

NOTE ON A RADICAL OF DIVINSKY

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In his book [1] Divinsky refers to eight radicals as classical. In [6] radicals were considered such that the radical of each one-sided ideal of a ring may be expressed as the intersection of a left ideal and a right ideal of the ring. From results obtained there it was deduced that seven of these eight radicals have this property. The purpose of this note is to give a proof that this property also holds for the remaining one of these classical radicals.

Divinsky [1] defines this radical to be the lower radical generated by the nil radicals of rings with descending chain condition. Thus it is the smallest radical to coincide with the classical radical on rings with d.c.c. By using results of Fuchs [2], Gardner [3] has shown that a ring R belongs to this radical \mathcal{D} of Divinsky if and only if R belongs to the lower Baer radical β and also R/T is divisible, as an additive abelian group, where T is the maximal torsion subgroup of R . Thus if δ denotes the divisible radical, τ the torsion radical and α is the lower radical generated by $\delta \cup \tau$ then $\mathcal{D} = \beta \cap \alpha$. Since δ and τ are A -radicals it follows that α is an A -radical [4]. Hence \mathcal{D} is a normal radical and so is left and right strong and principally left and right hereditary [5].

It is shown in [6, Theorem 8] that, for any A -radical α , the radical $\beta \cap \alpha$ has the desired property if and only if $R \in \alpha$ implies $\beta(R) \in \alpha$. We shall prove that this property holds for the above radical α .

Let $R \in \alpha$ and let $S = R/\alpha(\beta(R))$. Then $S \in \alpha$ and $\beta(S) = \beta(R)/\alpha(\beta(R))$ belongs to \mathcal{S}_α , where \mathcal{S}_α denotes the semisimple class of the radical α . Thus to show that $R \in \alpha$ implies $\beta(R) \in \alpha$ it suffices to show that $R \in \alpha, \beta(R) \in \mathcal{S}_\alpha$ implies $\beta(R) = 0$.

So now let R be a ring with $R \in \alpha$ and $\beta(R) \in \mathcal{S}_\alpha$. Let $\tau(R) = T$ be the maximal torsion subgroup of R . Then $\beta(T) \in \alpha$ and $\beta(R) \in \mathcal{S}_\alpha$ implies $\beta(T) = 0$. For each prime p let T_p denote the p -component of T and $T[p] = \{x \in T \mid px = 0\}$. Then, as $T[p]pR = 0$, $T[p] \cap pR$ is a nilpotent ideal of T . $\beta(T) = 0$ then implies $T[p] \cap pR = 0$. So $p^2x = 0$ implies $px \in T[p] \cap pR = 0$. It follows that $T_p = T[p]$. Since R/T is divisible we have $T_p + pR = R$. Now $T = \sum T_q$, where the summation is taken over all primes q . For $q \neq p$ we have $T_q = pT_q \subseteq pR$. It follows that $T[p] + pR = R$. $T[p] \cap pR = 0$ then implies $\beta(R) = \beta(T[p]) + \beta(pR) = \beta(pR)$. Also $pR = pT[p] + p^2R = p^2R$. Hence, by Lemma 16 of [4], applied to pR we have $\beta(pR) = p\beta(pR)$. This gives $\beta(R) = p\beta(R)$ and so $\beta(R)$ is divisible. Thus $\beta(R) \in \alpha$. However $\beta(R) \in \mathcal{S}_\alpha$ and so $\beta(R) = 0$, as required.

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