The projective invariants of four medials

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Introduction

The theory of four particular linear forms, or matrices of k columns and $2k$ rows, occurred to me many years ago in an attempt to study the invariants of any number of compound linear forms, or subspaces within a space of n dimensions. In what follows, the invariant theory is given, and its significance for a study of the general matrix of k rows and columns is suggested. The collineation used in § 4 was considered by Mr J. H. Grace[1], who emphasized the importance of the k cross ratios upon transversal lines of four $[k-1]$'s in $[2k-1]$. It seemed appropriate to examine these cross ratios which are irrational invariants μ_i of the figure of four such spaces, and to work out their relation to the known rational invariants X_i. The main result is given in § 5 (7). In § 5 (10) it is shewn that the harmonic section of a line transversal of the four spaces exists when a *linear* relation holds between the invariants.

This work is supplementary to a geometrical treatment[2] of matrix pencils by means of the intersection of one $[k-1]$ and a certain locus V_k^k described by all line transversals of the other three $[k-1]$'s.

The projective invariants of four medials

§ 1. By a medial is meant a linear space $[k-1]$ in the space $[2k-1]$; so that, on taking $k=1$, 2, 3 etc. in succession, a point is a medial on a straight line, a straight line is a medial in three dimensions, a plane is a medial in five dimensions, and so on. Within the hierarchy of subordinate linear spaces, ranging from point [0] to prime $[2k-2]$, a medial stands midway, and is self dual. It is characteristic of odd-dimensional space.

Analytically a medial is defined, with reference to a simplex of $2k$ points $\{1, 0, \ldots, 0\}$, $\{0, 1, 0 \ldots, 0\}$, etc. in $2k$ homogeneous coordinates, by a matrix A of $2k$ rows and k columns: let us say

$$A = [a_1 a_2 \ldots a_k],$$

[1] "Extension of a geometrical porism and other theorems," *Proc. Camb. Phil. Soc.*, **25** (1929), 421-432.

[2] *Phil. Trans. Royal Soc.*, **A 239** (1942), 233-267 (No. 805).

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where each column a_j denotes a point of the medial, and these k points are linearly independent. Similarly for further medials B, C, D, each defined by k such points b, c, or d. Let

$$AB = [a_1 a_2 \ldots . a_k b_1 b_2 \ldots . b_k], \qquad (AB) = (a_1 \ldots . a_k b_1 \ldots . b_k),$$

respectively denote the matrix and determinant of $2k$ rows and columns, arranged in the specified order. Then (AB) is the only irreducible projective invariant of two medials. It vanishes if the medials have a point in common, but not otherwise. When $k = 2$ it is the mutual moment of two lines.

To prove this we take any such rational integral invariant as the sum of non-zero terms T, each of which is a product of factors $(e_1 e_2 \ldots . e_{2k})$, where each e is either an a or a b, this being an immediate consequence of the Fundamental Theorem[1]. The same theorem shews that the medial A is given by the ground form $(A\pi) = \Sigma A_{i \ldots} \pi_{\ldots j}$, where π denotes a variable medial which replaces B in (AB): that is, the k points a are *convolved* in the ground form and also in every invariant (or more generally, concomitant) of the medial A. Similarly for B. Such convolution is either explicit[2] or implicit in each term of T. But by a fundamental identity we can convolve A explicitly in the first factor of each term of T, so accounting for k of the $2k$ places e_j. The remaining k places can only belong to B; and since there are in all $2k$ different columns a or b, the factor is non-zero only when it is (AB) or a mere derangement of this. This proves the result.

Three medials A, B, C have no essentially new type of invariant. For if T has one factor it can only be (AB) or (BC) or (CA): if two factors, one of the medials must be repeated and gives, say, terms $(AB'C')(AB''C'')$, where $B'B''$, and $C'C''$ are convolutions of B and C respectively. By the fundamental identity B can be convolved explicitly in the first factor at the expense of symbols of A or C'. Since the removal of any a from the first to the second factor yields

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[1] Turnbull, *Theory of Determinants, Matrices and Invariants* (Blackie, 1928), 203.

[2] The symbol A is convolved *explicitly* when all its k partial symbols a occur in one bracket factor $(e_1 \ldots\ldots e_{2k})$, and *implicitly* when they occur distributed over two or more such factors. In the latter case the expression involves several terms T, the sum of which is necessarily unaltered by determinantal permutation of the partial symbols a (cf. *loc. cit.* p. 46).

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a zero result (with duplicates a in the second factor), only C' can be removed: and the term becomes $(AB)(AC)$, which reduces. For three or more factors in T, the same kind of result follows by convolving an A in each of the first, second, ... factors until all A are exhausted, and then convolving a B in the last, last but one, ... factors.

Four medials A, B, C, D have many new types of invariant, with remarkable properties. Only those which are linear in each medial are discussed here. Such invariants are of degree $4k$ in the symbols a, b, c, d, and by the fundamental theorem must be an aggregate of terms

$$(a_j \dots b_l \dots c_m \dots d_n \dots)\,(a_p \dots b_q \dots c_r \dots d_s \dots)$$

involving exactly k symbols of each kind a, b, c, d within two determinantal factors of $2k$ columns each. As before, each set A, B, C or D is implicitly or explicitly convolved in the aggregate. Hence one set B may be explicitly convolved in the first factor, another D in the second, and the invariant may be taken to be

$$\left.\begin{aligned} X_i &= \Sigma\,(A_iB_kC_l)\,(C_iD_kA_l), \qquad k = i + l, \\ &= (\dot{A}_iB_k\ddot{C}_l)\,(\ddot{C}_iD_k\dot{A}_l) \end{aligned}\right\}. \tag{1}$$

Here $A_i = a_1a_2 \dots a_i$, that is the matrix of the first i symbols a, B_k denotes all the symbols b, $C_l = c_{i+1}c_{i+2}\dots c_k$, the last l symbols c, and so on. We have in fact partitioned both A and C into i and l columns:

$$A = A_k = A_iA_l = A_iA_{k-i}, \qquad C = C_k = C_iC_l = C_iC_{k-i}. \tag{2}$$

The summation Σ extends to $\binom{k}{i}^2$ terms due to separate determinantal permutation of A and C. It is also indicated by the dots beside A_i, A_l (giving $\binom{k}{i}$ terms), and the double dots beside C_l, C_i (giving $\binom{k}{i}$ terms).

For example, if $k = 2$, $i = 1$,

$$X_1 = (a_1Bc_2)\,(c_1Da_2) - (a_2Bc_2)\,(c_1Da_1) + (a_2Bc_1)\,(c_2Da_1) - (a_1Bc_1)\,(c_2Da_2).$$

It will be proved that, of all such X_i, due to permutations of A, B, C, D and to the range of values i from 0 to k inclusive, exactly $k-2$ invariants are irreducible. Within the determinant, by derangement of columns, we have

$$A_iA_l = (-)^{il}A_lA_i, \qquad (A_iBC_l) = (-)^{il+k}\,(C_lBA_i). \tag{3}$$

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Hence on writing $X_i = (ABCD)_i$ and then interchanging the factors and deranging the columns we find that

$$X_i = (ABCD)_i = (ADCB)_l = (CDAB)_i = (CBAD)_l. \qquad (4)$$

In fact a term of $(ADCB)_l$ is $(A_l D_k C_i)\,(C_l B_k A_i)$, which deranges to the original term $(A_i B_k C_l)\,(C_i D_k A_l)$ of (1) with an even number of changes in sign.

It is natural to continue such derangement of the letters. What happens, for example, when A and B are interchanged? We find that

$$X_i = \sum_{j=i}^{k} (-)^j \binom{j}{i} (BACD)_j.$$

Proof. Consider the expression $(\dot{A}_i B_k C_l)\,(C_i D_k \dot{A}_l)$, where at present A alone is permuted. This consists of $\binom{k}{i}$ terms all belonging to X_i. By a fundamental identity [*loc. cit.* p. 45] A can be convolved explicitly in the first factor; this brings all A_l into the factor and displaces l letters b or c in every possible different way. Suppose then that

$$B_k = B_j B_{k-j}, \qquad C_l = C_{j-i} C_{k-j},$$

where $j = i, i+1, \ldots, k$ in turn. Thus A_l displaces $k - j$ of the b and $j - i$ of the c; and the whole operation yields the identity

$$(\dot{A}_i B_k C_l)\,(C_i D_k \dot{A}_l) = \sum_{j=i}^{k} (A_i \dot{B}_j A_l \ddot{C}_{k-j})\,(C_i D_k \dot{B}_{k-j} \ddot{C}_{j-i}).$$

The right hand expression is a triple summation, indicated by the dots, the double dots and the Σ. The C summation (indicated by double dots) evidently has $\binom{l}{j-i}$ terms.

Now permute all k of the letters c determinantally (that is, with change of sign for each interchange of two letters), so forming $k!$ such identities, including this one; and add the results. Briefly, operate on each side of the identity with Young's negative symmetric group operator $\{c_1 c_2 \ldots . c_k\}'$. On the left we have at once

$$l!\,i!\,(\dot{A}_i B_k \ddot{C}_l)\,(\ddot{C}_i D_k \dot{A}_l)$$

which is $l!\,i!\,X_i$. On the right fix j and B provisionally, and consider the effect on the $\binom{l}{j-i}$ terms of the C summation. The operation gives $k!\binom{l}{j-i}$ such terms. But since the operation effects *all* the

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c's, the terms must be identical, apart from sign, with those of

$$\Gamma = (\ldots . \dot{C}_{k-j})\,(\dot{C}_i \ldots \dot{C}_{j-i})$$

which has $\binom{k}{j}$ terms. Since all the operations on the c's are determinantal it follows that, whenever they produce the same term of Γ, they must produce it with the same sign. Again, since all the k letters are permuted determinantally, then if one term of Γ is produced, every term with its correct sign is produced. Thus the right hand member of the identity can only be a multiple of Γ, which is found by counting all the terms on the right and dividing by the number of terms in Γ. This gives

$$k!\binom{l}{j-i} \div \binom{k}{j} = \frac{l!\,j!}{(j-i)!}.$$

On dividing throughout by $l!\,i!$ and restoring the summations for j and B we have

$$X_i = \sum_{j=i}^{k} \binom{j}{i} (A_i \dot{B}_j A_l \ddot{C}_{k-j})\ (\ddot{C}_i D_k \dot{B}_{k-j} \ddot{C}_{j-i})$$

where the double dots now indicate the permutation of C_{k-j} with $C_j\,(=C_i C_{j-i})$. On rearranging the matrices within the two determinants of a term we obtain the arrangement

$$(B_j A_i A_l C_{k-j})\ (C_i C_{j-i} D_k B_{k-j})$$

which involves $N = ij + (2k-j)(j-i)$ changes of sign. Since $N \equiv -j^2 \equiv j \bmod 2$, we now have

$$X_i = \sum_{j=i}^{k} (-)^j \binom{j}{i} (\dot{B}_j A_k \ddot{C}_{k-j})\ (\ddot{C}_j D_k \dot{B}_l) = \sum_{j=i}^{k} (-)^j \binom{j}{i} (BACD)_j, \quad (5)$$

which proves the result.

On writing $(-)^j \binom{j}{i} = \lambda_j$ and combining the results (4) and (5), we have

$$X_i = (ABCD)_i = \Sigma\lambda_j (BACD)_j = \Sigma\lambda_j (CDBA)_j = (DCBA)_i = (BADC)_i.$$

Hence we may interchange any pair of A, B, C, D, together with the complementary pair, without altering the value of X_i. This means

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that the 24 permutations of A, B, C, D fall into three groups of eight, X, Y and Z say, of which the X group is

$$\begin{aligned} X_i &= (ABCD)_i = (BADC)_i = (CDAB)_i = (DCBA)_i \\ &= (ADCB)_l = (BCDA)_l = (CBAD)_l = (DABC)_l \end{aligned} \tag{6}$$

where $i + l = k$. We may characterize these by writing $X_i = \{AC, BD\}_i$, the pair AC always separating the pair BD. Furthermore let $Y_i = (ADBC)_i$, $Z_i = (ACDB)_i$. Then

$$X_i = \{AC, BD\}_i, \qquad Y_i = \{AB, DC\}_i, \qquad Z_i = \{AD, CB\}_i. \tag{7}$$

Relation (5) expresses X_i linearly in terms of the set Z. By (6) we can interchange B with D and i with l, getting a new form of (5),

$$X_i = \sum_{j=l}^{k} (-)^j \binom{j}{l} (DACB)_j \tag{8}$$

which gives X_i linearly in terms of the set Y. Hence (5) and (8) give the result

$$X_i = \sum_{j=i}^{k} (-)^j \binom{j}{i} Z_{k-j}, \qquad\qquad X_i = \sum_{j=l}^{k} (-)^j \binom{j}{l} Y_j. \tag{9}$$

These can conveniently be written in matrix notation as $X = PZ$, $X = QY$, where $X = \{X_0, X_1, \ldots, X_k\}$, Y, Z are three column vectors, and P, Q are $(k+1)\times(k+1)$ matrices. By cyclic interchange, from (7), we have at once

$$X = QY, \quad Y = QZ, \quad Z = QX, \quad X = PZ, \quad Y = PX, \quad Z = PY, \tag{10}$$

so that $P^3 = Q^3 = I$ the unit matrix, and $P = Q^{-1}$.[1]

Another form of the relations between the sets X, Y, Z is obtained by taking an arbitrary variable μ and

$$\mu' = 1/(1-\mu), \quad \mu'' = 1 - 1/\mu.$$

It is readily verified by (9) that

$$\begin{aligned} &X_0 - X_1\mu + X_2\mu^2 - \ldots + (-)^k X_k \mu^k \\ &= (Y_0 - Y_1\mu'' + \ldots + (-)^k Y_k \mu''^k)(-\mu)^k \\ &= (Z_0 - Z_1\mu' + \ldots + (-)^k Z_k \mu'^k)(\mu - 1)^k \end{aligned} \tag{11}$$

for all values of μ. The close analogy with the theory of cross ratios is explained below (p. 67).

[1] This matrix, P of triangular form with binomial coefficients as elements, was studied in the *Journal London Math. Soc.*, 2 (1927), 242-4. *Cf.* Vaidyanathaswamy, "Integer roots of the unit matrix," *Journal London Math. Soc.*, 3 (1928), 121-4.

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Irreducibility of the Invariants

§ 2. THEOREM. *Exactly $k-2$ of the invariants X_i, Y_i, Z_i are irreducible.*

Proof. If $i = 0$, $i = k$ we have the forms

$$X_0 = (BC)\,(DA), \quad X_k = (AB)\,(CD)$$

which are obviously reducible. Hence all six $X_0, \ldots, Z_k$, of suffix 0 or k, reduce at once. Moreover each Y_i and Z_i is reducible to a sum of the X_i; also $X_0 - X_1 + X_2 - \ldots = Y_0$. Hence at most $k-2$ of the whole set are irreducible; $X_1, X_2, \ldots, X_{k-2}$, say.

Nor can any further relation $\sum_{h=0}^{k} \lambda_h X_h = 0$ hold between the X_i, where the λ_h are numerical and not all zero. For if $\lambda_i \neq 0$, $i > 1$, then we consider the special case for which

$$B_k = A_l C_i, \quad D_k = A_i C_l.$$

In this case, whenever $h > i$, every term of X_h, as given by the series § 1 (1), will contain more than k of the a's in the first factor, and therefore at least two equal a's. This causes X_h to vanish. Likewise the second factor causes X_h to vanish whenever $h < i$. Hence the assumed identity $\sum_{h=0}^{k} \lambda_h X_h = 0$ reduces to the single term $\lambda_i X_i$, with $h = i$ only, which is $\lambda_i (\dot{A}_i A_l C_i \dot{C}_l)\,(\dot{C}_i A_i C_l \dot{A}_l)$. But this is equal to $\lambda_i (AC)^2$ which does not vanish identically. This contradicts the assumption that a relation $\Sigma \lambda_i X_i = 0$ exists, and proves the theorem.

Matrix Outer Products

§ 3. Consider the covariant $(xPQR) = (xPQ')\,(Q''R)$ of three spaces P, Q, R and a point x, all in $[n-1]$. Let the four matrices x, P, Q, R have n rows, but 1, p, q, r columns respectively, where $1 + p + q' = q'' + r = n$, $q = q' + q''$.

Equated to zero this covariant, being linear in x, means that the point x lies in a certain prime π. Evidently each point of P belongs to π, since xP vanishes when x is in P. Furthermore by convolving Q in the first factor we have

$$(xPQR) = \Sigma\,(Qx\ldots)\,(\ldots R) + \Sigma\,(Q\ldots)\,(\ldots xR)$$

where the first summation represents a prime containing Q, and the second, one containing R.

Hence $(xPQR)$ represents the prime through P and the inter-

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section of Q and R. So too does (xPS) where S is this intersection, which is a $[q' - 1]$. Hence we can write

$$QR = Q'^{\cdot}(Q''^{\cdot}R) = S. \tag{1}$$

Equally well $S = (-)^{qr}R'^{\cdot}(R''^{\cdot}Q)$, by convolving Q in the last factor. Such products QR, RQ are virtually the *regressive outer products* of Grassmann[1]. They express the intersection $[q' - 1]$ of Q and R as a linear combination either of the $\binom{q}{q'}$ spaces such as Q', or of the $\binom{r}{q'}$ spaces R'.

When $P \ldots . QR$ is a set of several such matrices, of n rows and $p + \ldots . + q + r$ columns, let the columns in this order be partitioned into determinants (each of n rows and columns) from the right, with a residue of σ initial columns $(0 \leqq \sigma < n)$. Also let each matrix P etc., which happens to be broken by this process, undergo separate determinantal permutation.

We shall call the resulting expression a *matrix outer product* (of currency σ and weight w, where w is the number of determinantal factors).

For example

$$B_k C_k D_k A_l = B_k C_l^{\cdot} \, (C_i^{\cdot} D_k A_l) \tag{2}$$

is a matrix outer product of four factors as shewn. Here $\sigma = k + l$, $w = 1$, $n = 2k$. When $\sigma = 0$ the product is an invariant: for example

$$A_i B_k C_k D_k A_l = (A_i B_k C_l^{\cdot}) \, (C_i^{\cdot} D_k A_l) \tag{3}$$

is the outer product (and an invariant) of five spaces. But X_i, which involves derangement of the two spaces A_i, A_l also, is not an outer product.

When $\sigma > 0$ the outer product, if non-zero, is always a space $[\sigma - 1]$, as in the example QR above. For each factor in turn from the right can if necessary be merged into a simpler space by substitutions of type (1). Thus, in (2), $C_k D_k A_l$ denotes the intersection L, an $[l - 1]$, of C with the space DA, while the whole product denotes the $[k + l - 1]$ space BL. The method applies to any such case.

Partitioning the product $P \ldots . QR$ from right to left would equally well lead to a space $[\sigma - 1]$; but in general it would be a different space.

The case when $\sigma = 1$ is of special importance. Then the product

[1] Cf. Grassmann, *Ausdehnungslehre* (1862), § 113 p. 83. Forder, *Calculus of Extension* (Cambridge 1941), 217-249.

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determines a *point* of P. Consider, for example, the product PQR in four dimensions where

$$P = ab,\ Q = cd,\ R = ef$$

are three lines through pairs of points a, b, c, d, e, f. We have by definition

$$PQR = ab.\ cd.\ ef = a\,(bcdef) - b\,(acdef),$$

which is manifestly a point on the line ab, the point $\lambda a + \mu b$ in fact, where the numerical coefficients are

$$\lambda = (bcdef),\quad \mu = (cadef).$$

But the fundamental identity $a^{\cdot}\,(b^{\cdot}c^{\cdot}d^{\cdot}e^{\cdot}f^{\cdot}) = 0$, of six terms, at once gives

$$PQR + QRP + RPQ = 0, \tag{4}$$

where QRP is a point $c^{\cdot}\,(d^{\cdot}abef)$ of the line cd, and RPQ is a point of ef. Hence the identity, which shews that these three points are in line, gives the three collinear points on the transversal line of three lines in [4].

Similarly for any such PQR for which $\sigma = 1$. The corresponding identity of $n + 1$ terms renders the three points PQR, QRP, RPQ collinear. That is, the *combination* PQR represents the line transversal of three such spaces P, Q, R in $[n - 1]$, while the *permutation* denotes the point of intersection of the line and the space first named.

Four factors $PQRS$, with $\sigma = 1$, would give the four points of a transversal plane, by the corresponding identity. And so on.

The Point Collineation of Four Medials

§ 4. Associated with four medials there is a bilinear concomitant

$$\Phi_1 = (ua^{\cdot})\,(A''Bc^{:})\,(C''^{:}D\xi), \qquad aA' = A,\ cC' = C, \tag{1}$$

where ξ denotes a variable point and u a variable prime. This is a matrix outer product $uABCD\xi$, with $\sigma = 0$, $n = 2k$.

Clearly ξ, $CD\xi$, $ABCD\xi$ respectively, represent three points ξ, z, x say, where z is in C and x in A. We have $z = CD\xi$, $x = ABz$, which shew, by § 3 (4), that a unique line through a general point ξ of $[2k - 1]$ meets both C and D, meeting C at z, and again a unique line through z meets both A and B, meeting A at x. We assume that A, B, C, D are in general position. As z ranges over C, x ranges over A.

Thus Φ_1 gives a point to point collineation between points ξ of

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$[2k-1]$ and x of A, where x is given linearly in terms of ξ. Take therefore

$$\Phi_1 = u\Phi\xi = \sum_{p,q=1}^{2k} u_p\phi_{pq}\xi_q, \qquad x_p = \sum_{q=1}^{2k} \phi_{pq}\xi_q, \tag{2}$$

introducing a $2k \times 2k$ matrix $[\phi_{pq}]$, where the element ϕ_{pq} is the coefficient of $u_p\xi_q$ in the expression (1).

Since Φ_1 transforms the whole space $[2k-1]$ to a medial $[k-1]$, this matrix Φ is singular, and of rank k at most.

The absolute bilinear form of contragredience $u\xi = \sum_{p=1}^{2k} u_p\xi_p$ corresponds to the $2k \times 2k$ unit matrix, and to the identical collineation within the space $[2k-1]$.

The sum of the coefficients of $u_p\xi_p$ in (1) is the *trace* T_1 of the matrix Φ. This yields at once

$$T_1 = (A'^{\cdot}BCDa^{\cdot}) = X_{k-1}. \tag{3}$$

When A, B, C, D are in general position the singular matrix Φ has a $k \times k$ matrix M for its non-singular core. To prove this we note that the *latent points* of Φ occur whenever x and ξ coincide. Such points can only lie with x in the medial A. Also when ξ is any point of A it gives a unique x of A as before; but now the process may be reversed, and from a given x a unique ξ of A can be found. Hence a $k \times k$ non-singular matrix M exists, such that $x = M\xi$, $\xi = M^{-1}x$, where each of x and ξ is expressed by k (not $2k$) components, suitable for points within A.

Also when x coincides with ξ, a transversal line through x of A cuts all four medials, as the construction at once shews. There are therefore at most k such transversal lines of four medials in general position of $[2k-1]$; and they will occur when the latent roots of M, k in number, are distinct and non-zero, answering to the k latent points.

Since we are dealing with homogeneous coordinates, the same point x is obtained on multiplying by the invariant $(AB)(CD)$. The collineation is now

$$(AB)(CD)x = ABCD\xi = a_p^{\cdot}(A^{p\cdot}BCD\xi) \tag{4}$$

where $a_pA^p = A = a_1a_2\ldots\ldots a_k$, and p is any suffix 1, 2, ..., k, and A^p denotes the $k-1$ a's distinct from a_p. Let this be called the *normalized* form of the collineation.

But since ξ is assumed to be a point of A it can be given by

$$\xi = a_1\theta_1 + \ldots\ldots + a_k\theta_k \tag{5}$$

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in terms of the k points a_i and scalar components θ_i. Similarly let $x = \Sigma a_p \theta'_p$. From (4) we get $(AB)(CD)\theta'_p = (A^p BCD\xi)$. This gives the collineation $x \rightarrow \xi$, for points within A, in the form

$$\theta' = N\theta, \qquad N = [n_{pq}], \qquad (AB)(CD)n_{pq} = (A^p BCDa_q), \quad (6)$$

where both p and q range over $1, 2, \ldots, k$. Thus we have an explicit form for N the normalized matrix. The corresponding matrix M is then

$$M = [(A^p BCDa_q)]. \quad (7)$$

The latent roots $\mu_1, \ldots, \mu_k$ of N are given by the characteristic equation

$$|M - \mu(AB)(CD)I| = 0. \quad (8)$$

For a latent point we have $\theta' = \mu_i\theta$, that is

$$\sum_q (A^p BCDa_q)\,\theta_q = \mu_i (AB)(CD)\,\theta_p, \quad p = 1, 2, \ldots, k. \quad (9)$$

Compound Collineations

§ 5. From two points ξ, η we can derive image points x, y both in A, such that $x = ABCD\xi$, $y = ABCD\eta$. This at once causes the line xy to be the image of the line $\xi\eta$, The line coordinates xy are then given linearly in terms of the line coordinates $\xi\eta$ by the second compound of the matrix Φ. Similarly from l points $\xi, \eta, \ldots$ we derive an $[l-1]$, say Ξ_l, which has an image $[l-1]$ given by $xy\ldots$, obtained by the l^{th} compound of the matrix Φ.

Now we may denote these compound collineations by the expressions

$$\Phi_l = (U_l A_l^{\cdot})(A_i^{\cdot} BC_l^{\cdot})(C_i^{\cdot} D\Xi_l) = (UABCD\Xi)_l \quad (1)$$

where $i + l = k$, $n = 2k$, $l = 1, 2, 3, \ldots, k-1$. For completeness we also define the case $l = 0$ as

$$\Phi_0 = (AB)(CD). \quad (2)$$

In proof of (1) we consider the three products Ξ_l, $CD\Xi_l$, $ABCD\Xi_l$, each of which denotes an $[l-1]$. By § 4 if ξ is any point of Ξ_l, the second product gives that point z of C where the line transversal of C, D, Ξ_l meets C: and as ξ ranges over Ξ_l, z ranges over Ξ_c $(= CD\Xi_l)$. Similarly the third product gives the image of Ξ_l in A. Thus the expression (1) denotes the l^{th} compound.

The *matrix* $\Phi^{(l)}$ obtained by deleting the variables U and Ξ in (1) will have $\binom{2k}{l}$ rows and columns. To a scalar factor it must there-

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fore be the l^{th} compound of the matrix Φ. The *trace* of this new matrix is given by the sum of its principal diagonal elements; and this sum is obtained at once from (1) by deleting U and Ξ, and then permuting A_l with C_iD, getting (C_iDA_l). This gives, for the trace,

$$T_l = (A_i^{\cdot}BC_l^{\cdot})(C_i^{\cdot}DA_l^{\cdot}) = X_i, \tag{3}$$

which is the invariant already discussed.

The precise value of the l^{th} compound matrix $\Phi^{(l)}$ of (1) is given by the formula

$$M^{(l)} = \Phi_0^{l-1}\,\Phi^{(l)} \tag{4}$$

where $M^{(l)}$ is the l^{th} compound of the matrix M, and Φ_0 is the scalar factor $(AB)(CD)$. For M is the matrix of the bilinear form $\Phi_1(\xi)$ given in § 4 (1), and $M^{(2)}$ is that of $\Phi_1(\xi^{\cdot})\,\Phi_1(\eta^{\cdot}) \equiv \Phi_2(\xi\eta)$, say; and again $M^{(3)}$ is that of $\Phi_2(\xi^{\cdot}\eta^{\cdot})\,\Phi_1(\zeta^{\cdot})$; and so on. Now consider $\Phi_l(\Xi_l)\,\Phi_1(\xi)$ where the $l+1$ points $\Xi\xi$ are to be permuted. We have

$$\begin{aligned}&(U_lA_l)(A_iBC_l)(C_iD\Xi_l^{\cdot})(ua)(A'Bc^{\cdot})(C'^{\cdot}Dx^{\cdot})\\ &= (U_lA_l)(A_iBC_l)(C_{i-1}^{\cdot}xD\Xi)(ua)(A'Bc^{\cdot})(C'^{\cdot}Dc^{\cdot})\end{aligned}$$

where $C = C_iC_l = C_{i-1}cC_l = cC'$. The last factor, having k symbols c, can only be (CD) or else zero. This gives $(A'Bc^{\cdot})(C'Dc)$ for the two last factors. Hence, if C_lC_i are also convolved in the original expression we have

$$(U_lA_l)(A_iBC_l^{\cdot})(C_{i-1}^{\cdot}xD\Xi_l)(ua)(A'Bc^{\cdot})(C'Dc).$$

On convolving C_lc in the second factor we at once get $(A'Ba)(C'Dc)$ for the final factors. And when the a's are permuted in two sets of k this gives

$$\begin{aligned}&(U_lA_l^{\cdot})(A_{i-1}^{\cdot}c^{\cdot}BC_l^{\cdot})(C_{i-1}^{\cdot}xD\Xi)(ua^{\cdot})(AB)(CD)\\ &= (uUa^{\cdot}A_l^{\cdot})(A_{i-1}^{\cdot}c^{\cdot}BC_l^{\cdot})(C_{i-1}^{\cdot}xD\Xi)\,\Phi_0\\ &= (U_{l+1}A_{l+1}^{\cdot})(A_{i-1}^{\cdot}BC_{l+1}^{\cdot})(C_{i-1}^{\cdot}D\Xi_{l+1})\,\Phi_0\\ &= \Phi_{l+1}\,\Phi_0.\end{aligned} \tag{5}$$

Hence the factor Φ_0 enters at each step in passing from one compound to the next, so that (4) follows by induction.

Again the traces of the compounds $M^{(l)}$ are well known to be the coefficients of powers of λ in the characteristic function $|M - \lambda I|$ of M. Hence by (3) and (4) the characteristic equation of M can be written

$$\lambda^k - (A_{k-1}^{\cdot}BCDa^{\cdot})\,\lambda^{k-1} + (AB)(CD)(A_{k-2}^{\cdot}BCDA_2^{\cdot})\,\lambda^{k-2} - \ldots. = 0 \tag{6}$$

and that of $|N - \mu I| = 0$ can be written

$$X_k\mu^k - X_{k-1}\mu^{k-1} + X_{k-2}\mu^{k-2} - \ldots. + (-)^kX_0 = 0. \tag{7}$$

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Hence *the* $k+1$ *quadrilinear invariants* X *are the coefficients of the characteristic equation for the collineation* $x = ABCD\xi/\Phi_0$ *when taken in normalized form.*

The latent roots μ_i can be identified with cross ratios $\{ABCD\}_i$ in which the four medials cut their k common transversal lines, in the following way. If ϕ, ψ, χ denote sets of k coordinates for a point within B, C, D respectively, then the relations

$$A\theta' + B\phi + C\psi = 0,\ A\theta + C\psi + D\chi = 0 \tag{8}$$

hold when, and only when, the points $\theta'\ \phi\ \psi$ are collinear as also the points $\theta\ \psi\ \chi$. But these are precisely the geometrical conditions of the collineation $\theta' = N\theta$ within A. Now consider the latent point (when $A\theta$ coincides with the point $A\theta'$), corresponding to the latent root μ_i. Then $\theta' = \mu_i\theta$ and the lines $\theta'\phi\psi$, $\theta\psi\chi$ coincide and give four collinear points, one on each medial,

$$A\theta, \quad A\theta + C\psi, \quad C\psi, \quad \mu_i A\theta + C\psi,$$

or simply a, $a+c$, c, $\mu a + c$, in terms of the points $a = A\theta$, $c = C\psi$. The cross ratio of these points is μ_i. Hence *the latent root of the collineation is equal to the cross ratio* $\{ABCD\}$ *say, of the four collinear points where a transversal cuts the medials*: and there are generally k such transversals, one for each root and cross ratio.

We must note that the matrix N, whose latent roots give these cross ratios, has been normalized, in the sense that each of its elements is an absolute invariant of the points of the four medials, owing to the introduction of their common denominator $(AB)(CD)$. Geometrically the same collineation Φ_1 is given by any scalar multiple of N, so that only the ratios of the cross ratios are given if the point collineation of A is given.

The existence of the three sets of invariants X, Y, Z of four medials, and the cyclic relations between them, now fall into line with the well-known permutative properties of cross ratios. On writing the cross ratios of four points in the forms

$$\{ABCD\} = \mu, \quad \{ADBC\} = \mu', \quad \{ACDB\} = \mu'',$$

where $\mu' = 1/(1-\mu)$, $\mu'' = 1 - 1/\mu$, then the identities (11) of § 1 at once shew that the characteristic equation (7) for μ in terms of the X_i is capable of two further modes—for μ', μ'' in terms of the Z_i, Y_i respectively.

The six permutations of B, C, D consequently give rise to six collineations within A, changing a point x to ξ or to ξ' or to ξ'', and

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reciprocally ξ or ξ' or ξ'' to x. All six collineations evidently have the same latent points, k in number; and their six sets of latent roots correspond to the six sets of cross ratios μ_i, μ_i', μ_i'', μ_i^{-1}, etc.

Thus there are three characteristic equations of order k, with coefficients X_i, Y_i, Z_i respectively, for a set of four medials: and their three sets of roots correspond to the three pairs μ, μ^{-1} of cross ratios determined by four collinear points. If μ_i is a root of the X equation then those of Y and Z are $(\mu_i - 1)/\mu_i$ and $(1 - \mu_i)^{-1}$ respectively.

Interesting particular results follow. If $\mu_i = -1$ then the transversal is cut harmonically by the pairs AC, BD. Hence

$$X_0 + X_1 + \dots + X_k = 0, \tag{10}$$

is the necessary and sufficient condition for the medials AC to separate harmonically the medials BD upon one of the k transversal lines of all four medials in $[2k - 1]$.

Two of the ranges have equal cross ratios if the discriminant of the characteristic k-ic of X or Y or Z vanishes.

When all the roots are equal the four medials belong to the same system of ∞^1 generators of a certain locus $\mathfrak{R}$ of the type V_k^k, described by the ∞^k straight lines which meet all four medials. The collineation becomes the identical transformation, and the matrix M (or N) is scalar.

The quaternary case

§ 6. Here $k = 2$ and, let us say,

$$X_0 = (BC)(DA) = Y_2,$$
$$Y_0 = (CA)(BD) = Z_2,$$
$$Z_0 = (AB)(CD) = X_2.$$

Also $X_1 = (a^{\cdot}Bc'^{:})(c^{:}Da'^{\cdot})$, etc. The conditions § 1 (9) reduce to

$$X_1 = X_0 - Y_0 + Z_0, \quad Y_1 = X_0 + Y_0 - Z_0, \quad Z_1 = -X_0 + Y_0 + Z_0.$$

The quadratic equation

$$X_0 - X_1\mu + X_2\mu^2 = 0$$

gives the cross ratios of the transversals (now two in number) of the four lines A, B, C, D in [3]. The condition for AC to separate BD harmonically upon one transversal is $X_0 + X_1 + X_2 = 0$, which reduces to

$$2(BC)(DA) - (CA)(BD) + 2(AB)(CD) = 0. \tag{1}$$

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The two cross ratios are equal when $X_1^2 = 4\, X_0 X_2$, which gives

$$\sqrt{X_0} + \sqrt{Y_0} + \sqrt{Z_0} = 0, \qquad (2)$$

a well-known result. This happens when any one line D touches the quadric through the other three lines, so making the two transversals coincide; in particular when D is a generator of either system on this quadric surface.

Canonical Forms

§ 7. On taking A and C to be the frame of reference, with $[A, C]$ as a $2k \times 2k$ unit matrix, we can express any four skew medials in the form

$$A = \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad B = \begin{bmatrix} I \\ I \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad D = \begin{bmatrix} I \\ N \end{bmatrix}, \qquad (1)$$

where I is the $k \times k$ unit matrix, and N is at present a general $k \times k$ matrix. For, with these values of A and C, this is the form which B and D assume when the parameters, in the conditions $A\theta' + B\phi + C\psi = 0$ and $A\theta + C\psi + D\chi = 0$, are taken to satisfy $\theta' = -\phi = \psi = -\chi$, while $\theta' = N\theta$. This is legitimate since the choice of frame of reference *within* each medial is arbitrary.

The three points a_i, b_i, c_i of A, B, C respectively are now collinear, for $i = 1, 2, \ldots, k$: and if a_i is a latent point this line also contains the point d_i of D. When the collineation Φ_1 has k latent points, we may take them to be these a_i, for which $\theta' = \mu_i \theta$, so that

$$N = \text{diag}\,(\mu_1, \mu_2, \ldots, \mu_k). \qquad (2)$$

The invariants $X_i = (A_i^{\cdot} BCDA_l^{\cdot})$ can easily be calculated for these values of A, B, C, D in (1) and (2). The result is

$$X_0 = (-)^k \mu_1 \mu_2 \ldots \mu_k, \quad X_i = (-)^k \Sigma \mu_{p_1} \mu_{p_2} \ldots \mu_{p_l}, \quad X_k = (-1)^k. \qquad (3)$$

These X_i are in fact the elementary symmetric functions of the latent roots of N, and this gives another proof that the characteristic equation $|N - \mu I| = 0$ has the X_i for its coefficients. But the earlier proof is more general, since it would apply to any N, with or without repeated latent roots.

The original collineation $u\Phi\xi = (uABCD\xi)$, which has a singular $2k \times 2k$ matrix, can also be calculated in this case (1) above. It gives

$$\Phi = \begin{bmatrix} N & I \\ 0 & 0 \end{bmatrix}.$$

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For example, when $k = 3$, and N is in canonical form (2),

$$u\Phi\xi = u \begin{bmatrix} \mu_1 & . & . & 1 & . & . \\ . & \mu_2 & . & . & 1 & . \\ . & . & \mu_3 & . & . & 1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{bmatrix} \xi.$$

Manifestly the characteristic equation for Φ is

$$(\mu - \mu_1) \ldots . (\mu - \mu_k)\, \mu^k = 0$$

and the traces of Φ and all its compounds are the same as those of N. The *reduced* characteristic equation of Φ is the $(k+1)$-ic $\mu \Pi (\mu - \mu_i) = 0$.

If $A = a_1 A^1 = a_i A^i$, $i \neq 1$, and a_1 is a latent point then $(A^i BCDa_1) = 0$, $(A^1 BCDa_1) = \mu_1 (AB)(CD)$ as follows from the canonical form of § 6 (4). Similarly $a_1 a_2$ is a latent line if $(A^{ij} BCDa_1 a_2) = 0$ or $\mu_1 \mu_2 (AB)(CD)$, according as $i, j \neq$ or $= 1, 2$, where $A = a_i a_j A^{ij}$. And so on. These are the necessary and sufficient invariant conditions for latency of particular spaces within A. Here B, C, D can be taken in general coordinates.

The vanishing of X_i

§ 8. On writing A_l' for $BCDA_l$, the image of A_l in the collineation, we can express the invariant X_i as a sum of determinants $(A_i A_l')$, obtained as before by permutation of the k points $a_1, \ldots, a_k$. When these base points are so chosen that a_2 is the image of a_1, a_3 of a_2, and so on till a_k is reached, then all the terms of X_i vanish except one, that in fact for which $A_i = a_1 a_2 \ldots a_i$. This is because A_i and A_l' contain at least one a_r in common for every other term.

For example, if $k = 4$, $X_2 = \Sigma (12, 34) = \Sigma (123'4')$ by an obvious notation; and $1' = 2$, $2' = 3$, $3' = 4$ for the image points. The term $(123'4') = (1244') \neq 0$, whereas any one of the other five terms vanishes ($(132'4') = (1334') = 0$).

The last point a_k has for its image a linear combination of the earlier $k - 1$ points, say

$$a_{k+1} = a_k' = \sum_{m=1}^{k-1} \alpha_m a_m,$$

where the coefficient α_m is $(-)^{m+1} X_m / X_k$, since these values would

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give the requisite characteristic equation, on taking the k points a_r to be θ, $N\theta$, ..., $N^{k-1}\theta$ respectively[1].

Accordingly, *if* X_i *vanishes, the space* A_i *meets* A_l', *where* A_l' *is the image of a space* A_l *complementary to* A_i *in the simplex defining the whole space* A. This is a result due to Segre[2].

Alternatively we can state the condition thus: X_i *vanishes when the image of the* k^{th} *and final independent point of the chain* $a_1 a_2 \ldots.$ *lies in a* $[k-2]$ *through all but the point* a_{i+1} *of these* k *points.*

For we have by construction the relation

$$X_0 a_1 - X_1 a_2 + \ldots . + (-)^k X_k a_{k+1} = 0 \qquad (1)$$

which gives $a_{k+1} = a_k'$ as the first point of the chain to depend on previous points, where the coefficients in such a relation are necessarily those of the characteristic equation. Hence a_{k+1} cannot depend on a_{i+1} if $X_i = 0$.

The above conditions are necessary and sufficient for the case when the characteristic equation cannot be reduced to one of lower degree, in fact for the case of one invariant factor. Segre shewed that the above geometrical condition suffices to cause X_i to vanish in all cases, and is necessary also in the four cases when $i = 1, 2, k-2, k-1$. For the case of two or more invariant factors the above set $a_1, \ldots ., a_k$ no longer holds, and (1) is replaced by a relation of lower degree. It is easy to express X_i in terms of a corresponding (and more complicated) basis, but the geometrical statement is more elaborate.

Extensionals to even dimensions

§ 9. Corresponding to X_i in $[2k-1]$ there is a quadric primal of $[2k]$ given by $(xABCDx)_i = 0$, where x has $2k+1$ elements, and each of A, B, C, D denote k columns each of $2k+1$ elements. Such a quadric is the locus of a point x in $[2k]$ through which four $[k]$'s xA, xB, xC, xD are drawn to cut an arbitrary prime $[2k-1]$, not containing x, in four $[k-1]$'s whose invariant X_i vanishes.

More interesting is the *harmonic quadric primal* of four $[k-1]$'s in $[2k]$, namely the locus

$$\sum_{i=1}^{k} (xABCDx)_i = 0 \qquad (1)$$

[1] Cf. Turnbull and Aitken, *Canonical Matrices* (Blackie, 1932), p. 48.

[2] *Mehrdimensionale Räume*, Encyk. Math. Wissenschaften, III. C 7, 841, where an earlier result by Segre (*Math. Annalen*, 24 (1884), 152-156), is utilized, which concerns two quadrics. This is relevant since any collineation can be resolved into successive reciprocation in two quadrics by a theorem of Frobenius.

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which is the extensional of $\Sigma X_i = 0$. This means that A, B, C, D are four arbitrary $[k-1]$'s in $[2k]$. They are met by a plane $abcd$ in a single point of each. This plane contains a unique conic through $abcd$, for which a and c separate b and d harmonically. The locus of points of this conic as the plane takes all possible positions is a quadric primal.

Three such harmonic quadrics exist for the four given $[k-1]$'s, corresponding to the pairings ab, cd; ac, bd; ad, bc.

In proof of this take any point x on the conic $abcd$, and project these points, and the spaces A, B, C, D from x to an arbitrary prime. The result will be four collinear points $a'b'c'd'$, one on each of four $[k-1]$'s in this prime $[2k-1]$. Also $a'b'c'd'$ is a harmonic range, for which $\Sigma X = 0$. Hence (1) is true of $[2k]$.

Through an arbitrary point x of $[2k]$ just k planes can be drawn which meet four skew $[k-1]$'s, corresponding to the k lines traversing four medials in $[2k-1]$. Two of the k planes coincide when x satisfies the extensional of the discriminantal equation: that is, the discriminant of the characteristic equation $\Sigma \pm X_i \mu^i = 0$ being formed, $\Delta(X_i)$, say, then the locus of x is $\Delta((xABCDx)_i) = 0$. Since the discriminant is of degree $2(k-1)$ in the coefficients X_i, each of which corresponds to two of the x, the locus of x is of order $4(k-1)$. Hence the locus of a point, through which two of the k possible transversal planes of four $[k-1]$'s in $[2k]$ coincide, is a primal of order $4(k-1)$.

When $k = 2$ this gives Segre's quartic manifold in four dimensions.

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