_1 = \omega_2 = \omega_3 = \mathfrak{a}_{-3}(1_X)|0\rangle$.

Proof. We will only prove (4.7) when the triple $(\omega_1, \omega_2, \omega_3)$ is

$$\Big(\mathfrak{a}_{-2}(1_X)\mathfrak{a}_{-1}(\alpha_i)|0\rangle,\ \mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(\alpha_j)|0\rangle,\ \mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(\alpha_k)|0\rangle\Big)$$

since the proof of (4.7) for other triples is similar.

To show that $\langle\omega_1, \omega_2, \omega_3\rangle_{0,d} = 0$, let $C_i, C_j, C_k \subset X$ be real 2-dimensional cycles representing the cohomology classes $\alpha_i, \alpha_j, \alpha_k \in H^2(X, \mathbb{C})$, respectively, such that $C_i, C_j, C_k, D_0, D_\infty$ are in general position. By Lemma 3.1, the classes

$$\mathfrak{a}_{-2}(1_X)\mathfrak{a}_{-1}(\alpha_i)|0\rangle,\ \mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(\alpha_j)|0\rangle,\ \mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(\alpha_k)|0\rangle$$

are geometrically represented by the closures W_1, W_2, W_3 of the subsets

$$W(2, X; 1, C_i)^0, \quad W(1, X; 2, C_j)^0, \quad W(1, X; 2, C_k)^0, \tag{4.8}$$

respectively. By Lemma 3.3, it suffices to prove that

$$\Lambda_\theta \cap \mathrm{ev}_1^{-1}(W_1) \cap \mathrm{ev}_2^{-1}(W_2) \cap \mathrm{ev}_3^{-1}(W_3) = \emptyset.$$

Assume $[\varphi : (\Sigma; \tilde{p}_1, \tilde{p}_2, \tilde{p}_3) \to X^{[3]}] \in \Lambda_\theta \cap \mathrm{ev}_1^{-1}(W_1) \cap \mathrm{ev}_2^{-1}(W_2) \cap \mathrm{ev}_3^{-1}(W_3)$. Since $[\varphi : (\Sigma; \tilde{p}_1, \tilde{p}_2, \tilde{p}_3) \to X^{[3]}] \in \mathrm{ev}_1^{-1}(W_1)$ and $\rho_3(\varphi(\Sigma))$ is a single point in $X^{(3)}$, we see from (4.8) that $\rho_3(\varphi(\Sigma))$ is of the form

$$\rho_3(\varphi(\Sigma)) = 2x_1 + x_2 \tag{4.9}$$

for some (possibly the same) points $x_1, x_2 \in X$ such that $x_2 \in C_i$. Since

$$[\varphi : (\Sigma; \tilde{p}_1, \tilde{p}_2, \tilde{p}_3) \to X^{[3]}] \in \mathrm{ev}_1^{-1}(W_2),$$

we must have $x_1 \in C_j$. Similarly, $x_1 \in C_k$. So $x_1 \in C_j \cap C_k$. Since $C_i, C_j, C_k, D_0, D_\infty$ are in general position, $x_1 \notin C_i \cup D_0 \cup D_\infty$ and $x_1 \neq x_2$. Let $\varphi = (\varphi_1, \varphi_2, \cdots)$ be the standard decomposition of φ. Without loss of generality, we assume that φ_1 is not a constant map. By Lemma 3.3, $\mathrm{Spt}(\varphi_1) \in D_0 \cup D_\infty$. So $x_2 = \mathrm{Spt}(\varphi_1) \in D_0 \cup D_\infty$ and $x_2 \in C_i \cap (D_0 \cup D_\infty)$. Therefore, $\varphi_1 : \Sigma \to X$ is the constant map $\varphi_1(\Sigma) = x_2$, contradicting the assumption that φ_1 is not constant. □

In the rest of this section, we will compute $\langle\omega_1, \omega_2, \omega_3\rangle_{0,d}$ when the unordered triple $(\omega_1, \omega_2, \omega_3)$ is one of those listed in Lemma 4.2 (i), (ii) and (iii). Lemma 4.3 below deals with the unordered triple in Lemma 4.2 (i), and its proof uses a geometric argument similar to the proof of Lemma 4.2.

Lemma 4.3. *Let* $\{\alpha_1, \ldots, \alpha_s\}$ *be a linear basis of* $H^2(X, \mathbb{C})$*. Let* $d \geq 1$ *and the unordered triple* $(\omega_1, \omega_2, \omega_3)$ *be from Lemma 4.2 (i). Then,*

$$\langle\omega_1, \omega_2, \omega_3\rangle_{0,d} = 8\langle\alpha_i, \alpha_j\rangle\, \langle K_X, \alpha_k\rangle. \tag{4.10}$$

Proof. Let $C_i, C_j, C_k \subset X$ be real 2-dimensional cycles representing the cohomology classes $\alpha_i, \alpha_j, \alpha_k$, respectively, such that $C_i, C_j, C_k, D_0, D_\infty$ are in general position. By Lemma 3.1, the cohomology classes

$$\mathfrak{a}_{-2}(1_X)\mathfrak{a}_{-1}(\alpha_i)|0\rangle, \mathfrak{a}_{-2}(1_X)\mathfrak{a}_{-1}(\alpha_j)|0\rangle, \mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(\alpha_k)|0\rangle \tag{4.11}$$

https://doi.org/10.1017/fms.2023.17 Published online by Cambridge University Press

are geometrically represented by the closures W_1, W_2, W_3 of the subsets

$$W(2, X; 1, C_i)^0, \quad W(2, X; 1, C_j)^0, \quad W(1, X; 2, C_k)^0, \tag{4.12}$$

respectively. Let $[\varphi : (\Sigma; \tilde{p}_1, \tilde{p}_2, \tilde{p}_3) \to X^{[3]}] \in \Lambda_\theta \cap \mathrm{ev}_1^{-1}(W_1) \cap \mathrm{ev}_2^{-1}(W_2) \cap \mathrm{ev}_3^{-1}(W_3)$. Since $[\varphi : (\Sigma; \tilde{p}_1, \tilde{p}_2, \tilde{p}_3) \to X^{[3]}] \in \mathrm{ev}_3^{-1}(W_3)$ and $\rho_3(\varphi(\Sigma))$ is a single point in $X^{(3)}$, we see from (4.12) that $\rho_3(\varphi(\Sigma))$ is of the form

$$\rho_3(\varphi(\Sigma)) = x_1 + 2x_2 \tag{4.13}$$

for some (possibly the same) points $x_1, x_2 \in X$ such that $x_2 \in C_k$. By Lemma 3.3, since $[\varphi : (\Sigma; \tilde{p}_1, \tilde{p}_2, \tilde{p}_3) \to X^{[3]}] \in \Lambda_\theta$ and φ is not a constant map, $x_2 \in D_0 \cup D_\infty$. So $x_2 \in C_k \cap (D_0 \cup D_\infty)$. Since C_i, C_j, C_k, D_0 and D_∞ are in general position, $x_2 \notin C_i \cup C_j$. Since $[\varphi : (\Sigma; \tilde{p}_1, \tilde{p}_2, \tilde{p}_3) \to X^{[3]}] \in \mathrm{ev}_1^{-1}(W_1) \cap \mathrm{ev}_2^{-1}(W_2)$, we see from (4.13) that $x_1 \in C_i \cap C_j$. It follows that $x_1 \neq x_2$, and the standard decomposition of φ (see (3.13)) is of the form $\varphi = (\varphi_1, \varphi_2)$ with $\mathrm{Spt}(\varphi_1) = x_1$ and $\mathrm{Spt}(\varphi_2) = 2x_2$. In particular, $\varphi_1 : \Sigma \to X$ is the constant map sending Σ to x_1. Hence, we obtain

$$\langle \omega_1, \omega_2, \omega_3 \rangle_{0,d} = \langle \alpha_i, \alpha_j \rangle \left\langle \mathfrak{a}_{-2}(1_X)|0\rangle, \mathfrak{a}_{-2}(1_X)|0\rangle, \mathfrak{a}_{-2}(\alpha_k)|0\rangle \right\rangle_{0,d}$$

by splitting off the factors $\mathfrak{a}_{-1}(\alpha_i), \mathfrak{a}_{-1}(\alpha_j), \mathfrak{a}_{-1}(1_X)$ from the three classes in (4.11), respectively. It is known (see [38, (1.35)]) that

$$\langle B_n, \beta_n \rangle = -2 \tag{4.14}$$

for $n \geq 2$. In particular, $\left\langle \mathfrak{a}_{-2}(1_X)|0\rangle, \beta_2 \right\rangle = -2$. So by the Divisor Axiom (2.6),

$$\langle \omega_1, \omega_2, \omega_3 \rangle_{0,d} = 4d^2 \langle \alpha_i, \alpha_j \rangle \left\langle \mathfrak{a}_{-2}(\alpha_k)|0\rangle \right\rangle_{0,d}.$$

Finally, by Lemma 3.2 with $n = 2$, $\left\langle \mathfrak{a}_{-2}(\alpha_k)|0\rangle \right\rangle_{0,d} = 2\langle K_X, \alpha_k \rangle / d^2$. Therefore,

$$\langle \omega_1, \omega_2, \omega_3 \rangle_{0,d} = 8 \langle \alpha_i, \alpha_j \rangle \, \langle K_X, \alpha_k \rangle.$$

□

To handle the unordered triples $(\omega_1, \omega_2, \omega_3)$ listed in Lemma 4.2 (ii), we will now prove three technical lemmas. For simplicity, in the rest of this section, we let

$$c_1 = -\frac{1}{2} B_3 = -\frac{1}{2} \mathfrak{a}_{-2}(1_X)\mathfrak{a}_{-1}(1_X)|0\rangle \in H^2(X^{[3]}, \mathbb{C}).$$

Recall from (4.14) that $\langle B_3, \beta_3 \rangle = -2$. It follows that

$$\langle c_1, \beta_3 \rangle = 1. \tag{4.15}$$

The self-intersection c_1^2 via Heisenberg operators is given by the lemma below.

Lemma 4.4. *Let $c_1 = -B_3/2$. Then, c_1^2 is equal to*

$$\mathfrak{a}_{-3}(1_X)|0\rangle - \frac{1}{2} \mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(K_X)|0\rangle - \frac{1}{2} \mathfrak{a}_{-1}(1_X) \cdot \mathfrak{a}_{-1}\mathfrak{a}_{-1}(\tau_{2*}1_X)|0\rangle. \tag{4.16}$$

Proof. Recall Lehn's boundary operator $\mathfrak{d}$ from (3.10). By definition,

$$c_1^2 = -\frac{1}{2} \mathfrak{d}\mathfrak{a}_{-2}(1_X)\mathfrak{a}_{-1}(1_X)|0\rangle.$$

https://doi.org/10.1017/fms.2023.17 Published online by Cambridge University Press

Moving $\mathfrak{d}$ all the way to the right and using $\mathfrak{d}|0\rangle = 0$, we get

$$c_1^2 = -\frac{1}{2}\big\{\mathfrak{a}'_{-2}(1_X)\mathfrak{a}_{-1}(1_X)|0\rangle + \mathfrak{a}_{-2}(1_X)\mathfrak{a}'_{-1}(1_X)|0\rangle\big\}.$$

By (3.11), $\mathfrak{a}'_{-2}(1_X) = -2\mathfrak{L}_{-2}(1_X) + \mathfrak{a}_{-2}(K_X)$. So by (3.9) and (3.8),

$$\begin{aligned}&\mathfrak{a}'_{-2}(1_X)\mathfrak{a}_{-1}(1_X)|0\rangle\\ =&-2\mathfrak{a}_{-3}(1_X)|0\rangle - 2\mathfrak{a}_{-1}(1_X)\mathfrak{L}_{-2}(1_X)|0\rangle + \mathfrak{a}_{-2}(K_X)\mathfrak{a}_{-1}(1_X)|0\rangle\\ =&-2\mathfrak{a}_{-3}(1_X)|0\rangle + \mathfrak{a}_{-1}(1_X)\cdot\mathfrak{a}_{-1}\mathfrak{a}_{-1}(\tau_{2*}1_X)|0\rangle + \mathfrak{a}_{-2}(K_X)\mathfrak{a}_{-1}(1_X)|0\rangle.\end{aligned}$$

Similarly, $\mathfrak{a}'_{-1}(1_X)|0\rangle = -\mathfrak{L}_{-1}(1_X)|0\rangle = 0$. Therefore, c_1^2 is equal to

$$\mathfrak{a}_{-3}(1_X)|0\rangle - \frac{1}{2}\mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(K_X)|0\rangle - \frac{1}{2}\mathfrak{a}_{-1}(1_X)\cdot\mathfrak{a}_{-1}\mathfrak{a}_{-1}(\tau_{2*}1_X)|0\rangle. \qquad \square$$

Since X is simply connected, if $\{\alpha_1, \ldots, \alpha_s\}$ is a linear basis of $H^2(X, \mathbb{C})$, then

$$\tau_{2*}1_X = x\otimes 1_X + 1_X\otimes x + \sum_{1\le j\le k\le s} b_{j,k}\,\alpha_j\otimes\alpha_k \tag{4.17}$$

for some $b_{j,k}\in\mathbb{C}$, via the Künneth decomposition of $H^*(X^2, \mathbb{C})$.

In the next two lemmas, we will compute certain special 1-point and 2-point genus-0 extremal Gromov-Witten invariants which will appear in our applications of the composition law (2.7) and involve the class $\omega = \mathfrak{a}_{-3}(1_X)|0\rangle$.

Lemma 4.5. *Let X be a simply connected smooth projective surface. Let $d\ge 1$, $\omega = \mathfrak{a}_{-3}(1_X)|0\rangle$ and $\tilde{\omega} = \mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(\alpha)|0\rangle$ for some $\alpha\in H^2(X,\mathbb{C})$. Then,*

$$\langle\omega\tilde{\omega}\rangle_{0,d} = -\frac{12}{d^2}\langle K_X, \alpha\rangle. \tag{4.18}$$

Proof. Let $\{\alpha_1, \ldots, \alpha_s\}$ be a linear basis of $H^2(X, \mathbb{C})$. By (4.16) and (4.17),

$$\begin{aligned}\omega &= c_1^2 + \frac{1}{2}\mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(K_X)|0\rangle + \frac{1}{2}\mathfrak{a}_{-1}(1_X)\cdot\mathfrak{a}_{-1}\mathfrak{a}_{-1}(\tau_{2*}1_X)|0\rangle\\ &= c_1^2 + \frac{1}{2}\mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(K_X)|0\rangle + \mathfrak{a}_{-1}(1_X)^2\mathfrak{a}_{-1}(x)|0\rangle\\ &\quad+\sum_{1\le j\le k\le s} b_{j,k}\mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-1}(\alpha_j)\mathfrak{a}_{-1}(\alpha_k)|0\rangle.\end{aligned} \tag{4.19}$$

In view of the linear basis (4.4), $\mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(K_X)|0\rangle\cdot\tilde{\omega}$ is a linear combination of $\mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-1}(x)^2|0\rangle$, $\mathfrak{a}_{-1}(\alpha_j)\mathfrak{a}_{-1}(\alpha_k)\mathfrak{a}_{-1}(x)|0\rangle$, $\mathfrak{a}_{-1}(\alpha_j)\mathfrak{a}_{-2}(x)|0\rangle$ and $\mathfrak{a}_{-3}(x)|0\rangle$. Hence, $\langle\mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(K_X)|0\rangle\cdot\tilde{\omega}\rangle_{0,d} = 0$ by Lemma 3.2. Similarly,

$$\langle\mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-1}(\alpha_j)\mathfrak{a}_{-1}(\alpha_k)|0\rangle\cdot\tilde{\omega}\rangle_{0,d} = 0$$

whenever $1\le j, k\le s$. Note that $\mathfrak{a}_{-1}(1_X)^2\mathfrak{a}_{-1}(x)|0\rangle\cdot\tilde{\omega} = 2\mathfrak{a}_{-1}(x)\mathfrak{a}_{-2}(\alpha)|0\rangle$. Combining with (4.19) and Lemma 3.2, we conclude that

$$\langle\omega\tilde{\omega}\rangle_{0,d} = \langle c_1^2\tilde{\omega}\rangle_{0,d} + \langle\mathfrak{a}_{-1}(1_X)^2\mathfrak{a}_{-1}(x)|0\rangle\cdot\tilde{\omega}\rangle_{0,d} = \langle c_1^2\tilde{\omega}\rangle_{0,d} + \frac{4}{d^2}\langle K_X, \alpha\rangle.$$

https://doi.org/10.1017/fms.2023.17 Published online by Cambridge University Press

As in the proof of Lemma 4.4, using (3.11) and (3.9), we get

$$\begin{aligned} c_1^2\tilde{\omega} &= \mathfrak{d}\mathfrak{d}\mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(\alpha)|0\rangle \\ &= \mathfrak{d}\big\{2\mathfrak{a}_{-3}(\alpha) + \langle K_X, \alpha\rangle \mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(x) + 2\mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-1}(\alpha)\mathfrak{a}_{-1}(x)\big\}|0\rangle \\ &= -8\mathfrak{a}_{-1}(x)\mathfrak{a}_{-2}(\alpha)|0\rangle + \gamma \end{aligned}$$

where γ is a term satisfying $\langle\gamma\rangle_{0,d} = 0$. By Lemma 3.2 again,

$$\langle \omega\tilde{\omega}\rangle_{0,d} = -8\big\langle \mathfrak{a}_{-1}(x)\mathfrak{a}_{-2}(\alpha)|0\rangle\big\rangle_{0,d} + \frac{4}{d^2}\langle K_X, \alpha\rangle = -\frac{12}{d^2}\langle K_X, \alpha\rangle. \qquad \square$$

Lemma 4.6. *Let X be simply connected. Let $d \ge 1$, $c_1 = -B_3/2$, $\omega = \mathfrak{a}_{-3}(1_X)|0\rangle$ and $\tilde{\omega} = \mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(\alpha)|0\rangle$ for some $\alpha \in H^2(X, \mathbb{C})$. Then,*

(i) $\langle c_1\omega, \tilde{\omega}\rangle_{0,d} = -12\langle K_X, \alpha\rangle/d$.
(ii) $\langle \omega, c_1\tilde{\omega}\rangle_{0,d} = -2\ \langle K_X, \alpha\rangle\ c_{3,d}$ *where $c_{3,d}$ is from (3.16).*
(iii) $\langle c_1\omega, \omega\rangle_{0,d} = 3K_X^2\ c_{3,d}$.

Proof. (i) Since $c_1\omega = \mathfrak{d}\mathfrak{a}_{-3}(1_X)|0\rangle$, we see from (3.11) that

$$c_1\omega = \mathfrak{a}'_{-3}(1_X)|0\rangle = \big(-3\mathfrak{L}_{-3}(1_X) + 3\mathfrak{a}_{-3}(K_X)\big)|0\rangle.$$

By (3.8), $\mathfrak{L}_{-3}(1_X)|0\rangle = -\mathfrak{a}_{-1}\mathfrak{a}_{-2}(\tau_{2*}1_X)|0\rangle$. Thus, we have

$$c_1\omega = 3\mathfrak{a}_{-1}\mathfrak{a}_{-2}(\tau_{2*}1_X)|0\rangle + 3\mathfrak{a}_{-3}(K_X)|0\rangle. \tag{4.20}$$

By Lemma 3.4, $\big\langle \mathfrak{a}_{-3}(K_X)|0\rangle, \tilde{\omega}\big\rangle_{0,d} = 0$. Therefore,

$$\langle c_1\omega, \tilde{\omega}\rangle_{0,d} = 3\big\langle \mathfrak{a}_{-1}\mathfrak{a}_{-2}(\tau_{2*}1_X)|0\rangle, \tilde{\omega}\big\rangle_{0,d}.$$

By (4.17), Lemma 3.4 and $c_{2,d} = -4/d$, we obtain

$$\langle c_1\omega, \tilde{\omega}\rangle_{0,d} = 3\big\langle \mathfrak{a}_{-1}(x)\mathfrak{a}_{-2}(1_X)|0\rangle, \tilde{\omega}\big\rangle_{0,d} = 3\langle K_X, \alpha\rangle\cdot c_{2,d} = -\frac{12}{d}\langle K_X, \alpha\rangle.$$

(ii) Similarly, by (3.11) and (3.8), we conclude that

$$c_1\tilde{\omega} = \mathfrak{a}'_{-1}(1_X)\mathfrak{a}_{-2}(\alpha)|0\rangle + \mathfrak{a}_{-1}(1_X)\mathfrak{a}'_{-2}(\alpha)|0\rangle = -2\mathfrak{a}_{-3}(\alpha)|0\rangle + \gamma$$

where γ is a term satisfying $\langle \omega, \gamma\rangle_{0,d} = 0$. By Lemma 3.4,

$$\langle \omega, c_1\tilde{\omega}\rangle_{0,d} = -2\ \big\langle \mathfrak{a}_{-3}(1_X)|0\rangle, \mathfrak{a}_{-3}(\alpha)|0\rangle\big\rangle_{0,d} = -2\ \langle K_X, \alpha\rangle\ c_{3,d}.$$

(iii) We see from (4.20) that $\langle c_1\omega, \omega\rangle_{0,d}$ is equal to

$$3\big\langle \mathfrak{a}_{-1}\mathfrak{a}_{-2}(\tau_{2*}1_X)|0\rangle, \mathfrak{a}_{-3}(1_X)|0\rangle\big\rangle_{0,d} + 3\big\langle \mathfrak{a}_{-3}(K_X)|0\rangle, \mathfrak{a}_{-3}(1_X)|0\rangle\big\rangle_{0,d}.$$

By (4.17) and Lemma 3.4, we have $\big\langle \mathfrak{a}_{-1}\mathfrak{a}_{-2}(\tau_{2*}1_X)|0\rangle, \mathfrak{a}_{-3}(1_X)|0\rangle\big\rangle_{0,d} = 0$ and $\big\langle \mathfrak{a}_{-3}(K_X)|0\rangle, \mathfrak{a}_{-3}(1_X)|0\rangle\big\rangle_{0,d} = K_X^2\ c_{3,d}$. Hence, $\langle c_1\omega, \omega\rangle_{0,d} = 3K_X^2\ c_{3,d}$. $\square$

Our next proposition determines the invariant $\langle \omega_1, \omega_2, \omega_3\rangle_{0,d}$ when the unordered triple $(\omega_1, \omega_2, \omega_3)$ is from Lemma 4.2 (ii). Its proof involves the composition law (2.7) and the linear basis $\{\Delta_a\}$ from (4.6).

https://doi.org/10.1017/fms.2023.17 Published online by Cambridge University Press

Proposition 4.7. *Let $\{\alpha_1, \dots, \alpha_s\}$ be a linear basis of $H^2(X, \mathbb{C})$. Let $d \ge 1$ and $\omega_1 = \omega_2 = \mathfrak{a}_{-3}(1_X)|0\rangle$. Let $\omega_3 = \mathfrak{a}_{-2}(1_X)\mathfrak{a}_{-1}(\alpha_i)|0\rangle$ or $\mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(\alpha_i)|0\rangle$. Then,*

$$\big\langle \omega_1, \omega_2, \omega_3 \big\rangle_{0,d} = -2\langle K_X, \alpha_i\rangle d c_{3,d} \tag{4.21}$$

where $c_{3,d}$ is the universal constant from (3.16).

Proof. The proof of (4.21) for $\omega_3 = \mathfrak{a}_{-2}(1_X)\mathfrak{a}_{-1}(\alpha_i)|0\rangle$ is similar to the proof of (4.21) for $\omega_3 = \mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(\alpha_i)|0\rangle$. So we will only prove

$$Q \overset{\text{def}}{=} \big\langle \omega_1, \omega_2, \omega_3 \big\rangle_{0,d} = -2\ \langle K_X, \alpha_i\rangle\ d c_{3,d}. \tag{4.22}$$

for $\omega_3 = \mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(\alpha_i)|0\rangle$ Let $c_1 = -B_3/2$. Apply the composition law (2.7) to

$$\gamma_1 = \gamma_2 = c_1, \quad \gamma_3 = \omega_2 = \mathfrak{a}_{-3}(1_X)|0\rangle, \quad \gamma_4 = \omega_3 = \mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(\alpha_i)|0\rangle.$$

We will prove (4.22) by comparing both sides of (2.7).

First of all, the left-hand side of (2.7) is equal to

$$\langle c_1^2, \omega_2, \omega_3\rangle_{0,d} + \langle c_1, c_1, \omega_2\omega_3\rangle_{0,d} + \sum_{\substack{d_1+d_2=d\\ d_1,d_2>0}} \sum_a \langle c_1, c_1, \Delta_a\rangle_{0,d_1} \langle \Delta^a, \omega_2, \omega_3\rangle_{0,d_2}. \tag{4.23}$$

By (4.16) and Lemma 4.2, $\langle c_1^2, \omega_2, \omega_3\rangle_{0,d} = Q$. Since $\langle c_1, \beta_3\rangle = 1$ by (4.15), we get

$$\langle c_1, c_1, \omega_2\omega_3\rangle_{0,d} = d^2\ \langle \omega_2\omega_3\rangle_{0,d},$$
$$\langle c_1, c_1, \Delta_a\rangle_{0,d_1} = d_1^2\ \langle \Delta_a\rangle_{0,d_1}$$

in view of (2.6). By Lemma 3.2, $\langle \Delta_a\rangle_{0,d_1} \ne 0$ only when $\Delta_a = \mathfrak{a}_{-2}(\alpha_j)\mathfrak{a}_{-1}(x)|0\rangle$ for some j. If $\Delta_a = \mathfrak{a}_{-2}(\alpha_j)\mathfrak{a}_{-1}(x)|0\rangle$, then we see from (4.4) that Δ^a is a linear combination of the classes $\mathfrak{a}_{-2}(\alpha_k)\mathfrak{a}_{-1}(1_X)|0\rangle$, $1 \le k \le s$. So $\langle \Delta^a, \omega_2, \omega_3\rangle_{0,d_2} = 0$ by Lemma 4.2. It follows from (4.23) that the left-hand side of (2.7) is equal to $Q + d^2\ \langle \omega_2\omega_3\rangle_{0,d}$. By Lemma 4.5, we see that the left-hand side of (2.7) is equal to

$$Q - 12\langle K_X, \alpha_i\rangle. \tag{4.24}$$

Next, the right-hand side of (2.7) is equal to

$$\langle c_1\omega_2, c_1, \omega_3\rangle_{0,d} + \langle c_1, \omega_2, c_1\omega_3\rangle_{0,d} + \sum_{d_1+d_2=d,\, d_1,d_2>0} \sum_a \langle c_1, \omega_2, \Delta_a\rangle_{0,d_1} \langle \Delta^a, c_1, \omega_3\rangle_{0,d_2}.$$

By Lemma 4.6, we have $\langle c_1\omega_2, c_1, \omega_3\rangle_{0,d} = d\langle c_1\omega_2, \omega_3\rangle_{0,d} = -12\langle K_X, \alpha_i\rangle$ and

$$\langle c_1, \omega_2, c_1\omega_3\rangle_{0,d} = d\langle \omega_2, c_1\omega_3\rangle_{0,d} = -2\ \langle K_X, \alpha_i\rangle\ d c_{3,d}.$$

Therefore, the right-hand side of (2.7) is equal to

$$-12\langle K_X, \alpha_i\rangle - 2\ \langle K_X, \alpha_i\rangle\ d c_{3,d} + \sum_{\substack{d_1+d_2=d\\ d_1,d_2>0}} \sum_a d_1 d_2 \langle \omega_2, \Delta_a\rangle_{0,d_1} \langle \Delta^a, \omega_3\rangle_{0,d_2}. \tag{4.25}$$

By the list (4.5) of the basis $\mathfrak{B}^6$ and Lemma 3.4, $\langle \omega_2, \Delta_a\rangle_{0,d_1} \ne 0$ only if $\Delta_a = \mathfrak{a}_{-3}(\alpha_j)|0\rangle$ for some j with $1 \le j \le s$. If $\Delta_a = \mathfrak{a}_{-3}(\alpha_j)|0\rangle$, then Δ^a is a linear combination of $\mathfrak{B}^6 - \big\{\mathfrak{a}_{-2}(1_X)\mathfrak{a}_{-1}(x)|0\rangle\big\}$.

https://doi.org/10.1017/fms.2023.17 Published online by Cambridge University Press

So $\langle \Delta^a, \omega_3\rangle_{0,d_2} = 0$ by Lemma 3.4 again. By (4.25), the right-hand side of (2.7) is equal to

$$-12\langle K_X, \alpha_i\rangle - 2\ \langle K_X, \alpha_i\rangle\ dc_{3,d}. \tag{4.26}$$

Finally, combining (4.24) and (4.26) yields (4.22). □

We are left with the computation of the invariant $\langle \omega_1, \omega_2, \omega_3\rangle_{0,d}$ when the triple $(\omega_1, \omega_2, \omega_3)$ is from Lemma 4.2 (iii), that is, when $\omega_1 = \omega_2 = \omega_3 = \mathfrak{a}_{-3}(1_X)|0\rangle$. This will be done in Proposition 4.9 below. We prove a technical lemma first.

Lemma 4.8. *Let X be simply connected. Let $d \geq 1$ and $\omega = \mathfrak{a}_{-3}(1_X)|0\rangle$. Then,*

$$\langle \omega^2\rangle_{0,d} = \frac{18K_X^2}{d^2} \tag{4.27}$$

Proof. Let $\{\alpha_1, \ldots, \alpha_s\}$ be a linear basis of $H^2(X, \mathbb{C})$. By (4.19),

$$\omega^2 = c_1^2 \cdot \omega + \frac{1}{2}\mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(K_X)|0\rangle \cdot \omega + \mathfrak{a}_{-1}(1_X)^2\mathfrak{a}_{-1}(x)|0\rangle \cdot \omega$$
$$+ \sum_{1\leq j\leq k\leq s} b_{j,k}\mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-1}(\alpha_j)\mathfrak{a}_{-1}(\alpha_k)|0\rangle \cdot \omega.$$

The cup products $\mathfrak{a}_{-1}(1_X)^2\mathfrak{a}_{-1}(x)|0\rangle \cdot \omega$ and $\mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-1}(\alpha_j)\mathfrak{a}_{-1}(\alpha_k)|0\rangle \cdot \omega$ are scalar multiples of $\mathfrak{a}_{-3}(x)|0\rangle$. Therefore, we conclude from Lemma 3.2 that

$$\langle \omega^2\rangle_{0,d} = \langle c_1^2 \cdot \omega\rangle_{0,d} + \frac{1}{2}\big\langle \mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(K_X)|0\rangle \cdot \omega\big\rangle_{0,d}$$
$$= \langle \mathfrak{d}c_1\omega\rangle_{0,d} + \frac{1}{2}\big\langle \omega \cdot \mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(K_X)|0\rangle\big\rangle_{0,d}. \tag{4.28}$$

By (4.20), $c_1\omega = 3\mathfrak{a}_{-1}\mathfrak{a}_{-2}(\tau_{2*}1_X)|0\rangle + 3\mathfrak{a}_{-3}(K_X)|0\rangle$. So

$$\langle \mathfrak{d}c_1\omega\rangle_{0,d} = 3\big\langle \mathfrak{d}\mathfrak{a}_{-1}\mathfrak{a}_{-2}(\tau_{2*}1_X)|0\rangle\big\rangle_{0,d} + 3\big\langle \mathfrak{d}\mathfrak{a}_{-3}(K_X)|0\rangle\big\rangle_{0,d} = \frac{24K_X^2}{d^2} \tag{4.29}$$

by (4.17), (3.11), (3.9), (3.8) and Lemma 3.2. Similarly, by (4.19) and Lemma 3.2,

$$\big\langle \omega \cdot \mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(K_X)|0\rangle\big\rangle_{0,d}$$
$$= \big\langle c_1^2 \cdot \mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(K_X)|0\rangle\big\rangle_{0,d} + \big\langle \mathfrak{a}_{-1}(1_X)^2\mathfrak{a}_{-1}(x)|0\rangle \cdot \mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(K_X)|0\rangle\big\rangle_{0,d}$$
$$= \big\langle \mathfrak{d}^2\mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(K_X)|0\rangle\big\rangle_{0,d} + \frac{4K_X^2}{d^2}. \tag{4.30}$$

By (3.11), (3.9) and (3.8), we see that $\mathfrak{d}\mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(K_X)|0\rangle$ is equal to

$$-2\mathfrak{a}_{-3}(K_X)|0\rangle + 2\mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-1}(x)\mathfrak{a}_{-1}(K_X)|0\rangle + K_X^2\mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(x)|0\rangle.$$

Applying (3.11), (3.9), (3.8) and Lemma 3.2 repeatedly, we get

$$\big\langle \mathfrak{d}^2\mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(K_X)|0\rangle\big\rangle_{0,d}$$
$$= -2\big\langle \mathfrak{d}\mathfrak{a}_{-3}(K_X)|0\rangle\big\rangle_{0,d} + 2\big\langle \mathfrak{d}\mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-1}(x)\mathfrak{a}_{-1}(K_X)|0\rangle\big\rangle_{0,d}$$

https://doi.org/10.1017/fms.2023.17 Published online by Cambridge University Press

$$
\begin{aligned}
&+ K_X^2 \big\langle \mathfrak{d}\mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(x)|0\rangle \big\rangle_{0,d} \\
&= -\frac{12K_X^2}{d^2} - \frac{4K_X^2}{d^2} \\
&= -\frac{16K_X^2}{d^2}.
\end{aligned}
$$

Combining with (4.30), we have $\big\langle \omega \cdot \mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(K_X)|0\rangle \big\rangle_{0,d} = -12K_X^2/d^2$. Together with (4.28) and (4.29), we obtain $\langle \omega^2 \rangle_{0,d} = 18K_X^2/d^2$. □

Proposition 4.9. *Let X be a simply connected projective surface. Let $d \geq 1$. Then, $\big\langle \mathfrak{a}_{-3}(1_X)|0\rangle, \mathfrak{a}_{-3}(1_X)|0\rangle, \mathfrak{a}_{-3}(1_X)|0\rangle \big\rangle_{0,d}$ is equal to*

$$
-18K_X^2 + 5K_X^2\, dc_{3,d} - \ 2K_X^2 \sum_{i=1}^{d-1} ic_{3,i} + \frac{1}{3}K_X^2 \sum_{i=1}^{d-1} ic_{3,i}\ (d-i)c_{3,d-i} \tag{4.31}
$$

where $c_{3,d}$ is the universal constant from (3.16).

Proof. For simplicity, let $\omega = \mathfrak{a}_{-3}(1_X)|0\rangle$ and $Q' = \big\langle \omega, \omega, \omega \big\rangle_{0,d}$. Our idea to compute Q' is the same as in the proof of Proposition 4.7. Let $c_1 = -B_3/2$. We apply the composition law (2.7) to $\gamma_1 = \gamma_2 = c_1$ and $\gamma_3 = \gamma_4 = \omega$.

First of all, notice that the left-hand-side of (2.7) is equal to

$$
\langle c_1^2, \omega, \omega \rangle_{0,d} + \langle c_1, c_1, \omega^2 \rangle_{0,d} + \sum_{d_1+d_2=d,\, d_1,d_2>0} \ \sum_a \ \langle c_1, c_1, \Delta_a \rangle_{0,d_1} \langle \Delta^a, \omega, \omega \rangle_{0,d_2}.
$$

By (4.16), Lemma 4.2 and Proposition 4.7, we have $\langle c_1^2, \omega, \omega \rangle_{0,d} = Q' + K_X^2\, dc_{3,d}$. By (2.6) and Lemma 4.8, we get $\langle c_1, c_1, \omega^2 \rangle_{0,d} = d^2 \langle \omega^2 \rangle_{0,d} = 18K_X^2$. Next, note from (4.3) and (4.4) that if $\Delta_a = \mathfrak{a}_{-2}(\alpha_i)\mathfrak{a}_{-1}(x)|0\rangle$, then $\Delta^a = -1/2 \cdot \mathfrak{a}_{-2}(\alpha^i)\mathfrak{a}_{-1}(1_X)|0\rangle$ where $\{\alpha^1, \ldots, \alpha^s\} \subset H^2(X, \mathbb{C})$ is the dual basis of $\{\alpha_1, \ldots, \alpha_s\}$ with respect to the pairing of X. So by Lemma 3.2 and Proposition 4.7, we obtain

$$
\begin{aligned}
&\sum_a \ \langle c_1, c_1, \Delta_a \rangle_{0,d_1} \langle \Delta^a, \omega, \omega \rangle_{0,d_2} \\
&= \sum_{i=1}^{s} \ d_1^2 \big\langle \mathfrak{a}_{-2}(\alpha_i)\mathfrak{a}_{-1}(x)|0\rangle \big\rangle_{0,d_1} \cdot \left(-\frac{1}{2}\right) \big\langle \mathfrak{a}_{-2}(\alpha^i)\mathfrak{a}_{-1}(1_X)|0\rangle, \omega, \omega \big\rangle_{0,d_2} \\
&= \sum_{i=1}^{s} \ 2\langle K_X, \alpha_i \rangle \cdot \langle K_X, \alpha^i \rangle \ d_2 c_{3,d_2}.
\end{aligned}
$$

Since $\sum_{i=1}^s \langle K_X, \alpha_i \rangle \cdot \langle K_X, \alpha^i \rangle = K_X^2$, we conclude that

$$
\sum_a \ \langle c_1, c_1, \Delta_a \rangle_{0,d_1} \langle \Delta^a, \omega, \omega \rangle_{0,d_2} = 2K_X^2 \ d_2 c_{3,d_2}.
$$

In summary, we see that the left-hand side of (2.7) is equal to

$$
(Q' + K_X^2\, dc_{3,d}) + 18K_X^2 + 2K_X^2 \sum_{0<d_2<d} d_2 c_{3,d_2}. \tag{4.32}
$$

https://doi.org/10.1017/fms.2023.17 Published online by Cambridge University Press

Next, the right-hand side of (2.7) is equal to

$$\langle c_1\omega, c_1, \omega\rangle_{0,d} + \langle c_1, \omega, c_1\omega\rangle_{0,d} + \sum_{\substack{d_1+d_2=d \\ d_1,d_2>0}} \sum_a \langle c_1, \omega, \Delta_a\rangle_{0,d_1} \langle \Delta^a, c_1, \omega\rangle_{0,d_2}$$
$$= 6K_X^2\, dc_{3,d} + \sum_{d_1+d_2=d,\, d_1,d_2>0} \sum_a d_1\langle \omega, \Delta_a\rangle_{0,d_1}\, d_2\langle \Delta^a, \omega\rangle_{0,d_2}$$

by Lemma 4.6 (iii). If $\Delta_a = \mathfrak{a}_{-3}(\alpha_i)|0\rangle$, then $\Delta^a = 1/3 \cdot \mathfrak{a}_{-3}(\alpha^i)|0\rangle$. Therefore,

$$\begin{aligned}
&\sum_a \langle \omega, \Delta_a\rangle_{0,d_1} \langle \Delta^a, \omega\rangle_{0,d_2} \\
&= \sum_{i=1}^s \big\langle \mathfrak{a}_{-3}(1_X)|0\rangle, \mathfrak{a}_{-3}(\alpha_i)|0\rangle\big\rangle_{0,d_1} \cdot \frac{1}{3}\big\langle \mathfrak{a}_{-3}(\alpha^i)|0\rangle, \mathfrak{a}_{-3}(1_X)|0\rangle\big\rangle_{0,d_2} \\
&= \sum_{i=1}^s \langle K_X, \alpha_i\rangle c_{3,d_1} \cdot \frac{1}{3}\langle K_X, \alpha^i\rangle c_{3,d_2} \\
&= \frac{1}{3}K_X^2\; c_{3,d_1}\; c_{3,d_2}
\end{aligned}$$

by Lemma 3.4. In summary, we see that the right-hand side of (2.7) is equal to

$$6K_X^2\, dc_{3,d} + \frac{1}{3}K_X^2 \sum_{d_1+d_2=d,\, d_1,d_2>0} d_1c_{3,d_1}\; d_2c_{3,d_2}. \tag{4.33}$$

Finally, comparing (4.32) and (4.33) yields (4.31). □

The results in this section are summarized into a theorem.

Theorem 4.10. *Let X be a simply connected smooth projective surface. Assume that $\{\alpha_1, \ldots, \alpha_s\}$ is a linear basis of $H^2(X,\mathbb{C})$, and let $\mathfrak{B}^4$ stand for the linear basis of $H^4(X^{[3]},\mathbb{C})$ from (4.4). Let $d \geq 1$ and $\omega_1, \omega_2, \omega_3 \in \mathfrak{B}^4$. Then,*

$$\langle \omega_1, \omega_2, \omega_3\rangle_{0,d\beta_3} = 0$$

if the unordered triple $(\omega_1, \omega_2, \omega_3)$ is not one of the following four cases:

(i) $\Big(\mathfrak{a}_{-2}(1_X)\mathfrak{a}_{-1}(\alpha_i)|0\rangle, \mathfrak{a}_{-2}(1_X)\mathfrak{a}_{-1}(\alpha_j)|0\rangle, \mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(\alpha_k)|0\rangle\Big)$*;*
(ii) $\omega_1 = \omega_2 = \mathfrak{a}_{-3}(1_X)|0\rangle$*, and* $\omega_3 = \mathfrak{a}_{-2}(1_X)\mathfrak{a}_{-1}(\alpha_i)|0\rangle$ *or* $\mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(\alpha_i)|0\rangle$*;*
(iii) $\omega_1 = \omega_2 = \omega_3 = \mathfrak{a}_{-3}(1_X)|0\rangle$.

Moreover, $\langle \omega_1, \omega_2, \omega_3\rangle_{0,d\beta_3} = 8\langle \alpha_i, \alpha_j\rangle\, \langle K_X, \alpha_k\rangle$ in case (i), and

$$\langle \omega_1, \omega_2, \omega_3\rangle_{0,d\beta_3} = -2\, \langle K_X, \alpha_i\rangle\, dc_{3,d}$$

in case (ii), where $c_{3,d}$ is the universal constant from (3.16). In case (iii),

$$\langle \omega_1, \omega_2, \omega_3\rangle_{0,d\beta_3} = \left(-18 + 5dc_{3,d} - \ 2\sum_{i=1}^{d-1} ic_{3,i} + \frac{1}{3}\sum_{i=1}^{d-1} ic_{3,i}\; (d-i)c_{3,d-i}\right)K_X^2.$$

Proof. Follows from Lemmas 4.2 and 4.3 and Propositions 4.7 and 4.9. □

Via a representation theoretic approach, [28] presents a complicated proof of Ruan's Cohomological Crepant Resolution Conjecture for the Hilbert-Chow morphism $\rho_n : X^{[n]} \to X^{(n)}$ for all $n \geq 1$. As an

https://doi.org/10.1017/fms.2023.17 Published online by Cambridge University Press

application of Theorem 4.10 (together with the results in [23, 27] about the 1-point and 2-point genus-0 extremal Gromov-Witten invariants of $X^{[n]}$), we now give a direct (but tedious) proof of this conjecture when $n = 3$.

Corollary 4.11. *Let X be a simply connected smooth projective surface. Then Ruan's Cohomological Crepant Resolution Conjecture for the Hilbert-Chow morphism $\rho_3 : X^{[3]} \to X^{(3)}$ holds (i.e., the Chen-Ruan cohomology ring of $X^{(3)}$ is isomorphic to the quantum corrected cohomology ring of $X^{[3]}$).*

Proof. First of all, we briefly recall from [39] and [38, Chapter 16] that the Cohomological Crepant Resolution Conjecture for $\rho_n : X^{[n]} \to X^{(n)}$ asserts that there exists a ring isomorphism

$$\Psi_n : H^*_{CR}(X^{(n)}, \mathbb{C}) \to H^*_{\rho_n}(X^{[n]}, \mathbb{C})$$

where $H^*_{CR}(X^{(n)}, \mathbb{C})$ is the Chen-Ruan cohomology of $X^{(n)}$, and $H^*_{\rho_n}(X^{[n]}, \mathbb{C})$ is the cohomology $H^*(X^{[n]}, \mathbb{C})$ together with the quantum corrected ring product $\cdot_{\rho_n}$. For $w_1, w_2 \in H^*(X^{[n]}, \mathbb{C})$, the product $w_1 \cdot_{\rho_n} w_2$ is defined by putting

$$\langle w_1 \cdot_{\rho_n} w_2, w_3 \rangle = \langle w_1, w_2, w_3 \rangle_{\rho_n}(-1)$$

where $w_3 \in H^*(X^{[n]}, \mathbb{C})$, $\langle \cdot, \cdot \rangle$ on the left-hand side is the pairing on $H^*(X^{[n]}, \mathbb{C})$, and

$$\langle w_1, w_2, w_3 \rangle_{\rho_n}(q) = \sum_{d \ge 0} \langle w_1, w_2, w_3 \rangle_{0, d\beta_n} \cdot q^d$$

with q being a variable. Put

$$\mathcal{F}_X = \bigoplus_{n \ge 0} H^*_{CR}(X^{(n)}, \mathbb{C}).$$

By [38, Theorem 10.1], the space $\mathcal{F}_X$ is an irreducible representation of the Heisenberg algebra generated by the operators $\mathfrak{p}_m(\alpha) \in \mathrm{End}(\mathcal{F}_X)$, $m \in \mathbb{Z}$ and $\alpha \in H^*(X, \mathbb{C})$ with the commutation relation

$$[\mathfrak{p}_m(\alpha), \mathfrak{p}_n(\beta)] = m \cdot \delta_{m,-n} \cdot \langle \alpha, \beta \rangle \cdot \mathrm{Id}_{\mathcal{F}_X}$$

and with the vacuum vector $|0\rangle = 1 \in H^*(pt, \mathbb{C}) \cong \mathbb{C}$. Define Ψ_n by putting

$$\Psi_n\Big(\sqrt{-1}^{\,n_1+\ldots+n_s-s} \mathfrak{p}_{-n_1}(\alpha_1) \cdots \mathfrak{p}_{-n_s}(\alpha_s)|0\rangle\Big) = \mathfrak{a}_{-n_1}(\alpha_1) \cdots \mathfrak{a}_{-n_s}(\alpha_s)|0\rangle.$$

Then, $\Psi_n : H^*_{CR}(X^{(n)}, \mathbb{C}) \to H^*_{\rho_n}(X^{[n]}, \mathbb{C})$ is an isomorphism of vector spaces. To show Ψ_n is a ring isomorphism, we must prove that for all $w_1, w_2, w_3 \in H^*_{\rho_n}(X^{[n]}, \mathbb{C})$,

$$\langle \Psi_n^{-1}(w_1), \Psi_n^{-1}(w_2), \Psi_n^{-1}(w_3) \rangle_{CR} = \langle w_1, w_2, w_3 \rangle_{\rho_n}(-1). \tag{4.34}$$

In the rest of the proof, we assume $n = 3$. To prove (4.34), it suffices to prove it as w_1, w_2, w_3 run over the linear basis (4.6) of $H^*_{\rho_n}(X^{[3]}, \mathbb{C})$. We will only prove (4.34) for the case

$$\omega_1 = \omega_2 = \mathfrak{a}_{-3}(1_X)|0\rangle, \ \ \omega_3 = \mathfrak{a}_{-2}(1_X)\mathfrak{a}_{-1}(\alpha_i)|0\rangle \tag{4.35}$$

with $\alpha_i \in H^2(X, \mathbb{C})$ since the remaining cases are similar, long and tedious. By Theorem 4.10 (ii) and (3.16), $\langle w_1, w_2, w_3 \rangle_{\rho_n}(q)$ is equal to

$$\langle w_1, w_2, w_3 \rangle - 2\langle K_X, \alpha_i \rangle \sum_{d \ge 1} d c_{3,d} q^d = \langle w_1, w_2, w_3 \rangle + 18\langle K_X, \alpha_i \rangle \left(\frac{3q^3}{q^3 + 1} - \frac{q}{q+1} \right).$$

https://doi.org/10.1017/fms.2023.17 Published online by Cambridge University Press

So $\langle w_1, w_2, w_3\rangle_{\rho_n}(-1) = \langle w_1, w_2, w_3\rangle + 18\langle K_X, \alpha_i\rangle$. We claim that

$$\langle w_1, w_2, w_3\rangle = -18\langle K_X, \alpha_i\rangle. \tag{4.36}$$

Indeed, we see from the first five lines in the proof of Lemma 4.8 that

$$\langle w_1, w_2, w_3\rangle = \langle w_1^2, w_3\rangle = \langle c_1^2 \cdot w_1, w_3\rangle + \frac{1}{2}\langle \mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(K_X)|0\rangle \cdot w_1, w_3\rangle.$$

Note that $\langle c_1^2 \cdot w_1, w_3\rangle = \langle c_1 w_1, c_1 w_3\rangle$. By (4.20), $\langle w_1, w_2, w_3\rangle$ is equal to

$$3\langle \mathfrak{a}_{-1}\mathfrak{a}_{-2}(\tau_{2*}1_X)|0\rangle, c_1 w_3\rangle + 3\langle \mathfrak{a}_{-3}(K_X)|0\rangle, c_1 w_3\rangle \\ + \frac{1}{2}\langle \mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(K_X)|0\rangle \cdot w_1, w_3\rangle.$$

Similar to (4.20), $c_1 w_3$ is equal to

$$\mathfrak{a}_{-1}(\alpha_i)\mathfrak{a}_{-1}\mathfrak{a}_{-1}(\tau_{2*}1_X)|0\rangle - 2\mathfrak{a}_{-3}(\alpha_i)|0\rangle + \mathfrak{a}_{-2}(K_X)\mathfrak{a}_{-1}(\alpha_i)|0\rangle.$$

Together with $\langle \mathfrak{a}_{-3}(K_X)|0\rangle, \mathfrak{a}_{-3}(\alpha_i)|0\rangle\rangle = 3\langle K_X, \alpha_i\rangle$, we see that

$$\langle w_1, w_2, w_3\rangle = -24\langle K_X, \alpha_i\rangle + \frac{1}{2}\langle \mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(K_X)|0\rangle \cdot w_1, w_3\rangle.$$

By a similar calculation, $\langle \mathfrak{a}_{-1}(1_X)\mathfrak{a}_{-2}(K_X)|0\rangle \cdot w_1, w_3\rangle = 12\langle K_X, \alpha_i\rangle$. So we get $\langle w_1, w_2, w_3\rangle = -24\langle K_X, \alpha_i\rangle + 6\langle K_X, \alpha_i\rangle = -18\langle K_X, \alpha_i\rangle$. This proves (4.36). Thus,

$$\langle w_1, w_2, w_3\rangle_{\rho_n}(-1) = \langle w_1, w_2, w_3\rangle + 18\langle K_X, \alpha_i\rangle = 0. \tag{4.37}$$

On the orbifold side, the calculation of $\langle \Psi_3^{-1}(w_1), \Psi_3^{-1}(w_2), \Psi_3^{-1}(w_3)\rangle_{CR}$ is similar but much simpler. Indeed, by the results in [38],

$$\langle \Psi_3^{-1}(w_1), \Psi_3^{-1}(w_2), \Psi_3^{-1}(w_3)\rangle_{CR}$$

can be read from its counterpart $\langle w_1, w_2, w_3\rangle$ by replacing every term $\langle K_X, \alpha_i\rangle$ by 0. Hence, $\langle \Psi_3^{-1}(w_1), \Psi_3^{-1}(w_2), \Psi_3^{-1}(w_3)\rangle_{CR} = 0$ by (4.36). Together with (4.37), we conclude that (4.34) holds for the case (4.35). □

5. Genus-1 extremal Gromov-Witten invariants of $X^{[3]}$

In this section, we determine the genus-1 extremal Gromov-Witten invariants of the Hilbert scheme $X^{[3]}$. First of all, we show that Conjecture 1.3 holds for $n = 3$.

Lemma 5.1. *Let X be a smooth projective surface, and let $d \geq 1$. Then,*

$$\langle\rangle_{1,d\beta_3} = (a_d + b_d \cdot \chi(X)) \cdot K_X^2$$

where a_d and b_d are universal constants depending only on d.

Proof. Let $[\varphi : D \to X^{[3]}] \in \overline{\mathfrak{M}}_{1,0}(X^{[3]}, d\beta_3)$. Then, $\varphi(D)$ is contracted by the Hilbert-Chow morphism $\rho_3 : X^{[3]} \to X^{(3)}$. So either $\rho_3(\varphi(D)) = x_1 + 2x_2$ for some points $x_1 \neq x_2$, or $\rho_3(\varphi(D)) = 3x$ for some point $x \in X$. When $\rho_3(\varphi(D)) = x_1 + 2x_2$, every image $\varphi(p)$ is of the form $x_1 + \xi(p)$ for some $\xi(p) \in M_2(x_2)$. Define

$$\varphi' : D \to X^{[2]}$$

by $\varphi'(p) = \xi(p)$. Then φ' is a stable map, and gives rise to an element

$$[\varphi' : D \to X^{[2]}] \in \overline{\mathfrak{M}}_{1,0}(X^{[2]}, d\beta_2).$$

The stable map φ' is one of the two components in the standard decomposition of φ (see (3.13)). The other component in the standard decomposition of φ is the constant map $D \to x_1 \in X$. Conversely, given a point $x_1 \in X$ and an element

$$[\varphi' : D \to X^{[2]}] \in \overline{\mathfrak{M}}_{1,0}(X^{[2]}, d\beta_2)$$

with $\{x_1\} \neq \mathrm{Supp}\big(\rho_2(\varphi'(D))\big)$, we can define a unique stable map

$$[\varphi : D \to X^{[3]}] \in \overline{\mathfrak{M}}_{1,0}(X^{[3]}, d\beta_3)$$

by $\varphi(p) = x_1 + \varphi'(p)$. Using the arguments in [28], we conclude that

$$\langle\rangle_{1,d\beta_3} = (a_d + b_d \cdot \chi(X)) \cdot K_X^2$$

for some universal constants a_d and b_d depending only on d. □

In the rest of this section, we will determine the universal constants a_d and b_d in Lemma 5.1. We will let X be a smooth projective toric surface and use torus actions and virtual localizations as in [8, 12, 20, 30] to compute $\langle\rangle_{1,d\beta_3}$.

5.1. *The contracted* $(\mathbb{C}^*)^2$*-invariant curves in* $X^{[3]}$ *for a toric surface X*

Let X be a smooth projective toric surface. In this subsection, we will write down all the invariant curves contracted by the Hilbert-Chow morphism $\rho_3 : X^{[3]} \to X^{(3)}$.

We begin with some standard setups. The surface X is determined by a fan Σ which is a finite collection of strongly convex rational polyhedral cones σ contained in $N = \mathrm{Hom}(M, \mathbb{Z})$, where $M \cong \mathbb{Z}^2$. So X is obtained by gluing together affine toric varieties X_σ and X_τ along $X_{\sigma\cap\tau}$ for $\sigma, \tau \in \Sigma$. The coordinate ring of X_σ is $\mathbb{C}[\sigma^\vee \cap M]$, which is the $\mathbb{C}$-algebra with generators χ^m for $m \in \sigma^\vee \cap M$ and multiplication defined by $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$. By definition, $\sigma^\vee \cap M$ is the set of elements $m \in M$ satisfying $v(m) \geq 0$ for all $v \in \sigma$. The torus

$$\mathbb{T} = (\mathbb{C}^*)^2$$

acts on X with finitely many fixed points $x_1, \ldots, x_{\chi(X)}$. For each i, the point x_i lies in $U_i := X_{\sigma_i}$ for some $\sigma_i \in \Sigma$. As X is smooth and U_i possesses a unique fixed point x_i, U_i is isomorphic to the affine plane with x_i corresponding to the origin. Let u_i, v_i be the affine coordinates of U_i. Assume that

$$(s,t)(u_i, v_i) = (\lambda_i(s,t)u_i, \mu_i(s,t)v_i)$$

for $(s,t) \in \mathbb{T}$, where $\lambda_i(s,t)$ and $\mu_i(s,t)$ are two independent characters of $\mathbb{T}$. Denote the weights of $\lambda_i(s,t)$ and $\mu_i(s,t)$ by w_i and z_i, respectively, that is,

$$w_i = c_1\big(\lambda_i(s,t)\big), \quad z_i = c_1\big(\mu_i(s,t)\big)$$

in the equivariant Chow group $A_*^{\mathbb{T}}(pt)$. By the Atiyah-Bott localization formula,

$$K_X^2 = \int_X c_1(T_X)^2 = \sum_{i=1}^{\chi(X)} \frac{(w_i + z_i)^2}{w_i z_i} \tag{5.1}$$

noting that $T_{x_i,X} = \big(\lambda_i(s,t)\big)^{-1} + \big(\mu_i(s,t)\big)^{-1}$ as representations.

https://doi.org/10.1017/fms.2023.17 Published online by Cambridge University Press

The $\mathbb{T}$-action on the toric surface X induces a $\mathbb{T}$-action on the Hilbert scheme $X^{[3]}$ with a finite number of fixed points. The $\mathbb{T}$-fixed points in $X^{[3]}$ are enumerated as follows. For each $1 \le i \le \chi(X)$, there are three $\mathbb{T}$-fixed points

$$Q_{i,0}, \qquad Q_{i,1}, \qquad Q_{i,2}$$

in $M_3(x_i) \subset X^{[3]}$ corresponding, respectively, to the partitions $(2,1)$, (3) and $(1,1,1)$ of 3. The corresponding ideals are $(v_i^2, v_i u_i, u_i^2)$, (v_i^3, u_i) and (v_i, u_i^3). Also for each ordered pair (i,j) with $i, j \in \{1, \dots, \chi(X)\}$ and $i \ne j$, we have two fixed points

$$R_{i,j}^{(1)} = \xi_{i,1} + x_j, \qquad R_{i,j}^{(2)} = \xi_{i,2} + x_j$$

in $X^{[3]}$, where $\xi_{i,1}, \xi_{i,2} \in M_2(x_i)$ correspond to the ideals $(v_i^2, u_i), (v_i, u_i^2)$, respectively. Furthermore, whenever $i, j, k \in \{1, \dots, \chi(X)\}$ are mutually distinct, $x_i + x_j + x_k \in X^{[3]}$ is a $\mathbb{T}$-fixed point in $X^{[3]}$. Denote the tangent space of $X^{[3]}$ at $\xi \in X^{[3]}$ by T_ξ. As representations of $\mathbb{T}$, we have the decompositions (see [9]):

$$T_{Q_{i,0}} = 2\lambda_i^{-1} + 2\mu_i^{-1} + \lambda_i^{-2}\mu_i + \lambda_i\mu_i^{-2}, \tag{5.2}$$

$$T_{Q_{i,1}} = \lambda_i^{-1}\mu_i^2 + \lambda_i^{-1}\mu_i + \lambda_i^{-1} + \mu_i^{-3} + \mu_i^{-2} + \mu_i^{-1}, \tag{5.3}$$

$$T_{Q_{i,2}} = \lambda_i^{-3} + \lambda_i^{-2} + \lambda_i^{-1} + \lambda_i^2\mu_i^{-1} + \lambda_i\mu_i^{-1} + \mu_i^{-1}, \tag{5.4}$$

$$T_{R_{i,j}^{(1)}} = \lambda_i^{-1}\mu_i + \lambda_i^{-1} + \mu_i^{-2} + \mu_i^{-1} + \lambda_j^{-1} + \mu_j^{-1}, \tag{5.5}$$

$$T_{R_{i,j}^{(2)}} = \lambda_i^{-2} + \lambda_i^{-1} + \lambda_i\mu_i^{-1} + \mu_i^{-1} + \lambda_j^{-1} + \mu_j^{-1}. \tag{5.6}$$

There are exactly three $\mathbb{T}$-invariant curves $C_{0,1}^{(i)}$, $C_{0,2}^{(i)}$ and $C_{1,2}^{(i)}$ in $M_3(x_i)$. Namely, $C_{0,1}^{(i)}$ goes through $Q_{i,0}$ and $Q_{i,1}$, and is the fixed locus of $\ker(\lambda_i\mu_i^{-2})$; $C_{0,2}^{(i)}$ goes through $Q_{i,0}$ and $Q_{i,2}$, and is the fixed locus of $\ker(\lambda_i^{-2}\mu_i)$; $C_{1,2}^{(i)}$ goes through $Q_{i,1}$ and $Q_{i,2}$, and is the fixed locus of $\ker(\lambda_i^{-1}\mu_i)$. The following is from [8].

Lemma 5.2. *There are exactly $\chi(X)(\chi(X)+2)$ $\mathbb{T}$-invariant curves contracted by the Hilbert-Chow morphism $\rho_3 : X^{[3]} \to X^{(3)}$. They are described as follows:*

(i) *the curves $C_{i,j} = M_2(x_i) + x_j$ where $1 \le i, j \le \chi(X)$ and $i \ne j$;*
(ii) *the curves $C_{k,\ell}^{(i)} \subset M_3(x_i)$ where $1 \le i \le \chi(X)$ and $0 \le k < \ell \le 2$.*

Moreover, $C_{i,j} \sim C_{0,1}^{(i)} \sim C_{0,2}^{(i)} \sim \beta_3$ and $C_{1,2}^{(i)} \sim 3\beta_3$ for every $1 \le i \ne j \le \chi(X)$.

Next, let $f : \mathbb{P}^1 \to X^{[3]}$ be a degree-d morphism such that the image is one of the $\mathbb{T}$-invariant curves in Lemma 5.2 and f is totally ramified at the two $\mathbb{T}$-fixed points in $f(\mathbb{P}^1)$. The Euler characteristic $\chi(f^*T_{X^{[3]}})$ (as a representation) has been computed in [8]. When $f(\mathbb{P}^1) = C_{0,1}^{(i)}$, we have

$$\begin{aligned}\chi(f^*T_{X^{[3]}}) &= (1 + \lambda_i^{-1}\mu_i^2 + \lambda_i\mu_i^{-2} + \lambda_i^{-1}\mu_i + \mu_i^{-1} + \lambda_i^{-1} + \mu_i^{-1} - \lambda_i^{-1}\mu_i^{-1}) \\ &\quad + (\lambda_i^{-1}\mu_i^2 + 1 + \lambda_i^{-1}\mu_i - \lambda_i^{-2}\mu_i - \lambda_i^{-1}\mu_i^{-1} - \lambda_i^{-1})\Theta_{0,1}^{(i)}\end{aligned} \tag{5.7}$$

https://doi.org/10.1017/fms.2023.17 Published online by Cambridge University Press

where $\Theta_{0,1}^{(i)} = \sum_{m=1}^{d-1}(\lambda_i\mu_i^{-2})^{m/d}$ ($\Theta_{0,1}^{(i)} = 0$ when $d = 1$). If $f(\mathbb{P}^1) = C_{0,2}^{(i)}$, then

$$\chi(f^*T_{X^{[3]}}) = (1 + \mu_i^{-1}\lambda_i^2 + \mu_i\lambda_i^{-2} + \mu_i^{-1}\lambda_i + \lambda_i^{-1} + \mu_i^{-1} + \lambda_i^{-1} - \mu_i^{-1}\lambda_i^{-1}) \\ + (\mu_i^{-1}\lambda_i^2 + 1 + \mu_i^{-1}\lambda_i - \mu_i^{-2}\lambda_i - \mu_i^{-1}\lambda_i^{-1} - \mu_i^{-1})\Theta_{0,2}^{(i)} \tag{5.8}$$

where $\Theta_{0,2}^{(i)} = \sum_{m=1}^{d-1}(\mu_i\lambda_i^{-2})^{m/d}$ ($\Theta_{0,2}^{(i)} = 0$ when $d = 1$). Let $\Theta_{1,2}^{(i)} = \sum_{m=1}^{d-1}(\lambda_i\mu_i^{-1})^{m/d}$ with $\Theta_{1,2}^{(i)} = 0$ when $d = 1$. If $f(\mathbb{P}^1) = C_{1,2}^{(i)}$, then $\chi(f^*T_{X^{[3]}})$ is equal to

$$(1 + \lambda_i^{-1}\mu_i^2 + \mu_i + \lambda_i + \lambda_i^{-1}\mu_i - \lambda_i^{-2} - \lambda_i^{-1}\mu_i^{-1} - \lambda_i^{-3} - \lambda_i^{-2}\mu_i^{-1} - \lambda_i^{-1}\mu_i^{-2})\Theta_{1,2}^{(i)} \\ + (\lambda_i^{-1} + \mu_i^{-1} + \lambda_i^{-1}\mu_i + 1 + \lambda_i\mu_i^{-1} + \lambda_i^{-1}\mu_i^2 + \mu_i + \lambda_i + \lambda_i^2\mu_i^{-1} \\ - \lambda_i^{-1}\mu_i^{-1} - \lambda_i^{-2}\mu_i^{-1} - \lambda_i^{-1}\mu_i^{-2}). \tag{5.9}$$

Finally, when $f(\mathbb{P}^1) = C_{i,j}$, then $\chi(f^*T_{X^{[3]}})$ is equal to

$$(1 + \lambda_i^{-1}\mu_i + \lambda_i\mu_i^{-1} + \lambda_i^{-1} + \mu_i^{-1} + \lambda_j^{-1} + \mu_j^{-1} - \lambda_i^{-1}\mu_i^{-1}) \\ + (1 + \lambda_i^{-1}\mu_i - \lambda_i^{-2} - \lambda_i^{-1}\mu_i^{-1})\Theta_{1,2}^{(i)}. \tag{5.10}$$

5.2. *$\mathbb{T}$-invariant stable maps, stable graphs and localizations*

Let X be a smooth projective toric surface and $d \geq 1$. For simplicity, put

$$\overline{\mathfrak{M}}_{g,r,d} = \overline{\mathfrak{M}}_{g,r}(X^{[3]}, d\beta_3).$$

In this subsection, using virtual localization formula, we will express the genus-1 extremal Gromov-Witten invariant $\langle\rangle_{1,d\beta_3}$ in terms of stable graphs.

As in [12, 19], if $[f : C \to X^{[3]}] \in \overline{\mathfrak{M}}_{1,0,d}$ is $\mathbb{T}$-invariant, then all the nodes, contracted components and ramification points are mapped into the $\mathbb{T}$-fixed point set $(X^{[3]})^{\mathbb{T}}$. Moreover, if $\widetilde{C} \subset C$ is a noncontracted component, then $\widetilde{C} = \mathbb{P}^1$, $f(\widetilde{C})$ is one of the $\mathbb{T}$-invariant curves in Lemma 5.2, and $f|_{\widetilde{C}}$ is of the form

$$(z_0, z_1) \mapsto (z_0^{\tilde{d}}, z_1^{\tilde{d}})$$

where $\tilde{d} = \deg(f|_{\widetilde{C}})$. Therefore, to each stable map $[f : C \to X^{[3]}] \in (\overline{\mathfrak{M}}_{1,0,d})^{\mathbb{T}}$, we can associate a stable graph Γ as follows. The stable graph Γ has one vertex for each connected component of $f^{-1}((X^{[3]})^{\mathbb{T}})$ and one edge for every noncontracted component. The edge e is marked with the degree d_e of f restricted to that noncontracted component $C_e = \mathbb{P}^1$, and the connected component corresponding to a vertex v is denoted by C_v. Let $V(\Gamma)$ (respectively, $E(\Gamma)$) denote the set of vertices (respectively, edges) of Γ. Define the labeling map

$$\mathfrak{L} : \quad V(\Gamma) \longrightarrow (X^{[3]})^{\mathbb{T}}$$

by putting $\mathfrak{L}(v) = f(C_v)$. The vertices have an additional labeling $g(v)$ which is the arithmetic genus of C_v ($g(v) = 0$ if C_v is a point) and satisfies the identity

$$1 - |V(\Gamma)| + |E(\Gamma)| + \sum_{v \in V(\Gamma)} g(v) = g = 1.$$

The valence of v, denoted by $\mathrm{val}(v)$, is the number of edges connected to v. Define a flag F of the graph Γ to be an incident edge-vertex pair (e, v). Put

$$i(F) = \mathfrak{L}(v).$$

A flag $F = (e, v)$ is defined to be *stable* if $2g(v) + \mathrm{val}(v) \geq 3$. Since $\mathrm{val}(v) \geq 1$, F is not stable if $g(v) = 0$ and $\mathrm{val}(v) = 1$ or 2 (in these cases, the component C_v is simply a point). Let $F(\Gamma)$ (respectively, $F(\Gamma)^{\mathrm{sta}}$) be the set of flags (respectively, stable flags) in Γ. The edge e in $F = (e, v)$ is incident to one other vertex v'. Define $j(F) = \mathfrak{L}(v')$. If $\mathrm{val}(v) = 1$, let $F(v)$ be the unique flag containing v; if $\mathrm{val}(v) = 2$, let $F_1(v)$ and $F_2(v)$ denote the two flags containing v.

Now the connected components of $(\overline{\mathfrak{M}}_{1,0,d})^{\mathbb{T}}$ are indexed by stable graphs corresponding to stable maps whose images are unions of the $\mathbb{T}$-invariant curves in Lemma 5.2 and whose contracted components and special points are mapped into $(X^{[3]})^{\mathbb{T}}$. We use Γ to denote these stable graphs. So we have

$$(\overline{\mathfrak{M}}_{1,0,d})^{\mathbb{T}} = \coprod_{\Gamma} \overline{\mathfrak{M}}_{\Gamma} \tag{5.11}$$

where $\overline{\mathfrak{M}}_{\Gamma}$ denotes the connected component of $(\overline{\mathfrak{M}}_{1,0,d})^{\mathbb{T}}$ indexed by Γ. Let $\overline{M}_{g,n}$ be the moduli space of n-pointed genus-g stable curves. Put

$$\overline{M}_{\Gamma} = \prod_{v \in V(\Gamma)} \overline{M}_{g(v),\mathrm{val}(v)}$$

($\overline{M}_{0,1}$ and $\overline{M}_{0,2}$ are treated as points in this product). Then there is a finite map $\overline{M}_{\Gamma} \to \overline{\mathfrak{M}}_{\Gamma}$ such that $\overline{\mathfrak{M}}_{\Gamma} = \overline{M}_{\Gamma}/\mathbf{A}_{\Gamma}$ where $\mathbf{A}_{\Gamma}$ fits in the exact sequence

$$0 \to \prod_{e \in E(\Gamma)} \mathbb{Z}/d_e\mathbb{Z} \to \mathbf{A}_{\Gamma} \to \mathrm{Aut}(\Gamma) \to 0. \tag{5.12}$$

Since a stable curve is connected, we see from the description of the $\mathbb{T}$-invariant curves in Lemma 5.2 that a summation over all the stable graphs Γ breaks up as

$$\sum_{\Gamma} = \sum_{1 \leq i \neq j \leq \chi(X)} \sum_{\Gamma \in \mathcal{S}_{d,i,j}} + \sum_{i=1}^{\chi(X)} \sum_{\Gamma \in \mathcal{T}_{d,i}} \tag{5.13}$$

where $\mathcal{S}_{d,i,j}$ is the set of all stable graphs Γ such that $f(C) = C_{i,j}$ for every $[f : C \to X^{[3]}] \in \overline{\mathfrak{M}}_{\Gamma}$, and $\mathcal{T}_{d,i}$ is the set of all stable graphs Γ such that $f(C) \subset C_{0,1}^{(i)} \cup C_{0,2}^{(i)} \cup C_{1,2}^{(i)}$ for every $[f : C \to X^{[3]}] \in \overline{\mathfrak{M}}_{\Gamma}$.

By the virtual localization formula of [12], we have

$$\langle\rangle_{1,d\beta_3} = \int_{[\overline{\mathfrak{M}}_{1,0,d}]^{\mathrm{vir}}} 1 = \sum_{\Gamma} \frac{1}{|\mathbf{A}_{\Gamma}|} \int_{[\overline{M}_{\Gamma}]^{\mathrm{vir}}} \frac{1}{e(N_{\Gamma}^{\mathrm{vir}})}. \tag{5.14}$$

Here, $[\overline{M}_{\Gamma}]^{\mathrm{vir}}$ is the pullback of $[\overline{\mathfrak{M}}_{\Gamma}]^{\mathrm{vir}}$ to M_{Γ} via the finite map $\overline{M}_{\Gamma} \to \overline{\mathfrak{M}}_{\Gamma}$, and $e(N_{\Gamma}^{\mathrm{vir}})$ is the pullback of the Euler class of the moving part $N_{\Gamma}^{\mathrm{vir}}$ of the tangent-obstruction complex. Let $\mathcal{T}^1$ and $\mathcal{T}^2$ be the cohomology sheaves of the restriction of the tangent-obstruction complex on $\overline{\mathfrak{M}}_{1,0,d}$ to $\overline{\mathfrak{M}}_{\Gamma}$. The fibers of $\mathcal{T}^1$ and $\mathcal{T}^2$ at a point associated to a stable map $[f : C \to X^{[3]}] \in \overline{\mathfrak{M}}_{\Gamma}$ fit into the exact sequence

$$\begin{aligned} 0 &\to \mathrm{Ext}^0(\Omega_C, \mathcal{O}_C) \to H^0(C, f^*T_{X^{[3]}}) \to \mathcal{T}^1 \\ &\to \mathrm{Ext}^1(\Omega_C, \mathcal{O}_C) \to H^1(C, f^*T_{X^{[3]}}) \to \mathcal{T}^2 \to 0. \end{aligned}$$

To understand $H^i(C, f^*T_{X^{[3]}})$, consider the normalization sequence resolving the nodes of C coming from all the intersections $x_F := C_v \cap C_e$:

$$0 \to \mathcal{O}_C \to \bigoplus_v \mathcal{O}_{C_v} \oplus \bigoplus_e \mathcal{O}_{C_e} \to \bigoplus_F \mathcal{O}_{x_F} \to 0.$$

Tensoring by $f^*T_{X^{[3]}}$ and taking cohomology, we obtain an exact sequence

$$0 \to H^0(C, f^*T_{X^{[3]}}) \to \bigoplus_v T_{\mathfrak{L}(v)} \oplus \bigoplus_e H^0(C_e, f^*T_{X^{[3]}}) \to \bigoplus_F T_{i(F)}$$
$$\to H^1(C, f^*T_{X^{[3]}}) \to \bigoplus_v H^1(C_v, f^*T_{X^{[3]}}) \oplus \bigoplus_e H^1(C_e, f^*T_{X^{[3]}}) \to 0. \tag{5.15}$$

Note that $H^1(C_v, f^*T_{X^{[3]}}) = H^1(C_v, \mathcal{O}_{C_v}) \otimes T_{\mathfrak{L}(v)}$ where $H^1(C_v, \mathcal{O}_{C_v})$ forms the dual of the Hodge bundle $\mathcal{H}_{g(v)}$ over $\overline{M}_{g(v),\mathrm{val}(v)}$. By the five formulas (5.2)-(5.6), the fixed parts of $T_{i(F)}$ and $H^1(C_v, f^*T_{X^{[3]}})$ vanish. Examining the terms in the four formulas (5.7)-(5.10) which carry negative signs, we see that the fixed part of $H^1(C_e, f^*T_{X^{[3]}})$ also vanishes. By (5.15), the fixed part of $H^1(C, f^*T_{X^{[3]}})$ vanishes. Thus, $\mathcal{T}^{2,f} = 0$, and the fixed stack is smooth with tangent bundle $\mathcal{T}^{1,f}$. Hence, $[\overline{\mathfrak{M}}_\Gamma]^{\mathrm{vir}} = [\overline{\mathfrak{M}}_\Gamma]$ and $[\overline{M}_\Gamma]^{\mathrm{vir}} = [\overline{M}_\Gamma]$. By (5.14), we obtain

$$\langle\rangle_{1,d\beta_3} = \sum_\Gamma \frac{1}{|\mathbf{A}_\Gamma|} \int_{[\overline{M}_\Gamma]} \frac{1}{e(N_\Gamma^{\mathrm{vir}})}.$$

In view of the splitting (5.13), the invariant $\langle\rangle_{1,d\beta_3}$ can be written as

$$\sum_{1 \le i \ne j \le \chi(X)} \sum_{\Gamma \in \mathcal{S}_{d,i,j}} \frac{1}{|\mathbf{A}_\Gamma|} \int_{[\overline{M}_\Gamma]} \frac{1}{e(N_\Gamma^{\mathrm{vir}})} + \sum_{i=1}^{\chi(X)} \sum_{\Gamma \in \mathcal{T}_{d,i}} \frac{1}{|\mathbf{A}_\Gamma|} \int_{[\overline{M}_\Gamma]} \frac{1}{e(N_\Gamma^{\mathrm{vir}})} \tag{5.16}$$

5.3. *Reformulation of* $\sum_{1 \le i \ne j \le \chi(X)} \sum_{\Gamma \in \mathcal{S}_{d,i,j}}$

In this subsection, we will reformulate the summation $\sum_{1 \le i \ne j \le \chi(X)} \sum_{\Gamma \in \mathcal{S}_{d,i,j}}$ in (5.16) by a suitable genus-1 Gromov-Witten invariant of $X \times X^{[2]}$. It allows us to reduce the computation of $\langle\rangle_{1,d\beta_3}$ to the local affine charts $U_i \ni x_i$.

For $1 \le i \le \chi(X)$ and $1 \le k \le 2$, let $R_{i,i}^{(k)} = (x_i, \xi_{i,k}) \in X \times X^{[2]}$ and

$$C_{i,i} = \{x_i\} \times M_2(x_i) \subset X \times X^{[2]}.$$

For $1 \le i \ne j \le \chi(X)$, regard the curve $C_{i,j} \subset X^{[3]}$ in Lemma 5.2 (i) as the curve $\{x_j\} \times M_2(x_i) \subset X \times X^{[2]}$. The $\mathbb{T}$-action on X induces a $\mathbb{T}$-action on $X \times X^{[2]}$. The $\mathbb{T}$-fixed point set $(X \times X^{[2]})^{\mathbb{T}}$ consists of the points $R_{i,j}^{(k)}$ with $1 \le i, j \le \chi(X)$ and $1 \le k \le 2$. The $\mathbb{T}$-invariant curves in $X \times X^{[2]}$ contracted by

$$\mathrm{Id} \times \rho_2 : \ X \times X^{[2]} \to X \times X^{(2)}$$

are precisely the curves $C_{i,j}$ with $1 \le i, j \le \chi(X)$. The decompositions of the tangent spaces of $X \times X^{[2]}$ at the points $R_{i,j}^{(k)}$ are given by the right-hand sides of (5.5) and (5.6). So we keep using $T_{R_{i,j}^{(k)}}$ to denote the tangent space of $X \times X^{[2]}$ at $R_{i,j}^{(k)}$. Similarly, if $f : \mathbb{P}^1 \to X \times X^{[2]}$ is a degree-d morphism such that $f(\mathbb{P}^1) = C_{i,j}$ and f is totally ramified at the two $\mathbb{T}$-fixed points in $f(\mathbb{P}^1)$, then the Euler characteristic $\chi(f^*T_{X \times X^{[2]}})$ is given by the right-hand side of (5.10).

Regard $\beta_2 \in H_2(X^{[2]})$ as in $H_2(X \times X^{[2]})$. Apply localization to the moduli space

$$\overline{\mathfrak{M}}_{1,0}(X \times X^{[2]}, d\beta_2)$$

whose expected dimension is equal to 0. The connected components of the $\mathbb{T}$-fixed point set $\left(\overline{\mathfrak{M}}_{1,0}(X \times X^{[2]}, d\beta_2)\right)^{\mathbb{T}}$ are indexed by stable graphs Γ. For $1 \le i, j \le \chi(X)$, let $\mathcal{S}_{d,i,j}$ be the set of all stable

https://doi.org/10.1017/fms.2023.17 Published online by Cambridge University Press

graphs Γ such that $f(C) = C_{i,j}$ for every stable map $[f : C \to X \times X^{[2]}]$ in the connected component $\overline{\mathfrak{M}}_\Gamma$ indexed by Γ. Note that when $i \neq j$, $\mathcal{S}_{d,i,j}$ can be identified with the set $\mathcal{S}_{d,i,j}$ introduced in (5.13). Moreover, for $\Gamma \in \mathcal{S}_{d,i,j}$ with $i \neq j$, $\overline{\mathfrak{M}}_\Gamma$ can be identified with the connected component $\overline{\mathfrak{M}}_\Gamma$ introduced in (5.11). By the virtual localization formula,

$$\int_{[\overline{\mathfrak{M}}_{1,0}(X\times X^{[2]},d\beta_2)]^{\text{vir}}} 1 \quad = \quad \sum_{1\le i,j\le \chi(X)} \sum_{\Gamma\in\mathcal{S}_{d,i,j}} \frac{1}{|\mathbf{A}_\Gamma|} \int_{[\overline{M}_\Gamma]} \frac{1}{e(N_\Gamma^{\text{vir}})}. \tag{5.17}$$

Note that for each graph $\Gamma \in \mathcal{S}_{d,i,j}$ with $i \neq j$, the summand $\dfrac{1}{|\mathbf{A}_\Gamma|} \displaystyle\int_{[\overline{M}_\Gamma]} \frac{1}{e(N_\Gamma^{\text{vir}})}$ in (5.17) is equal to the corresponding summand in (5.16).

Lemma 5.3. $\displaystyle\int_{[\overline{\mathfrak{M}}_{1,0}(X\times X^{[2]},d\beta_2)]^{\text{vir}}} 1 \; = \; \frac{1}{12d} \cdot \chi(X) \cdot K_X^2.$

Proof. We have $\overline{\mathfrak{M}}_{1,0}(X \times X^{[2]}, d\beta_2) \cong X \times \overline{\mathfrak{M}}_{1,0}(X^{[2]}, d\beta_2)$. By the results in [15], the moduli space $\overline{\mathfrak{M}}_{1,0}(X^{[2]}, d\beta_2)$ is smooth (as a stack) with dimension $(2d + 2)$, and the obstruction sheaf $\mathcal{O}b = R^1(f_{1,0})_*\text{ev}_1^*T_{X^{[2]}}$ on $\overline{\mathfrak{M}}_{1,0}(X^{[2]}, d\beta_2)$ is locally free of rank $(2d + 2)$ where $f_{1,0}$ (respectively, ev_1) denotes the forgetful (respectively, evaluation) map on $\overline{\mathfrak{M}}_{1,1}(X^{[2]}, d\beta_2)$. Moreover, we have

$$\langle\rangle_{1,d\beta_2} = \deg c_{2d+2}(\mathcal{O}b) = \frac{1}{12d} \cdot K_X^2. \tag{5.18}$$

Let ϕ_1 and ϕ_2 be the two projections on $X \times \overline{\mathfrak{M}}_{1,0}(X^{[2]}, d\beta_2)$. Let $\mathcal{H}_1$ be the (rank-1) Hodge bundle over the moduli space $\overline{\mathfrak{M}}_{1,0}(X^{[2]}, d\beta_2)$. A direct computation shows that the obstruction sheaf over $\overline{\mathfrak{M}}_{1,0}(X \times X^{[2]}, d\beta_2)$ is isomorphic to

$$(\phi_1^*T_X \otimes \phi_2^*\mathcal{H}_1^\vee) \oplus \phi_2^*\mathcal{O}b$$

which is locally free of rank $(2d + 4)$. Therefore, we conclude that

$$\begin{aligned}[\overline{\mathfrak{M}}_{1,0}(X \times X^{[2]}, d\beta_2)]^{\text{vir}} &= c_{2d+4}\big((\phi_1^*T_X \otimes \phi_2^*\mathcal{H}_1^\vee) \oplus \phi_2^*\mathcal{O}b\big) \\ &= c_2\big(\phi_1^*T_X \otimes \phi_2^*\mathcal{H}_1^\vee\big) \cdot c_{2d+2}\big(\phi_2^*\mathcal{O}b\big).\end{aligned}$$

Combining this with (5.18), we immediately verify our lemma. □

From (5.16), (5.17) and Lemma 5.3, we conclude that

$$\langle\rangle_{1,d\beta_3} = \frac{1}{12d} \cdot \chi(X) \cdot K_X^2 \; + \; \sum_{i=1}^{\chi(X)} \left(\sum_{\Gamma\in\mathcal{T}_{d,i}} - \sum_{\Gamma\in\mathcal{S}_{d,i,i}} \right) \frac{1}{|\mathbf{A}_\Gamma|} \int_{[\overline{M}_\Gamma]} \frac{1}{e(N_\Gamma^{\text{vir}})}. \tag{5.19}$$

Note that $\sum_{\Gamma\in\mathcal{T}_{d,i}} - \sum_{\Gamma\in\mathcal{S}_{d,i,i}}$ depends only on the local chart $U_i \ni x_i$.

For simplicity, whenever S is a set of stable graphs, we use $\sum_{\Gamma\in S}$ to denote

$$\sum_{\Gamma\in S} \frac{1}{|\mathbf{A}_\Gamma|} \int_{[\overline{M}_\Gamma]} \frac{1}{e(N_\Gamma^{\text{vir}})}. \tag{5.20}$$

5.4. A reduction lemma

In this subsection, we will prove a reduction lemma which indicates that we may ignore most of the stable graphs in $\mathcal{T}_{d,i}$ and $\mathcal{S}_{d,i,i}$ when we evaluate the summation $\sum_{\Gamma\in\mathcal{T}_{d,i}} - \sum_{\Gamma\in\mathcal{S}_{d,i,i}}$ in (5.19).

Before we state the reduction lemma, we present the motivations. As we will see in the next two subsections, $\sum_{\Gamma\in\mathcal{T}_{d,i}}-\sum_{\Gamma\in\mathcal{S}_{d,i,i}}$ is of the form

$$(w_i+z_i)^2\cdot\frac{p_{1,1}(w_i,z_i)}{q_{1,1}(w_i,z_i)}+(w_i+z_i)^3\cdot\frac{p_{1,2}(w_i,z_i)}{q_{1,2}(w_i,z_i)}$$

where $p_{1,1}(w_i,z_i),q_{1,1}(w_i,z_i),p_{1,2}(w_i,z_i),q_{1,2}(w_i,z_i)\in\mathbb{Q}[w_i,z_i]$ are symmetric homogeneous polynomials independent of i and X, $(w_i+z_i)\nmid q_{1,1}(w_i,z_i)$, $(w_i+z_i)\nmid q_{1,2}(w_i,z_i)$, $\deg(q_{1,1})=\deg(p_{1,1})+2$, $\deg(q_{1,2})=\deg(p_{1,2})+3$ and all the roots of $q_{1,1}(w,1)$ and $q_{1,2}(w,1)$ are rational. Note that $p_{1,1}(w_i,z_i),q_{1,1}(w_i,z_i),p_{1,2}(w_i,z_i)$ and $q_{1,2}(w_i,z_i)$ can be expressed as polynomials in w_i+z_i and w_iz_i. So the summation $\sum_{\Gamma\in\mathcal{T}_{d,i}}-\sum_{\Gamma\in\mathcal{S}_{d,i,i}}$ can be rewritten as

$$(w_i+z_i)^2\cdot\frac{\tilde{a}\cdot(w_iz_i)^m}{q_{1,1}(w_i,z_i)}+(w_i+z_i)^3\cdot\frac{p_{2,2}(w_i,z_i)}{q_{1,2}(w_i,z_i)}$$

where $\tilde{a}$ and m are independent of i and X, and $p_{2,2}(w_i,z_i)$ is a symmetric homogeneous polynomial independent of i and X. Put $q_{1,1}(w_i,z_i)=\tilde{a}_0(w_iz_i)^{m+1}+\tilde{a}_1(w_iz_i)^m(w_i+z_i)^2+\ldots+\tilde{a}_{m+1}(w_i+z_i)^{2(m+1)}$. Then,

$$(w_i+z_i)^2\cdot\frac{\tilde{a}\cdot(w_iz_i)^m}{q_{1,1}(w_i,z_i)}=a\cdot\frac{(w_i+z_i)^2}{w_iz_i}-(w_i+z_i)^4\cdot\frac{p_{2,1}(w_i,z_i)}{w_iz_i\cdot q_{1,1}(w_i,z_i)}$$

where $a=\tilde{a}/\tilde{a}_0$, and $p_{2,1}(w_i,z_i)$ is a symmetric homogeneous polynomial independent of i and X. It follows that $\sum_{\Gamma\in\mathcal{T}_{d,i}}-\sum_{\Gamma\in\mathcal{S}_{d,i,i}}$ is of the form

$$a_d\cdot\frac{(w_i+z_i)^2}{w_iz_i}+(w_i+z_i)^3\cdot\frac{p_d(w_i,z_i)}{q_d(w_i,z_i)} \tag{5.21}$$

where $a_d(=a),p_d(w_i,z_i)$ and $q_d(w_i,z_i)$ are independent of i and X and depend only on d, $a_d\in\mathbb{Q}$, $p_d(w_i,z_i)$ and $q_d(w_i,z_i)$ are symmetric homogeneous polynomials in $\mathbb{Q}[w_i,z_i]$, $(w_i+z_i)\nmid q_d(w_i,z_i)$ and the roots of $q_d(w,1)$ are rational. Our reduction lemma below asserts that $p_d=0$.

Lemma 5.4. *The summation* $\sum_{\Gamma\in\mathcal{T}_{d,i}}-\sum_{\Gamma\in\mathcal{S}_{d,i,i}}$ *is of the form*

$$a_d\cdot\frac{(w_i+z_i)^2}{w_iz_i} \tag{5.22}$$

where $a_d\in\mathbb{Q}$ *is independent of* i *and* X *and depends only on* d*, and*

$$\langle\rangle_{1,d\beta_3}=\left(a_d+\frac{1}{12d}\cdot\chi(X)\right)K_X^2. \tag{5.23}$$

Proof. Note that (5.23) follows from (5.19), (5.22) and (5.1). In the following, we will prove (5.22) (i.e., we will show that $p_d=0$ in (5.21)). For convenience, we will simply write a,p,q instead of a_d,p_d,q_d. Assume $p\neq 0$. We will draw contradictions. We may further assume that $p(w_i,z_i)$ and $q(w_i,z_i)$ have no common factors of positive degrees and that $q(w,1)$ is monic.

First of all, we conclude from (5.19), (5.21) and (5.1) that

$$\sum_{i=1}^{\chi(X)}(w_i+z_i)^3\cdot\frac{p(w_i,z_i)}{q(w_i,z_i)}=\langle\rangle_{1,d\beta_3}-\frac{1}{12d}\cdot\chi(X)\cdot K_X^2-aK_X^2. \tag{5.24}$$

For simplicity, denote the right-hand side of (5.24) by $e(X,d)$. The symmetric polynomials $p(w_i,z_i)$ and $q(w_i,z_i)$ can be expressed as polynomials in (w_i+z_i) and w_iz_i. Since $(w_i+z_i)\nmid q(w_i,z_i)$, $q(w_i,z_i)$

is of the form

$$q(w_i, z_i) = (w_i z_i)^{n_0} \cdot \tilde{q}(w_i, z_i) = (w_i z_i)^{n_0} \cdot \prod_{j=1}^{k} \left((w_i + z_i)^2 + a_j w_i z_i\right)^{n_j} \tag{5.25}$$

where $n_0 \geq 0$, $k \geq 0$, $a_1, \dots, a_k$ are distinct and $a_j \neq 0$ and $n_j > 0$ for every j. So $\deg(q)$ is even, $\deg(p) = \deg(q) - 3$ is odd and $(w_i + z_i) | p(w_i, z_i)$. Put

$$p(w_i, z_i) = (w_i + z_i) \cdot \tilde{p}(w_i, z_i).$$

Being of even degree, the symmetric homogeneous polynomial $\tilde{p}(w_i, z_i)$ is a polynomial of $(w_i + z_i)^2$ and $w_i z_i$. By (5.24), we have

$$\sum_{i=1}^{\chi(X)} (w_i + z_i)^4 \cdot \frac{\tilde{p}(w_i, z_i)}{q(w_i, z_i)} = e(X, d). \tag{5.26}$$

For $X = \mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, the weights w_i and z_i are of the form

$$\begin{aligned} &\{(w_i, z_i) | 1 \leq i \leq \chi(X)\} \\ =&\begin{cases} \{(w, z), (w - z, -z), (z - w, -w)\}, & \text{if } X = \mathbb{P}^2 \\ \{(w, z), (-w, -z), (w, -z), (-w, z)\}, & \text{if } X = \mathbb{P}^1 \times \mathbb{P}^1. \end{cases} \end{aligned}$$

Set $z = 1$. Letting $X = \mathbb{P}^2$ and $X = \mathbb{P}^1 \times \mathbb{P}^1$ in (5.26), respectively, we obtain

$$(w + 1)^4 \, \frac{\tilde{p}(w, 1)}{q(w, 1)} + (w - 2)^4 \, \frac{\tilde{p}(w - 1, -1)}{q(w - 1, -1)} + (1 - 2w)^4 \, \frac{\tilde{p}(1 - w, -w)}{q(1 - w, -w)} = e_1, \tag{5.27}$$

$$(w + 1)^4 \cdot \frac{\tilde{p}(w, 1)}{q(w, 1)} + (-w + 1)^4 \cdot \frac{\tilde{p}(-w, 1)}{q(-w, 1)} = e_2 \tag{5.28}$$

where $e_1 = e(\mathbb{P}^2, d)$ and $e_2 = e(\mathbb{P}^1 \times \mathbb{P}^1, d)/2$. Since $p(w_i, z_i)$ and $q(w_i, z_i)$ have no common factor of positive degree, neither do $\tilde{p}(w, 1)$ and $\tilde{q}(w, 1)$. If $k \geq 1$, then by (5.28) and (5.25), $\tilde{q}(w, 1) | \tilde{q}(-w, 1)$. So $\tilde{q}(w, 1) = \tilde{q}(-w, 1)$ since they are monic and $\tilde{q}(w_i, z_i) = \tilde{q}(-w_i, z_i)$. Since the roots of $q(w, 1)$ are rational, $a_j \neq -2$ and

$$(w_i + z_i)^2 + a_j w_i z_i \neq (w_i - z_i)^2 - a_j w_i z_i$$

for j and i. Therefore, $(w_i + z_i)^2 + a_j w_i z_i$ and $(w_i - z_i)^2 - a_j w_i z_i$ are distinct factors in the decomposition (5.25) of $q(w_i, z_i)$, and $q(w_i, z_i)$ can be rewritten as

$$\begin{aligned} &(w_i z_i)^{n_0} \cdot \prod_{j=1}^{s} \left(((w_i + z_i)^2 + a_j w_i z_i)((w_i - z_i)^2 - a_j w_i z_i)\right)^{n_j} && (5.29) \\ =&(w_i z_i)^{n_0} \cdot \prod_{j=1}^{s} \left((w_i^2 + z_i^2)^2 - \tilde{a}_j (w_i z_i)^2\right)^{n_j} && (5.30) \end{aligned}$$

where $s = k/2 \geq 0$, $\tilde{a}_j = (2 + a_j)^2$ and $\tilde{a}_1, \dots, \tilde{a}_s$ are distinct. Since the roots of $(w + 1)^2 + a_j w$ are rational, $\tilde{a}_j \geq 4$. Since $(w_i + z_i) \nmid q(w_i, z_i)$, $\tilde{a}_j \neq 4$. So $\tilde{a}_j > 4$, and $a_j \neq 0, -4$. Let $n = \deg(q) = 2n_0 + 4(n_1 + \dots + n_s)$. Then $\deg(\tilde{p}) = n - 4$.

If n_0 is positive and even, then as a polynomial in $(w_i+z_i)^2$ and w_iz_i, $\tilde{p}(w_i, z_i)$ contains the monomial $(w_i+z_i)^{n-4}$ with nonzero coefficient. So $(w+1)^4\,\tilde{p}(w, 1)$ is a polynomial of degree n in w. Since $q(w, 1)$ is of degree $n-n_0$, letting $w \to \infty$ in (5.28), we get $\infty = e_2$. This is impossible since e_2 is a finite number.

If n_0 is odd with $n_0 \geq 3$, then write $\tilde{p}(w_i, z_i) = \sum_{j=0}^{n-4} h_j w_i^j z_i^{(n-4)-j}$. Since $\tilde{p}(w_i, z_i)$ is symmetric, $h_j = h_{(n-4)-j}$. Since $(w_iz_i) \nmid \tilde{p}(w_i, z_i)$, $h_0 \neq 0$. Since $a_j \notin \{0, -4\}$, we see that $w \nmid q(w-1, -1)$ and $w \nmid \tilde{q}(1-w, -w)$. Substitute (5.30) into (5.27) and (5.28). Expanding the left-hand sides of (5.27) and (5.28), we get

$$\big((n_0+8)h_0 + 2h_1\big)w^{-(n_0-1)} + O(w^{-(n_0-2)}) = e_1,$$
$$(8h_0 + 2h_1)w^{-(n_0-1)} + O(w^{-(n_0-3)}) = e_2$$

where $O(w^{-i})$ with $i > 0$ denotes a term such that as $w \to 0$, $|O(w^{-i})| \leq c|w^{-i}|$ for some constant c. The two coefficients of $w^{-(n_0-1)}$ cannot be 0 simultaneously. So letting $w \to 0$, we have either $\infty = e_1$ or $\infty = e_2$. This is absurd.

By the previous two paragraphs, $n_0 = 0$ or 1. Since $\deg(q) \geq 3$, $s \geq 1$. The roots of $(w^2+1)^2 - \tilde{a}_j w^2$ are $\alpha, \alpha^{-1}, -\alpha, -\alpha^{-1}$ for some rational number $\alpha \neq 0, 1$, and these four roots are mutually distinct. Let $\alpha_0, \alpha_0^{-1}, -\alpha_0, -\alpha_0^{-1}$ be the roots of $(w^2+1)^2 - \tilde{a}_1 w^2$. By symmetry, let $0 < \alpha_0 < 1$. If $(w+\alpha_0) \nmid \big(q(w-1, -1)q(1-w, -w)\big)$, then letting $w \to -\alpha_0$ in (5.27), we obtain the contradiction $\infty = e_1$. If $(w+\alpha_0) | q(w-1, -1)$, then $(\alpha_0+1) \neq 0$ is a root of $q(w, 1)$. Therefore, $1/(\alpha_0+1)$ is a root of $q(w, 1)$ as well. Similarly, if $(w+\alpha_0) | q(1-w, -w)$, then $-(\alpha_0+1)/\alpha_0$ is a root of $q(w, 1)$; in this case, $\alpha_0/(\alpha_0+1)$ is also a root of $q(w, 1)$. Note that $0 < 1/(\alpha_0+1), \alpha_0/(\alpha_0+1) < 1$. Define two functions

$$\phi_1(x) = 1/(x+1), \quad \phi_2(x) = x/(x+1).$$

So there exists $\psi_1 \in \{\phi_1, \phi_2\}$ such that $\psi_1(\alpha_0)$ is a root of $q(w, 1)$. Putting $\alpha_1 = \psi_1(\alpha_0)$ and repeating the above process, we see that $q(w, 1)$ has a sequence of roots

$$\alpha_k = \psi_k \cdots \psi_1(\alpha_0), \quad k \geq 1$$

where $\psi_1, \ldots, \psi_k \in \{\phi_1, \phi_2\}$. By induction, we get $0 < \alpha_k < 1$ for every $k \geq 0$.

Claim. *$\alpha_i \neq \alpha_k$ whenever $i, k \geq 0$ and $i \neq k$.*

Proof. Assume $\alpha_i = \alpha_k$ with $0 \leq i < k$. Then, $\alpha_k = \psi_k \cdots \psi_{i+1}(\alpha_i)$. So we may assume that $i = 0$, $\alpha_0 = \alpha_k$ and $\alpha_k = \psi_k \cdots \psi_1(\alpha_0)$. Since $\psi_1, \ldots, \psi_k \in \{\phi_1, \phi_2\}$, we see from induction that $\alpha_k = (a\alpha_0+b)/(c\alpha_0+d)$ for some integers $a \geq 0, b \geq 0, c \geq 1, d \geq 1$ satisfying $ad - bc = \pm 1$. So $\alpha_0 = (a\alpha_0+b)/(c\alpha_0+d)$, and we get

$$c\alpha_0^2 + (d-a)\alpha_0 - b = 0.$$

Since α_0 is a rational number, $(d-a)^2+4bc = f^2$ for some integer f. If $ad-bc = -1$, then $f^2 = (d+a)^2+4$, and so $d+a = 0$, which contradicts $a \geq 0$ and $d \geq 1$. If $ad-bc = 1$, then $f^2+4 = (d+a)^2$, and so $f = 0$ and $d+a = 2$. Since $a \geq 0, b \geq 0, c \geq 1, d \geq 1$ are integers satisfying $ad-bc = 1$, we must have $a = d = 1$, $b = 0$ and $\alpha_0 = 0$. This contradicts $\alpha_0 \neq 0$. □

We continue the proof of our lemma. By the above claim, the polynomial $q(w, 1)$ has infinitely many roots $\alpha_k, k \geq 0$ which are mutually distinct. This is absurd. □

In view of Lemma 5.4, we introduce the following notation.

Notation 5.5. We use $M((w+z)^n)$ to denote an expression of the form

$$(w+z)^n \cdot \frac{p(w, z)}{q(w, z)}$$

where $p(w,z)$ and $q(w,z)$ are polynomials in w and z with $(w+z) \nmid q(w,z)$, and all the roots of the polynomial $q(w,1)$ are rational numbers.

By Lemma 5.4, when we evaluate the summation $\sum_{\Gamma\in\mathcal{T}_{d,i}} - \sum_{\Gamma\in\mathcal{S}_{d,i,i}}$ in (5.19), we can ignore those stable graphs Γ in $\mathcal{T}_{d,i}$ and $\mathcal{S}_{d,i,i}$ satisfying

$$\frac{1}{|\mathbf{A}_\Gamma|}\int_{[\overline{M}_\Gamma]}\frac{1}{e(N_\Gamma^{\mathrm{vir}})} = M\big((w_i+z_i)^3\big).$$

5.5. Computation of $\sum_{\Gamma\in\mathcal{S}_{d,i,i}}$

For simplicity, in the rest of the paper, we put

$$w = w_i = c_1(\lambda_i),$$
$$z = z_i = c_1(\mu_i).$$

Also, define $P_1(a,b)=1$. For $n\geq 2$, we define

$$P_n(a,b) = (a+b)\cdots(a+(n-1)b). \tag{5.31}$$

Now let $\Gamma \in \mathcal{S}_{d,i,i}$. Similar to the formulas (4.18) to (4.21) in [8] (see also [30]), we have the decomposition

$$e(N_\Gamma^{\mathrm{vir}}) = e_\Gamma^{\mathrm{E}}\cdot e_\Gamma^{\mathrm{V}}\cdot e_\Gamma^{\mathrm{F}}. \tag{5.32}$$

Here, $e_\Gamma^{\mathrm{E}}, e_\Gamma^{\mathrm{V}}, e_\Gamma^{\mathrm{F}}$ denote the contributions of the edges, vertices and flags with

$$e_\Gamma^{\mathrm{E}} = \prod_{e\in E(\Gamma)}\frac{(-1)^{d_e-1}((d_e-1)!)^2w^2z^2(w-z)^{2d_e}}{(w+z)P_{d_e}(-2d_ew, w-z)P_{d_e}(-d_e(w+z), w-z)} \tag{5.33}$$

$$e_\Gamma^{\mathrm{V}} = \prod_{\substack{v\in V(\Gamma)\\ g(v)=0\\ \mathrm{val}(v)=2}}(\omega_{F_1(v)}+\omega_{F_2(v)})\cdot\prod_{\substack{v\in V(\Gamma)\\ g(v)=0\\ \mathrm{val}(v)=1}}\omega_{F(v)}^{-1}\cdot\prod_{v\in V(\Gamma)}\frac{e(T_{\mathfrak{L}(v)})}{e(\mathcal{H}_{g(v)}^\vee\otimes T_{\mathfrak{L}(v)})} \tag{5.34}$$

$$e_\Gamma^{\mathrm{F}} = \prod_{F\in F(\Gamma)^{\mathrm{sta}}}(\omega_F-\psi_F)\cdot\prod_{F\in F(\Gamma)}e(T_{i(F)})^{-1} \tag{5.35}$$

where (5.33) (which is the product of the equivariant Euler classes of the moving parts $\chi(((f|_{C_e})^*T_{X^{[3]}})^{\mathrm{mov}})$, $e\in E(\Gamma)$) follows from (5.10) by reading its nonconstant terms, $\omega_F = e(T_{i(F)}C_{i,i})/d_e$ for a flag $F=(v,e)$ and ψ_F denotes the first Chern class of the line bundle on $\overline{M}_\Gamma$ whose fiber is the cotangent space of the component associated to v at the point corresponding to F. Note from (5.5) and (5.6) that $T_{R_{i,i}^{(1)}}C_{i,i} = \lambda_i^{-1}\mu_i$ and $T_{R_{i,i}^{(2)}}C_{i,i} = \lambda_i\mu_i^{-1}$. Thus, we obtain

$$\omega_F = \begin{cases}(-w+z)/d_e, & \text{if } i(F) = R_{i,i}^{(1)}\\ (w-z)/d_e, & \text{if } i(F) = R_{i,i}^{(2)}.\end{cases} \tag{5.36}$$

In (5.34), when $g(v)=0$, $e(\mathcal{H}_{g(v)}^\vee\otimes T_{\mathfrak{L}(v)})$ is treated as 1; when $g(v)=1$, $\mathcal{H}_{g(v)}$ is the rank-1 Hodge bundle over $\overline{M}_{g(v),\mathrm{val}(v)}$. Let $\lambda = c_1(\mathcal{H}_1)$. It is known that

$$\lambda^2 = 0, \qquad \int_{\overline{M}_{1,1}}\lambda = \frac{1}{24}. \tag{5.37}$$

https://doi.org/10.1017/fms.2023.17 Published online by Cambridge University Press

For $1 \leq j \leq n$, let ψ_j be the first Chern class of the line bundle on $\overline{M}_{1,n}$ whose fiber at an n-pointed stable curve is the cotangent space of the curve at the j-th marked point. Then it is known (e.g., see [18]) that $\psi_1 = \lambda$ on $\overline{M}_{1,1}$ and

$$\int_{\overline{M}_{1,2}} \psi_1^2 = \int_{\overline{M}_{1,2}} \psi_2^2 = \int_{\overline{M}_{1,2}} \psi_1 \psi_2 = \frac{1}{24}. \tag{5.38}$$

Lemma 5.6. *Let $d \geq 1$, and let $\Gamma \in \mathcal{S}_{d,i,i}$. Then, we have*

$$\frac{1}{e(N_\Gamma^{\rm vir})} = M\big((w+z)^{|E(\Gamma)|}\big). \tag{5.39}$$

Proof. We see from (5.33) that $(w+z)^{|E(\Gamma)|}$ divides the denominator of $e_\Gamma^{\rm E}$. Moreover, $(w+z)$ does not divide the numerators in (5.33). So we have

$$\frac{1}{e_\Gamma^{\rm E}} = (w+z)^{|E(\Gamma)|} \cdot \frac{p_{\Gamma,1}(w,z)}{q_{\Gamma,1}(w,z)} \tag{5.40}$$

where $p_{\Gamma,1}(w,z)$ and $q_{\Gamma,1}(w,z)$ are polynomials in w and z with $(w+z) \nmid q_{\Gamma,1}(w,z)$, and all the roots of $q_{\Gamma,1}(w,1)$ are rational. By (5.34), (5.35), (5.36), (5.5) and (5.6),

$$\frac{1}{e_\Gamma^{\rm V} \cdot e_\Gamma^{\rm F}} = \frac{p_{\Gamma,2}(w,z)}{q_{\Gamma,2}(w,z)}$$

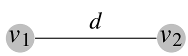
where $p_{\Gamma,2}(w,z)$ and $q_{\Gamma,2}(w,z)$ are polynomials in w and z with $(w+z) \nmid q_{\Gamma,2}(w,z)$, and all the roots of $q_{\Gamma,2}(w,1)$ are rational. By (5.32) and (5.40), we get (5.39). □

Lemma 5.7. *Let $d \geq 1$. Then, the summation $\sum_{\Gamma \in \mathcal{S}_{d,i,i}}$ is equal to*

$$\left(\frac{-d^2+d+16}{96d} + \frac{d}{48}\sum_{d_1=1}^{d-1}\frac{1}{d_1} - \frac{1}{48}\sum_{\delta \vdash d}\frac{d^2-d_1d_2}{d_1d_2 \cdot |{\rm Aut}(\delta)|}\right) \cdot \frac{(w+z)^2}{wz} + M\big((w+z)^3\big)$$

where $\delta = (d_1, d_2) \vdash d$ denotes a length-2 partition of d, $|{\rm Aut}(\delta)| = 1$ if $d_1 \neq d_2$ and $|{\rm Aut}(\delta)| = 2$ if $d_1 = d_2$.

Proof. By Lemma 5.6, we need only to consider those stable graphs $\Gamma \in \mathcal{S}_{d,i,i}$ with $|E(\Gamma)| = 1$ or 2. We begin with the case $|E(\Gamma)| = 1$ (i.e., $\Gamma \in \mathcal{S}_{d,i,i}$ has exactly one edge). There are exactly two such stable graphs:



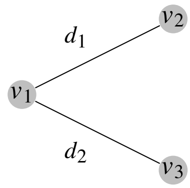
where $V(\Gamma) = \{v_1, v_2\}$, $\mathfrak{L}(v_1) = R_{i,i}^{(1)}$, $\mathfrak{L}(v_2) = R_{i,i}^{(2)}$, $g(v_1) \in \{1, 0\}$ and $g(v_2) \in \{1, 0\} - \{g(v_1)\}$. In both cases, $|\mathbf{A}_\Gamma| = d \cdot |{\rm Aut}(\Gamma)| = d$ by (5.12). Using (5.32)-(5.37) and noticing that (5.33) is unchanged when w and z are switched, we get

$$\sum_{\Gamma \in \mathcal{S}_{d,i,i},\, |E(\Gamma)|=1} = \frac{9-d}{48d} \cdot \frac{(w+z)^2}{wz} + M\big((w+z)^3\big). \tag{5.41}$$

Next, we consider the stable graphs $\Gamma \in \mathcal{S}_{d,i,i}$ with $|E(\Gamma)| = 2$. So $\Gamma \in \mathcal{S}_{d,i,i}$ has exactly two edges. Denoting the distributions of the degree d on the two edges by d_1 and d_2, we see that these stable graphs are of the form:

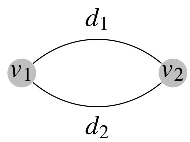
https://doi.org/10.1017/fms.2023.17 Published online by Cambridge University Press

(i)



where $\mathfrak{L}(v_1) \in \{R_{i,i}^{(1)}, R_{i,i}^{(2)}\}$, $\mathfrak{L}(v_2) = \mathfrak{L}(v_3) \in \{R_{i,i}^{(1)}, R_{i,i}^{(2)}\} - \{\mathfrak{L}(v_1)\}$, $g(v_1) \in \{1, 0\}$, $g(v_2) \in \{1, 0\} - \{g(v_1)\}$, $g(v_3) = 0$ and $\delta = (d_1, d_2) \vdash d$ is a partition of d. There are exactly 4 types of such graphs if we ignore the edge weights. By (5.12), $|\mathbf{A}_\Gamma| = d_1 d_2$ if $g(v_1) = 0$, while $|\mathbf{A}_\Gamma| = d_1 d_2 \cdot |\mathrm{Aut}(\delta)|$ if $g(v_1) = 1$.

(ii)



where $\mathfrak{L}(v_1) = R_{i,i}^{(1)}$, $\mathfrak{L}(v_2) = R_{i,i}^{(2)}$, $g(v_1) = g(v_2) = 0$ and $\delta = (d_1, d_2) \vdash d$ is a partition of d. We have $|\mathbf{A}_\Gamma| = d_1 d_2 \cdot |\mathrm{Aut}(\delta)|$.

A lengthy computation via (5.32)-(5.38) shows that $\sum_{\Gamma \in \mathcal{S}_{d,i,i},\, |E(\Gamma)|=2}$ is equal to

$$\left(\frac{-d^2 + 3d - 2}{96d} + \frac{d}{48} \sum_{d_1=1}^{d-1} \frac{1}{d_1} - \frac{1}{48} \sum_{\delta \vdash d} \frac{d^2 - d_1 d_2}{d_1 d_2 \cdot |\mathrm{Aut}(\delta)|} \right) \cdot \frac{(w+z)^2}{wz} + M\big((w+z)^3\big).$$

Summing this with (5.41), we complete the proof of our lemma. □

5.6. Computation of $\sum_{\Gamma \in \mathcal{T}_{d,i}}$

Let $\Gamma \in \mathcal{T}_{d,i}$. For an edge $e \in E(\Gamma)$ and for $0 \le j < k \le 2$, define $e \in E_{j,k}(\Gamma)$ if the component C_e is mapped to $C_{j,k}^{(i)}$. By Lemma 5.2, the curves $C_{0,1}^{(i)}$, $C_{0,2}^{(i)}$ and $C_{1,2}^{(i)}$ are homologous to β_3, β_3 and $3\beta_3$, respectively. Therefore,

$$\sum_{e \in E_{0,1}(\Gamma)} d_e + \sum_{e \in E_{0,2}(\Gamma)} d_e + \sum_{e \in E_{1,2}(\Gamma)} 3d_e = d. \tag{5.42}$$

Now formulas (5.32), (5.34) and (5.35) still hold with the understanding that

$$\omega_F = \begin{cases} (w-2z)/d_e, & \text{if } e \in E_{0,1}(\Gamma) \text{ and } i(F) = Q_{i,0} \\ (-w+2z)/d_e, & \text{if } e \in E_{0,1}(\Gamma) \text{ and } i(F) = Q_{i,1} \\ (-2w+z)/d_e, & \text{if } e \in E_{0,2}(\Gamma) \text{ and } i(F) = Q_{i,0} \\ (2w-z)/d_e, & \text{if } e \in E_{0,2}(\Gamma) \text{ and } i(F) = Q_{i,2} \\ (-w+z)/d_e, & \text{if } e \in E_{1,2}(\Gamma) \text{ and } i(F) = Q_{i,1} \\ (w-z)/d_e, & \text{if } e \in E_{1,2}(\Gamma) \text{ and } i(F) = Q_{i,2} \end{cases} \tag{5.43}$$

since $T_{Q_{i,0}} C_{0,1}^{(i)} = \lambda_i \mu_i^{-2}$, $T_{Q_{i,1}} C_{0,1}^{(i)} = \lambda_i^{-1} \mu_i^2$, $T_{Q_{i,0}} C_{0,2}^{(i)} = \lambda_i^{-2} \mu_i$, $T_{Q_{i,2}} C_{0,2}^{(i)} = \lambda_i^2 \mu_i^{-1}$, $T_{Q_{i,1}} C_{1,2}^{(i)} = \lambda_i^{-1} \mu_i$ and $T_{Q_{i,2}} C_{1,2}^{(i)} = \lambda_i \mu_i^{-1}$ in view of (5.2), (5.3) and (5.4). Moreover, we see from (5.7), (5.8) and (5.9) that

https://doi.org/10.1017/fms.2023.17 Published online by Cambridge University Press

the factor e^{E}_{Γ} in (5.32) is given by

$$\prod_{e\in E_{0,1}(\Gamma)}\Bigg(\frac{(-1)^{d_e-1}((d_e-1)!)^2(w-2z)^{2d_e}(w-z)wz^2}{(w+z)P_{d_e}(-d_e(2w-z),w-2z)}$$
$$\cdot\frac{P_{d_e}(-d_e(w-z),w-2z)}{P_{d_e}(-d_e(w+z),w-2z)P_{d_e}(-d_ew,w-2z)}\Bigg)$$
$$\cdot\prod_{e\in E_{0,2}(\Gamma)}\Bigg(\frac{(-1)^{d_e-1}((d_e-1)!)^2(z-2w)^{2d_e}(z-w)zw^2}{(z+w)P_{d_e}(-d_e(2z-w),z-2w)}$$
$$\cdot\frac{P_{d_e}(-d_e(z-w),z-2w)}{P_{d_e}(-d_e(z+w),z-2w)P_{d_e}(-d_ez,z-2w)}\Bigg)$$
$$\cdot\prod_{e\in E_{1,2}(\Gamma)}\Bigg(\frac{(-1)^{d_e}((d_e-1)!)^2(2w-z)(w-2z)(w-z)^{2d_e}w^2z^2}{(w+z)(2w+z)(w+2z)P_{d_e}(-2d_ew,w-z)P_{d_e}(-d_e(w+z),w-z)}$$
$$\cdot\frac{P_{d_e}(-d_e(w-2z),w-z)P_{d_e}(d_ez,w-z)P_{d_e}(d_ew,w-z)}{P_{d_e}(-3d_ew,w-z)P_{d_e}(-d_e(2w+z),w-z)P_{d_e}(-d_e(w+2z),w-z)}\Bigg). \tag{5.44}$$

Notation 5.8. Let $d\geq 1$, and let $\Gamma\in\mathcal{T}_{d,i}$. We use $V_0(\Gamma)$ to denote the subset of $V(\Gamma)$ consisting of all the vertices v of Γ such that

$$\mathfrak{L}(v)=Q_{i,0},\ \ g(v)=0,\ \ \mathrm{val}(v)=2,\ \ d_{e_1(v)}=d_{e_2(v)}$$

for the two edges $e_1(v)$ and $e_2(v)$ attaching to v, and $e_j(v)\in E_{0,j}(\Gamma)$ for $j=1,2$.

If $v_1,v_2\in V_0(\Gamma)$ are distinct, then $\mathfrak{L}(v_1)=Q_{i,0}=\mathfrak{L}(v_2)$. So none of the two edges attaching to v_1 coincide with any of the two edges attaching to v_2, and

$$2|V_0(\Gamma)|\leq|E(\Gamma)|. \tag{5.45}$$

Lemma 5.9. *Let $d\geq 1$ and $\Gamma\in\mathcal{T}_{d,i}$. Then,*

$$\frac{1}{e(N^{\mathrm{vir}}_{\Gamma})}=M\big((w+z)^3\big)$$

unless one of the following cases happens:

(i) $|V_0(\Gamma)|=2$ *and* $|E(\Gamma)|=4$*;*
(ii) $|V_0(\Gamma)|=1$ *and* $|E(\Gamma)|=2$*;*
(iii) $|V_0(\Gamma)|=1$ *and* $|E(\Gamma)|=3$*;*
(iv) $|V_0(\Gamma)|=0$ *and* $|E(\Gamma)|=1$*;*
(v) $|V_0(\Gamma)|=0$ *and* $|E(\Gamma)|=2$*.*

Proof. First of all, let us examine the factor e^{E}_{Γ}. If $e\in E_{0,1}(\Gamma)$, then we see from (5.31) that $(w+z)|P_{d_e}(-d_e(w-z),w-2z)$ if and only if $3|d_e$; moreover, if $3|d_e$, then $(w+z)^2\nmid P_{d_e}(-d_e(w-z),w-2z)$ and $(w+z)|P_{d_e}(-d_ew,w-2z)$. Applying a similar argument to $e\in E_{0,2}(\Gamma)$ and $e\in E_{1,2}(\Gamma)$, we conclude that

$$\frac{1}{e^{\mathrm{E}}_{\Gamma}}=M\big((w+z)^{|E(\Gamma)|}\big). \tag{5.46}$$

https://doi.org/10.1017/fms.2023.17 Published online by Cambridge University Press

Next, by (5.34), (5.35), (5.2), (5.3) and (5.4), the only possible factors in $1/(e_\Gamma^{\rm V} \cdot e_\Gamma^{\rm F})$ divisible by $(w+z)$ come from $(\omega_{F_1(v)} + \omega_{F_2(v)})$ with $g(v) = 0$ and $\mathrm{val}(v) = 2$. If such a factor $(\omega_{F_1(v)} + \omega_{F_2(v)})$ is divisible by $(w+z)$, then we see from (5.43) that $\mathfrak{L}(v) = Q_{i,0}$, $d_{e_1(v)} = d_{e_2(v)}$ for the two edges $e_1(v)$ and $e_2(v)$ attaching to v,

$$(\omega_{F_1(v)} + \omega_{F_2(v)}) = -\frac{1}{d_{e_j(v)}} \cdot (w+z),$$

and $e_j(v) \in E_{0,j}(\Gamma)$ for $j = 1, 2$. Hence, $v \in V_0(\Gamma)$. It follows that

$$\frac{1}{e_\Gamma^{\rm V} \cdot e_\Gamma^{\rm F}} = M\big((w+z)^{-|V_0(\Gamma)|}\big).$$

Combining this with (5.32) and (5.46), we conclude that

$$\frac{1}{e(N_\Gamma^{\rm vir})} = M\big((w+z)^{|E(\Gamma)|-|V_0(\Gamma)|}\big).$$

By (5.45), we have $|E(\Gamma)| - |V_0(\Gamma)| \ge |V_0(\Gamma)|$. Now our lemma follows. □

Lemma 5.10. *Let $d \ge 1$. Then, the summation $\sum_{\Gamma \in \mathcal{T}_{d,i}}$ is equal to*

$$f_d \cdot \frac{(w+z)^2}{wz} + M\big((w+z)^3\big) \tag{5.47}$$

where f_d is a universal constant depending only on d and is given by

$$\begin{aligned}
&\frac{4}{9d^2}\sum_{\delta\vdash d/2}\frac{d_1d_2}{|\mathrm{Aut}(\delta)|}\cdot\gamma_{d_1}^2\gamma_{d_2}^2 + \frac{1}{54d}\sum_{\delta\vdash d/2}\frac{d_1^2}{d_2}\cdot\gamma_{d_1}^2\gamma_{d_2}^2 - \frac{1}{108}\sum_{\delta\vdash d/2}\frac{d_1^2+d_1d_2+d_2^2}{d_1d_2\cdot|\mathrm{Aut}(\delta)|}\cdot\gamma_{d_1}^2\gamma_{d_2}^2\\
&+\frac{4d-49}{216d}\cdot\gamma_{d/2}^2 + \sum_{2d_1+3d_2=d}\left(\frac{d_1d_2}{d^2} - \frac{d^2}{432d_1d_2} + \frac{1}{72} + \frac{d}{16d_1^3} + \frac{d_1^2}{54dd_2}\right)\cdot\gamma_{d_1}^2\widetilde{\gamma}_{d_2}\\
&+\sum_{2d_1+d_2=d}\frac{(-1)^d(2d^2+d_2^2)}{216d_1(d_1+d_2)}\cdot\gamma_{d_1}^2\gamma_{d_2} + \frac{(-1)^d}{24}\left(\frac{5}{9} - \sum_{1\le m\le d-1, m\ne d/3}\frac{d}{d-3m}\right)\cdot\gamma_d\\
&+\frac{17(-1)^{d-1}}{54d}\cdot\gamma_d + \frac{49-7d}{144d}\cdot\widetilde{\gamma}_{d/3}\\
&+\sum_{\delta\vdash d}\frac{(-1)^d(-69d^4+77d^3d_1+307d^2d_1^2-704dd_1^3+384d_1^4)}{1728d^2d_1d_2\cdot|\mathrm{Aut}(\delta)|}\cdot\gamma_{d_1}\gamma_{d_2}\\
&+\frac{(-1)^d}{108}\sum_{d_1+d_2=d, d_1\ne d_2}\frac{d_1^2}{(d_1-d_2)d_2}\cdot\gamma_{d_1}\gamma_{d_2} + \sum_{d_1+3d_2=d}\frac{(-1)^{d_1}(2d^2+d_1^2)}{36(d^2-d_1^2)}\cdot\gamma_{d_1}\widetilde{\gamma}_{d_2}\\
&+\sum_{\delta\vdash d/3}\frac{3(d_1^4-d_1^3d_2+8d_1^2d_2^2-3d_1d_2^3-d_2^4)}{16d^2d_1d_2\cdot|\mathrm{Aut}(\delta)|}\cdot\widetilde{\gamma}_{d_1}\widetilde{\gamma}_{d_2}.
\end{aligned}$$

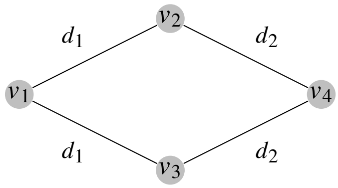
In the above, $d_1 > 0$, $d_2 > 0$, $\delta = (d_1, d_2)$ is a length-2 partition, $\gamma_{d_1} = -2$ if $3|d_1$ and $\gamma_{d_1} = 1$ if $3 \nmid d_1$, $\widetilde{\gamma}_{d_2} = 3$ if $2|d_2$ and $\widetilde{\gamma}_{d_2} = 1$ if $2 \nmid d_2$, and a summand containing $\sum_{\delta\vdash d/2}$ or $\gamma_{d/2}$ (respectively, $\sum_{\delta\vdash d/3}$) does not appear if $2 \nmid d$ (respectively, if $3 \nmid d$).

https://doi.org/10.1017/fms.2023.17 Published online by Cambridge University Press

Proof. By Lemma 5.9, the computation of $\sum_{\Gamma\in\mathcal{T}_{d,i}}$ is reduced to those stable graphs $\Gamma \in \mathcal{T}_{d,i}$ satisfying Lemma 5.9 (i), (ii), (iii), (iv) or (v).

To begin, the stable graphs $\Gamma \in \mathcal{T}_{d,i}$ satisfying Lemma 5.9 (i) are:

(i-1)



where $\{v_1, v_4\} = V_0(\Gamma)$ (so that $\mathfrak{L}(v_1) = Q_{i,0} = \mathfrak{L}(v_4)$), $\mathfrak{L}(v_2) = Q_{i,1}$, $\mathfrak{L}(v_3) = Q_{i,2}$, $g(v_j) = 0$ for every j, $2|d$ and $\delta = (d_1, d_2) \vdash d/2$ denotes a length-2 partition of $d/2$. By (5.12), $|\mathbf{A}_\Gamma| = d_1^2 d_2^2 \cdot |\mathrm{Aut}(\Gamma)| = d_1^2 d_2^2 \cdot |\mathrm{Aut}(\delta)|$. By (5.32), (5.44), (5.34), (5.35) and (5.43), we have

$$\sum_{\Gamma\in\mathcal{T}_{d,i},\,\text{Case (i-1)}} = \frac{4}{9d^2} \sum_{\delta\vdash d/2} \frac{d_1 d_2}{|\mathrm{Aut}(\delta)|} \cdot \gamma_{d_1}^2 \gamma_{d_2}^2 \cdot \frac{(w+z)^2}{wz} + M\big((w+z)^3\big)$$

where $\gamma_{d_1} = -2$ if $3|d_1$ and $\gamma_{d_1} = 1$ if $3 \nmid d_1$.

(i-2)

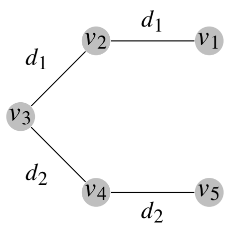


Figure (i-2)

where $\{v_2, v_4\} = V_0(\Gamma)$, $\mathfrak{L}(v_3) \in \{Q_{i,1}, Q_{i,2}\}$, $\mathfrak{L}(v_1) = \mathfrak{L}(v_5) \in \{Q_{i,1}, Q_{i,2}\} - \{\mathfrak{L}(v_3)\}$, $g(v_1) = 1$, $g(v_j) = 0$ for every $j \neq 1$, $2|d$ and $\delta = (d_1, d_2) \vdash d/2$ denotes a length-2 partition of $d/2$. There are exactly 2 types of such graphs if we ignore the edge weights. By (5.12), $|\mathbf{A}_\Gamma| = d_1^2 d_2^2$. By (5.32), (5.44), (5.34), (5.35) and (5.43) together with (5.37), we get

$$\sum_{\Gamma\in\mathcal{T}_{d,i},\,\text{Case (i-2)}} = \frac{1}{54d} \sum_{\delta\vdash d/2} \frac{d_1^2}{d_2} \cdot \gamma_{d_1}^2 \gamma_{d_2}^2 \cdot \frac{(w+z)^2}{wz} + M\big((w+z)^3\big).$$

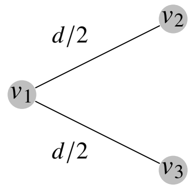
(i-3) Γ has the same shape as Figure (i-2) with $\{v_2, v_4\} = V_0(\Gamma)$, $\mathfrak{L}(v_3) \in \{Q_{i,1}, Q_{i,2}\}$, $\mathfrak{L}(v_1) = \mathfrak{L}(v_5) \in \{Q_{i,1}, Q_{i,2}\} - \{\mathfrak{L}(v_3)\}$, $g(v_3) = 1$, $g(v_j) = 0$ for every $j \neq 3$, and $2|d$. There are exactly 2 types of such graphs if we ignore the edge weights. By (5.12), $|\mathbf{A}_\Gamma| = d_1^2 d_2^2 \cdot |\mathrm{Aut}(\delta)|$. By (5.32), (5.44), (5.34), (5.35) and (5.43), together with (5.37) and (5.38), we obtain

$$\sum_{\Gamma\in\mathcal{T}_{d,i},\,\text{Case (i-3)}} = -\frac{1}{108} \sum_{\delta\vdash d/2} \frac{d_1^2 + d_1 d_2 + d_2^2}{d_1 d_2 \cdot |\mathrm{Aut}(\delta)|} \cdot \gamma_{d_1}^2 \gamma_{d_2}^2 \cdot \frac{(w+z)^2}{wz} + M\big((w+z)^3\big).$$

Next, the stable graphs $\Gamma \in \mathcal{T}_{d,i}$ satisfying Lemma 5.9 (ii) are:

https://doi.org/10.1017/fms.2023.17 Published online by Cambridge University Press

(ii)

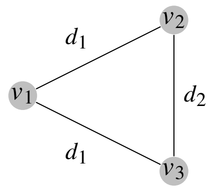


where $2|d$, $\{v_1\} = V_0(\Gamma)$, $\mathfrak{L}(v_2) = Q_{i,1}$, $\mathfrak{L}(v_3) = Q_{i,2}$, $g(v_j) = 1$ for some $j \in \{2,3\}$ and $g(v_k) = 0$ if $k \neq j$. There are 2 types of such graphs, and

$$\sum_{\Gamma \in \mathcal{T}_{d,i},\, \text{Case (ii)}} = \frac{4d-49}{216d} \cdot \gamma_{d/2}^2 \cdot \frac{(w+z)^2}{wz} + M\big((w+z)^3\big).$$

The stable graphs $\Gamma \in \mathcal{T}_{d,i}$ satisfying Lemma 5.9 (iii) are:

(iii-1)



where $\{v_1\} = V_0(\Gamma)$ (so that $\mathfrak{L}(v_1) = Q_{i,0}$), $\mathfrak{L}(v_2) = Q_{i,1}$, $\mathfrak{L}(v_3) = Q_{i,2}$, $g(v_j) = 0$ for every j and $2d_1 + 3d_2 = d$. We have

$$\sum_{\Gamma \in \mathcal{T}_{d,i},\, \text{Case (iii-1)}} = \sum_{2d_1+3d_2=d} \frac{d_1 d_2}{d^2} \cdot \gamma_{d_1}^2 \widetilde{\gamma}_{d_2} \cdot \frac{(w+z)^2}{wz} + M\big((w+z)^3\big)$$

where $\widetilde{\gamma}_{d_2} = 3$ if $2|d_2$ and $\widetilde{\gamma}_{d_2} = 1$ if $2 \nmid d_2$.

(iii-2)

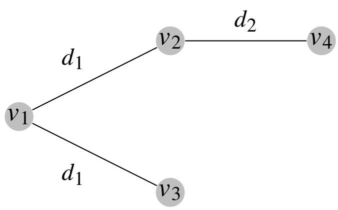


Figure (iii-2)

where $\{v_1\} = V_0(\Gamma)$, $\mathfrak{L}(v_2) = Q_{i,1}$, $\mathfrak{L}(v_3) = Q_{i,2}$, $\mathfrak{L}(v_4) = Q_{i,0}$ (respectively, $Q_{i,2}$), $2d_1 + d_2 = d$ (respectively, $2d_1 + 3d_2 = d$), $g(v_j) = 1$ for some $j \in \{2,3,4\}$ and $g(v_k) = 0$ if $k \neq j$. There are exactly 6 types of such graphs if we ignore the edge weights.

(iii-3) Γ has the same shape as Figure (iii-2) with $\{v_1\} = V_0(\Gamma)$, $\mathfrak{L}(v_2) = Q_{i,2}$, $\mathfrak{L}(v_3) = Q_{i,1}$, $\mathfrak{L}(v_4) = Q_{i,0}$ (respectively, $Q_{i,1}$), $2d_1 + d_2 = d$ (respectively, $2d_1 + 3d_2 = d$), $g(v_j) = 1$ for some

https://doi.org/10.1017/fms.2023.17 Published online by Cambridge University Press

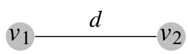
$j \in \{2,3,4\}$ and $g(v_k) = 0$ if $k \neq j$. There are exactly 6 types of such graphs if we ignore the edge weights. We see that $\sum_{\Gamma\in\mathcal{T}_{d,i},\,\text{Case (iii-2)}} + \sum_{\Gamma\in\mathcal{T}_{d,i},\,\text{Case (iii-3)}}$ is equal to

$$\left(\sum_{2d_1+3d_2=d}\left(-\frac{d^2}{432d_1d_2}+\frac{1}{72}+\frac{d}{16d_1^3}+\frac{d_1^2}{54dd_2}\right)\cdot\gamma_{d_1}^2\widetilde{\gamma}_{d_2}\right.$$

$$\left.+\sum_{2d_1+d_2=d}\frac{(-1)^d(2d^2+d_2^2)}{216d_1(d_1+d_2)}\cdot\gamma_{d_1}^2\gamma_{d_2}\right)\cdot\frac{(w+z)^2}{wz}+M\big((w+z)^3\big).$$

The stable graphs $\Gamma \in \mathcal{T}_{d,i}$ satisfying Lemma 5.9 (iv) are:

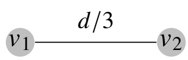
(iv-1)



where $\mathfrak{L}(v_1) = Q_{i,0}$, $\mathfrak{L}(v_2) \in \{Q_{i,1}, Q_{i,2}\}$, $g(v_1) \in \{1,0\}$ and $g(v_2) \in \{1,0\} - \{g(v_1)\}$. There are exactly 4 types of such graphs. We see that the summation $\sum_{\Gamma\in\mathcal{T}_{d,i},\,\text{Case (iv-1)}}$ is equal to

$$\left(\frac{(-1)^d}{24}\left(\frac{5}{9}-\sum_{1\le m\le d-1,\, m\neq d/3}\frac{d}{d-3m}\right)-\frac{17(-1)^d}{54d}\right)\gamma_d\cdot\frac{(w+z)^2}{wz}+M\big((w+z)^3\big).$$

(iv-2)

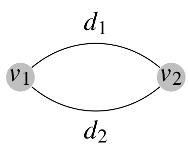


where $3|d$, $\mathfrak{L}(v_1) = Q_{i,1}$, $\mathfrak{L}(v_2) = Q_{i,2}$, $g(v_1) \in \{1,0\}$ and $g(v_2) \in \{1,0\} - \{g(v_1)\}$. There are exactly 2 types of such graphs. We obtain

$$\sum_{\Gamma\in\mathcal{T}_{d,i},\,\text{Case (iv-2)}} = \frac{49-7d}{144d}\cdot\widetilde{\gamma}_{d/3}\cdot\frac{(w+z)^2}{wz}+M\big((w+z)^3\big).$$

Finally, the stable graphs $\Gamma \in \mathcal{T}_{d,i}$ satisfying Lemma 5.9 (v) are:

(v-1)



where $\mathfrak{L}(v_1) = Q_{i,j_1}$ and $\mathfrak{L}(v_2) = Q_{i,j_2}$ for some $j_1, j_2 \in \{0,1,2\}$ with $j_1 < j_2$ and $g(v_1) = g(v_2) = 0$. There are exactly 3 types of such graphs if we ignore the edge weights.

https://doi.org/10.1017/fms.2023.17 Published online by Cambridge University Press

(v-2)

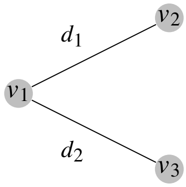


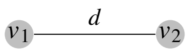
Figure (v-2)

where $\mathfrak{L}(v_1) = Q_{i,j_1}$ for some $j_1 \in \{0,1,2\}$, $\mathfrak{L}(v_2) = Q_{i,j_2}$ and $\mathfrak{L}(v_3) = Q_{i,j_3}$ for some $j_2, j_3 \in \{0,1,2\} - \{j_1\}$ with $j_2 \le j_3$, $g(v_1) = 1$ and $g(v_2) = g(v_3) = 0$. There are 9 types of such graphs if we ignore the edge weights.

(v-3) Γ has the same shape as Figure (v-2) with $\mathfrak{L}(v_1) = Q_{i,j_1}$ for some $j_1 \in \{0,1,2\}$, $\mathfrak{L}(v_2) = Q_{i,j_2}$ and $\mathfrak{L}(v_3) = Q_{i,j_3}$ for some $j_2, j_3 \in \{0,1,2\} - \{j_1\}$, $g(v_2) = 1$, $g(v_1) = g(v_3) = 0$ and $d_1 \neq d_2$ if $j_1 = 0$ and $j_2 \neq j_3$. There are exactly 12 types of such graphs if we ignore the edge weights.

We see that the summation $\sum_{\Gamma \in \mathcal{T}_{d,i}, \text{Case (v-1)}} + \sum_{\Gamma \in \mathcal{T}_{d,i}, \text{Case (v-2)}} + \sum_{\Gamma \in \mathcal{T}_{d,i}, \text{Case (v-3)}}$ is equal to the last three lines in the formula of f_d in our lemma. □

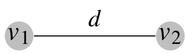
Example 5.11. Let $d = 1$. Then, we have $|E(\Gamma)| = 1$ and $|V(\Gamma)| = 2$ for every stable graph $\Gamma \in \mathcal{S}_{d,i,i} \cup \mathcal{T}_{d,i}$. If $\Gamma \in \mathcal{S}_{d,i,i}$, then Γ is one of the two stable graphs stated in the first paragraph in the proof of Lemma 5.7:



where $\mathfrak{L}(v_1) = R_{i,i}^{(1)}$, $\mathfrak{L}(v_2) = R_{i,i}^{(2)}$, $g(v_1) \in \{1,0\}$ and $g(v_2) \in \{1,0\} - \{g(v_1)\}$. An easy computation shows that $\sum_{\Gamma \in \mathcal{S}_{d,i,i}}$ is equal to

$$4 \int_{\overline{M}_{1,1}} \lambda \cdot \frac{(w+z)^2}{wz} = \frac{1}{6} \cdot \frac{(w+z)^2}{wz} \tag{5.48}$$

where w and z denote w_i and z_i, respectively. Similarly, if $\Gamma \in \mathcal{T}_{d,i}$, then Γ is one of the four stable graphs stated in Case (iv-1) in the proof of Lemma 5.10:



where $\mathfrak{L}(v_1) = Q_{i,0}$, $\mathfrak{L}(v_2) \in \{Q_{i,1}, Q_{i,2}\}$, $g(v_1) \in \{1,0\}$ and $g(v_2) \in \{1,0\} - \{g(v_1)\}$. A straightforward but lengthy computation shows that

$$\sum_{\Gamma \in \mathcal{T}_{d,i},\, g(v_1)=0} \frac{1}{|\mathbf{A}_\Gamma|} \int_{[\overline{M}_\Gamma]} \frac{1}{e(N_\Gamma^{\text{vir}})} = \frac{1}{24} \cdot \frac{(w+z)^2}{wz} \cdot \frac{8w^2 + 8z^2}{(w-2z)(2w-z)},$$

$$\sum_{\Gamma \in \mathcal{T}_{d,i},\, g(v_1)=1} \frac{1}{|\mathbf{A}_\Gamma|} \int_{[\overline{M}_\Gamma]} \frac{1}{e(N_\Gamma^{\text{vir}})} = \frac{1}{24} \cdot \frac{(w+z)^2}{wz} \cdot \frac{6w^2 - 35wz + 6z^2}{(w-2z)(2w-z)}.$$

It follows that $\sum_{\Gamma \in \mathcal{T}_{d,i}} = \sum_{\Gamma \in \mathcal{T}_{d,i},\, g(v_1)=0} + \sum_{\Gamma \in \mathcal{T}_{d,i},\, g(v_1)=1}$ is equal to

$$\frac{7}{24} \cdot \frac{(w+z)^2}{wz}.$$

https://doi.org/10.1017/fms.2023.17 Published online by Cambridge University Press

In particular, the constant f_1 in (5.47) is equal to $7/24$, as asserted by Lemma 5.10. Combining with (5.48), we conclude that $\sum_{\Gamma\in\mathcal{T}_{d,i}} - \sum_{\Gamma\in\mathcal{S}_{d,i,i}}$ is equal to

$$\frac{1}{8}\cdot\frac{(w+z)^2}{wz}.$$

Hence, we see from (5.19) that for a smooth projective toric surface X,

$$\langle\rangle_{1,\beta_3} = \frac{1}{12}\cdot\chi(X)\cdot K_X^2 + \frac{1}{8}\cdot K_X^2 = \left(\frac{1}{8}+\frac{1}{12}\cdot\chi(X)\right)\cdot K_X^2. \tag{5.49}$$

By Lemma 5.1, formula (5.49) holds for every smooth projective surface X.

It is unclear how to simplify the constant f_d in Lemma 5.10 for a general $d \geq 1$. Finally, we are able to determine the genus-1 extremal invariant $\langle\rangle_{1,d\beta_3}$.

Theorem 5.12. *Let X be a smooth projective surface. Let $d \geq 1$, and let f_d be the constant defined in Lemma 5.10. Then, $\langle\rangle_{1,d\beta_3}$ is equal to*

$$\left(f_d - \left(\frac{-d^2+d+16}{96d} + \frac{d}{48}\sum_{d_1=1}^{d-1}\frac{1}{d_1} - \frac{1}{48}\sum_{\delta\vdash d}\frac{d^2-d_1d_2}{d_1d_2\cdot|\mathrm{Aut}(\delta)|}\right) + \frac{1}{12d}\cdot\chi(X)\right)\cdot K_X^2$$

where $\delta = (d_1, d_2) \vdash d$ denotes a length-2 partition of d.

Proof. By Lemma 5.1, $\langle\rangle_{1,d\beta_3} = (a_d + b_d\cdot\chi(X))\cdot K_X^2$ where a_d and b_d are universal constants depending only on d. By (5.19), Lemma 5.4, Lemma 5.7 and Lemma 5.10, our theorem holds when X is a smooth projective toric surface. Therefore, the theorem holds for every smooth projective surface X. □

Acknowledgements. The authors would like to thank Professors Hua-Zhong Ke and Wei-Ping Li for valuable discussions and assistance. The authors also would like to thank the referees for carefully reading the paper and providing valuable comments which have greatly improved the exposition of the paper.

Conflict of Interest. The authors have no conflict of interest to declare.

Funding statement. The research of the first author was partially supported by NSFC Grants [11831017, 11890662].

References

[1] K. Behrend, ‘Gromov-Witten invariants in algebraic geometry’, *Invent. Math.* **127** (1997), 601–617.
[2] K. Behrend and B. Fantechi, ‘The intrinsic normal cone’, *Invent. Math.* **128** (1997), 45–88.
[3] J. Briancon, ‘Description de $\mathsf{Hilb}^n\mathbb{C}\{x, y\}$’, *Invent. Math.* **41** (1977), 45–89.
[4] J. Bryan and T. Graber, ‘The crepant resolution conjecture’ in *Proc. Sympos. Pure Math* vol. 80 (American Mathematical Society, Providence, RI, 2009), 23–42.
[5] T. Coates, A. Corti, H. Iritani and H.-H. Tseng, ‘Wall-crossings in toric Gromov-Witten theory I: crepant examples’, *Geom. Topol.* **13** (2009), 2675–2744.
[6] T. Coates and C. Manolache, ‘A splitting of the virtual class for genus one stable maps’, Preprint, arXiv:1809.04162.
[7] T. Coates and Y. Ruan, ‘Quantum cohomology and crepant resolutions: a conjecture’, *Ann. Inst. Fourier (Grenoble)* **63** (2013), 431–478.
[8] D. Edidin, W.-P. Li and Z. Qin, ‘Gromov-Witten invariants of the Hilbert scheme of 3-points on $\mathbb{P}^2$’, *Asian J. Math.* **7** (2003), 551–574.
[9] G. Ellingsrud and S.A. Strømme, ‘On the homology of the Hilbert scheme of points in the plane’, *Invent. Math.* **87** (1987), 343–352.
[10] J. Fogarty, ‘Algebraic families on an algebraic surface’, *Amer. J. Math.* **90** (1968), 511–521.
[11] W. Fulton and R. Pandharipande, ‘Notes on stable maps and quantum cohomology’, in *Proc. Sympos. Pure Math.* vol. 62 (American Mathematical Society, Providence, RI, 1997), 45–96.
[12] T. Graber and R. Pandharipande, ‘Localization of virtual classes’, *Invent. Math.* **135** (1999), 487–518.
[13] I. Grojnowski, ‘Instantons and affine algebras I: the Hilbert scheme and vertex operators’, *Math. Res. Lett.* **3** (1996), 275–291.

[14] A. Grothendieck, 'Techniques de construction et théorèmes d'existence en géométrie algébrique IV: Les schémas de Hilbert', *Sem. Bourbaki* **221** (1960–1961), 13.
[15] J. Hu, W.-P. Li and Z. Qin, 'The Gromov-Witten theory of the Hilbert schemes of points on surfaces with $p_g > 0$', *Internat. J. Math.* **26** (2015).
[16] A. Iarrobino, 'Punctual Hilbert schemes', *Bull. Amer. Math. Soc.* **78** (1972), 819–823.
[17] Y. Kiem and J. Li, 'Localizing virtual cycles by cosections', *J. Amer. Math. Soc.* **26** (2013), 1025–1050.
[18] J. Kock, 'Notes on psi classes', *Preprint*, 2001.
[19] M. Kontsevich, 'Enumeration of rational curves via torus actions', *in The Moduli Space of Curves*, (Birkhäuser, Boston, MA, 1995), 335–368.
[20] M. Kontsevich and Y. Manin, 'Gromov-Witten classes, quantum cohomology, and enumerative geometry', *Commun. Math. Phys.* **164** (1994), 525–562.
[21] M. Lehn, 'Chern classes of tautological sheaves on Hilbert schemes of points on surfaces', *Invent. Math.* **136** (1999), 157–207.
[22] J. Li, 'Dimension zero Donaldson-Thomas invariants of threefolds', *Geom. Topology* **10** (2006), 2117–2171.
[23] J. Li and W.-P. Li, 'Two point extremal Gromov-Witten invariants of Hilbert schemes of points on surfaces', *Math. Ann.* **349** (2011), 839–869.
[24] J. Li and G. Tian, 'Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties', *J. Amer. Math. Soc.* **11** (1998), 119–174.
[25] J. Li and G. Tian, 'Virtual moduli cycles and Gromov-Witten invariants of general symplectic manifolds', in *Topics in Symplectic* 4-Manifolds *(Irvine, CA, 1996)*, First International Press Lecture Series I, (International Press, Cambridge, MA, 1998), 47–83.
[26] J. Li and A. Zinger, 'On the genus-one Gromov-Witten invariants of complete intersections', *J. Differential Geom.* **82** (2009), 641–690.
[27] W.-P. Li and Z. Qin, 'On 1-point Gromov-Witten invariants of the Hilbert schemes of points on surfaces', *Turkish J. Math.* **26** (2002), 53–68.
[28] W.-P. Li and Z. Qin, 'The Cohomological Crepant Resolution Conjecture for the Hilbert-Chow morphisms', *J. Differ. Geom.* **104** (2016), 499–557.
[29] W.-P. Li, Z. Qin and Q. Zhang, 'Curves in the Hilbert schemes of points on surface', *Contemp. Math.* **322** (2003), 89–96.
[30] C.-C. M. Liu and A. Sheshmani, 'Equivariant Gromov-Witten invariants of algebraic GKM manifolds', *Symmetry, Integrability and Geometry: Methods and Applications (SIGMA)* **13** (2017), 1–21.
[31] D. Maulik, N. Nekrasov, A. Okounkov and R. Pandharipande, 'Gromov-Witten theory and Donaldson-Thomas theory, I', *Compos. Math.* **142** (2006), 1263–1285.
[32] D. Maulik, N. Nekrasov, A. Okounkov and R. Pandharipande, 'Gromov-Witten theory and Donaldson-Thomas theory, II', *Compos. Math.* **142** (2006), 1286–1304.
[33] D. Maulik and A. Oblomkov, 'Quantum cohomology of the Hilbert scheme of points on A_n-resolutions', *J. Amer. Math. Soc*. **22** (2009), 1055–1091.
[34] H. Nakajima, 'Heisenberg algebra and Hilbert schemes of points on projective surfaces', *Ann. Math.* **145** (1997), 379–388.
[35] G. Oberdieck, 'Gromov-Witten invariants of the Hilbert scheme of points of a K3 surface', *Geom. Topol.* **22** (2018), 323–437.
[36] A. Okounkov and R. Pandharipande, 'Quantum cohomology of the Hilbert schemes of points in the plane', *Invent. Math.* **179** (2010), 523–557.
[37] R. Pandharipande and H.-H. Tseng, 'Higher genus Gromov-Witten theory of $\mathsf{Hilb}^n(\mathbb{C}^2)$and CohFTs associated to local curves', *Forum of Math, Pi* **7** (2019), e4.
[38] Z. Qin, 'Hilbert schemes of points and infinite dimensional Lie algebras', in *Mathematical Surveys and Monographs* vol. 228 (American Mathematical Society, Providence, RI, 2018).
[39] Y. Ruan, 'The cohomology ring of crepant resolutions of orbifolds', *Contemp. Math.* **403** (2006), 117–126.
[40] R. Vakil and A. Zinger, 'A desingularization of the main component of the moduli space of genus-one stable maps into $\mathbb{P}^n$', *Geom. Topol.* **12** (2008), 1–95.
[41] A. Zinger, 'Standard versus reduced genus-one Gromov-Witten invariants', *Geom. Topol.* **12** (2008), 1203–1241.
[42] A. Zinger, 'Reduced genus-one Gromov-Witten invariants', *J. Differential Geom.* **83** (2009), 407–460.

https://doi.org/10.1017/fms.2023.17 Published online by Cambridge University Press