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Evaluation of the Dedekind Eta Function

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Abstract. We extend the methods of Van der Poorten and Chapman for explicitly evaluating the Dedekind eta function at quadratic irrationalities. Via evaluation of Hecke L-series we obtain new evaluations at points in imaginary quadratic number fields with class numbers 3 and 4. Further, we overcome the limitations of the earlier methods and via modular equations provide explicit evaluations where the class number is 5 or 7.

Introduction

In this paper we are concerned with special values of the Dedekind eta function. This function is defined for τ on the complex upper half plane by the q-series

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$$

where $q = e^{2\pi i\tau}$. As a function of τ it is a 24-th root of the discriminant function $\Delta(\tau)$ of an elliptic curve $\mathbb{C}/L$ coming from a lattice $L = \{a\tau + b \mid a, b \in \mathbb{Z}\}$.

In the literature, evaluations of eta at points in imaginary quadratic number fields apparently go back to Abel [1] who showed that $\eta(\sqrt{-5}/2)^2/2\eta(2\sqrt{-5})^2$ is a root of the palindromic quartic $x^4 - 2x^3 - 2x^2 - 2x + 1$. For this and some other facts related to the history of the subject see the introduction to [7].

More recently evaluations of quotients of the eta function have received new attention owing to the fact that they provide explicit generators of ring class fields (see [6]). These are of some importance to cryptography and computer primality tests (see [4]).

It was Chowla and Selberg in 1949 who announced the first formula for eta evaluations. Eventually they published [15], which relied on evaluating *Epstein zeta functions*

$$Z(s) = {\sum}'(ax^2 + bxy + cy^2)^{-s}.$$

This led to the following formula, (a slight rearrangement of [15, (2)]), which has been called the *Chowla–Selberg formula*:

$$\prod_{j=1}^{h} a_j^{-6}\Delta(\tau_j) = (2\pi|d|)^{-6h}\left[\prod_{m=1}^{|d|}\Gamma\left(\frac{m}{|d|}\right)^{\left(\frac{d}{m}\right)}\right]^{3w}. \tag{1}$$

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Here the product on the left is over a complete set of reduced binary quadratic forms (a_j, b_j, c_j) of a given fundamental discriminant $d < 0$ having class number h, and where for $1 \leq j \leq h$ the quantities $\tau_j = \frac{b_j+\sqrt{d}}{2a_j}$ are roots of the quadratic forms (a_j, b_j, c_j). On the right is a "product" of gamma functions with exponents given by Kronecker symbols $\left(\frac{d}{m}\right)$. Also w is the number of roots of unity in $\mathbb{Q}(\sqrt{d})$, the imaginary quadratic number field of discriminant d.

Taking 24-th roots and absolute values on both sides of (1) yields a formula involving a product of absolute values of eta functions. Clearly if there is only one form in the form class group, the formula provides an evaluation of the modulus of a single eta function in terms of gamma functions.

Through a long series of improvements, the Chowla–Selberg formula has been progressively extended. First Kaneko [11] and Nakajima and Taguchi [12] extended the formula to non-fundamental discriminants. Later Zhang and Williams [17] refined the formula so that the product on the left-hand side could be taken over a single genus of reduced forms of fundamental discriminant. Finally Huard, Kaplan and Williams [10] did the same thing for non-fundamental discriminant.

These formulae appear to have been inspired in part by Zucker and Robertson who have a series of papers [18, 19, 20, 21] on their technique of "solving" L-series, *i.e.,* they split various L-series into quadratic L-series and use the formulae of Dirichlet to provide explicit evaluations of them. However, even with these formulae there still existed no method to evaluate an isolated eta function corresponding to the root of a single reduced form, except in the cases where there was a single form per genus.

More recently however, van der Poorten and Williams [14] produced a formula breaking up the Chowla–Selberg formula completely. Later Chapman and van der Poorten [2] recognized various quantities which appeared in that work to be generalised class group characters, which led to the expression of eta evaluations in terms of Hecke L-series. Evaluating some of these L-series allowed various explicit evaluations of isolated eta functions to be given. In particular, a neat trick allowed the evaluation of a single eta function at a point in a quadratic field of fundamental discriminant $d = -31$, whose class number is 3.

Evaluations are also provided in [2] where the class number is 3 and the discriminant D is non-fundamental. In that analysis one relies on the class number being the same in the order of discriminant D as it is in the maximal order (the ring of integers) of the underlying quadratic number field.

Despite this progress, the methods of Williams, van der Poorten and Chapman provide only a small advance in actual explicit eta evaluations. Before that time a majority of the evaluations in the literature had the (implicit) restriction that the class group of the underlying quadratic field be 2-torsion. Adding some class number 3 cases represents worthwhile but limited progress beyond this.

In the present paper we overcome many of the former restrictions and manage to evaluate eta quotients in cases where the class group is not 2-torsion or is not of the specific form required by [2]. Our hope is that these new ideas may eventually lead to a more general theory.

In Section 1 we describe an explicit eta evaluation where the discriminant is $D = -44$ but where (unlike [2]) the class number of the non-maximal order of discriminant D is different from that of the maximal order of the associated quadratic number field $\mathbb{Q}(\sqrt{D})$. Section 2 uses minimal computational assistance to provide evaluations in a class number 4 ($d = -56$) case. In Section 3 we report on a solution for the case of fundamental discriminant and class number 5 ($d = -47$), making use of a known evaluation of Weber for one of his *Weber functions.* In Section 4 we use the results of [9] on Schläfli modular equations for new Weber-like functions to provide an eta evaluation where the class number is 7.

1 Evaluation of Eta for Non-fundamental Discriminant $D = -44$

Our first eta evaluations will be at roots of binary quadratic forms with discriminant $D = -44$. This is a non-fundamental discriminant of an order with class number 3. The underlying quadratic number field has discriminant $d = -11$ and class number 1, thus belonging to a situation not dealt with directly by the methods of [2]. However a relatively uncomplicated modification of those techniques will suffice to deal with this case.

The primitive reduced forms for $D = -44$ are $Q_1 = (1, 0, 11)$, $Q_2 = (3, 2, 4)$ and $Q_3 = (3, -2, 4)$. The sole reduced form of discriminant $d = -11$ is $Q = (1, 1, 3)$.

The generalised Chowla–Selberg formula [11, 12] yields the following product of eta values for the non-fundamental discriminant $D = -44$

$$(2) \qquad 3^{-\frac{1}{2}} \left|\eta(\sqrt{-11})\right| \left|\eta\Big(\frac{1+\sqrt{-11}}{3}\Big)\right|^2 = \Big(\frac{1}{88\pi}\Big)^{\frac{3}{4}} \left[\prod_{n=1}^{11} \Gamma\Big(\frac{n}{11}\Big)^{\left(\frac{11}{n}\right)}\right]^{\frac{3}{4}} 2^{\frac{1}{2}}.$$

Here $\left(\frac{11}{n}\right)$ is a Kronecker symbol and $\Gamma(z)$ is the gamma function. Note also that the product on the left has been simplified by application of the useful formula

$$(3) \qquad \left|\eta\Big(\frac{b+\sqrt{D}}{2a}\Big)\right| = \left|\eta\Big(\frac{-b+\sqrt{D}}{2a}\Big)\right|,$$

which follows trivially from the definition of $\eta(\tau)$.

Now, since we have the product of all the eta functions in question, it will suffice to determine pairwise quotients of them in order to find all their separate values. Thus it only remains to find the absolute value of the quotient of the two different eta values appearing in (2).

In order to evaluate this quotient we proceed in a manner similar to that of [2]. Letting ω be the cube root of unity $e^{2\pi i/3}$, we evaluate the sum $S(s, k)$

$$(4) \qquad \sum{}' \frac{1}{(x^2+11y^2)^s} + \sum{}' \frac{\omega^k}{(3x^2+2xy+4y^2)^s} + \sum{}' \frac{\omega^{2k}}{(3x^2-2xy+4y^2)^s}$$
$$= \sum_{m=1}^{\infty} \frac{N(Q_1, m)}{m^s} + \omega^k \sum_{m=1}^{\infty} \frac{N(Q_2, m)}{m^s} + \omega^{2k} \sum_{m=1}^{\infty} \frac{N(Q_3, m)}{m^s}$$

where $N(Q_i, m)$ is the number of representations of the integer m by the form Q_i, and where the dashes indicate that we sum only over pairs of integers (x, y) which are not both zero.

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Now we use the formula

$$N(Q_i, m) = \sum_{r|\kappa} N_R(Q_{i,r}, m/r^2)$$

from [2, §3.4] where κ is the conductor of the binary quadratic form Q_i, and the notation $N_R(Q', m')$ stands for zero if m' is not an integer or is not coprime to the conductor of Q', and for $N(Q', m')$ otherwise. The form $Q_{i,r}$ has discriminant D/r^2 and is obtained from Q_i by a canonical process described in [2, §§3.1, 3.2].

In our $D = -44$ example, $\kappa = 2$ for each of the quadratic forms Q_i. All three forms $Q_{i,2}$ turn out to be the same, namely the single primitive reduced form Q of discriminant $d = -11$. Also, in each case $Q_{i,1} = Q_i$.

Thus, our sum becomes

$$S(s,k) = \sum_{n=1}^{\infty} \frac{N_R(Q_1, m)}{m^s} + \omega^k \sum_{n=1}^{\infty} \frac{N_R(Q_2, m)}{m^s} + \omega^{2k} \sum_{n=1}^{\infty} \frac{N_R(Q_3, m)}{m^s} + (1 + \omega^k + \omega^{2k}) \sum_{n=1}^{\infty} \frac{N_R(Q, m/4)}{m^s}. \tag{5}$$

Note that for $k = 1, 2$ the fourth term of this expression is zero. These cases will suffice for our purposes.

The remaining three terms can be given an expression in terms of ideals as per [2, (3.1)]. Thus our sum is simply

$$S(s,k) = 2 \sum_{\substack{\mathfrak{a} \in I_2 \\ \text{integral}}} \frac{\chi^k(\mathfrak{a})}{(N\mathfrak{a})^s} \tag{6}$$

for some non-trivial character χ defined on ideals, where the sum is over all integral ideals of $\mathcal{O}_K$ which are coprime to 2 and where the factor of 2 which appears outside the sum is the number of roots of unity in the order of discriminant $D = -44$. Note that the other expressions r and $g(\kappa; r)$ that appear in [2, (3.1)] have the value 1 in this situation.

We would like to make all of this precise, naturally extending the ideas of [2]. We view the sum on the right-hand side of (6) as a primitive L-series of a certain class field extension H_f/K (for $f \in \mathbb{N}$) of the underlying quadratic field $K = \mathbb{Q}(\sqrt{-11})$, called the *ring class field of conductor f*.

A ring class field is a generalization of the Hilbert class field $H = H_1$, where this time, by class field theory, the Galois group of the extension H_f/K is now isomorphic to the class group of an order, of conductor f, of our quadratic field K. For more details on ring class fields see [3].

For our purposes it suffices to know that the class group of the non-maximal order of conductor f is also isomorphic to $G_f = I_f/P_{f,\mathbb{Z}}$, the $\mathcal{O}_K$ ideals coprime to f, modulo the principal such ideals generated by an element congruent to a rational

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integer modulo f. These *generalized ideal class groups* G_f are precisely those spoken of in [2].

In particular we now have that the generalized ideal class group G_f is isomorphic to the Galois group $G = G(H_f/K)$ of the ring class extension of conductor f. This isomorphism is given explicitly by the Artin map which, since H_f/K is Abelian, can be constructed from Frobenius automorphisms $\sigma_{\mathfrak{p}}$ for the primes $\mathfrak{p}$ of K.

Via this isomorphism, a character of the Galois group can be extended to a character on ideals of K. Specifically for a prime ideal $\mathfrak{p}$, if a character $\chi \in \hat{G}$ (where $\hat{G}$ is the character group of G) vanishes on the inertia group of $\mathfrak{p}$, then $\chi(\mathfrak{p})$ is set to equal $\chi(\sigma_{\mathfrak{p}})$, otherwise $\chi(\mathfrak{p})$ is set to zero.

For some non-trivial character χ of the kind just mentioned, our equation (6) can be written

$$S(s,\chi) = 2\,L(s,\chi) = 2\prod_{\mathfrak{p}}\Big(1 - \frac{\chi(\mathfrak{p})}{(N\mathfrak{p})^s}\Big)^{-1}. \tag{7}$$

In our very simple example, where we are investigating H_2/K, the formulation above simply guarantees that χ vanishes on all prime ideals of K which divide 2.

Now taking the limit as $s \to 1$ in (7) and applying the Kronecker limit formula to the same expression $S(s,k) = S(s,\chi)$, but as given by (4), we obtain

$$-\frac{8\pi}{\sqrt{44}}\log\left|\frac{\eta(\sqrt{-11})}{3^{-\frac{1}{4}}\eta\big(\frac{1+\sqrt{-11}}{3}\big)}\right| = 2L(1,\chi). \tag{8}$$

It remains only to calculate $L(1,\chi)$ for which we make use of the formula

$$\zeta_{H_2}(s) = \prod_{\chi\in\hat{G}} L(s,\chi),$$

expressing the Dedekind zeta function of the ring class field H_2 in terms of the primitive L-series of the extension H_2/K, one of which is the Dedekind zeta function of K. We are now in a situation analogous to the calculation of $L(1,\chi)$ that arose for the discriminant $d = -31$ in [2].

In fact, $L(1,\chi) = L(1,\chi^2)$ and the L-series we are after is given by

$$L(1,\chi)^2 = \lim_{s\to 1^+}\frac{\zeta_{H_2}(s)}{\zeta_K(s)}, \tag{9}$$

a quotient of zeta functions.

Now the analytic class number formula for a number field F states that

$$\lim_{s\to 1^+}(s-1)\zeta_F(s) = \frac{2^{s+t}\pi^t h_F R_F}{w_F\sqrt{|d_F|}} \tag{10}$$

where s and $2t$ are the numbers of real and complex embeddings of F, h_F is the class number, R_F is the regulator, w_F is the number of roots of unity and d_F is the discriminant of F.

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We can apply this result to the expression above for $L(1,\chi)^2$. However before the result becomes of particular use to us we need to express the regulator of the ring class field H_2 in terms of the regulator of its real subfield $N = \mathrm{Re}\{H_2\}$. To do this we use one of the Brauer relations which appear in [5]. First we need a definition.

Definition 1.1 For any number field K define the quantity

$$\xi(K) = \frac{h_K R_K}{w_K\sqrt{|d_K|}}$$

where h_k, R_K, w_K and d_K are the class number, regulator, number of roots of unity and discriminant of the field K, respectively.

With this definition, the result which interests us is given as [5, (7.32), §VIII.7], which we repeat here.

Theorem 1.2

$$\xi(\mathbb{Q})^2\xi(H_2) = \xi(N)^2\xi(K).$$

Substituting the expressions for ξ into this equation and simplifying we obtain

$$\frac{h_{H_2}R_{H_2}}{\sqrt{|d_{H_2}|}} = \frac{h_K R_N^2 h_N^2}{|d_N|\sqrt{|d_K|}}. \tag{11}$$

But in fact it is not difficut to prove (see for example [5, III.2]), that

$$d_{H_2} = d_N^2 d_K.$$

Similarly we can prove that in our case $d_N = -44$.

We also need h_N, but a short Pari [13] computation reveals that this is 1. The regulator of N is simply the logarithm of the fundamental unit, v say, of N.

Now substituting all the data that we have, in terms of v, into (9) and (8), we obtain

$$\left|\frac{\eta\left(\frac{1+\sqrt{-11}}{3}\right)}{\eta(\sqrt{-11})}\right|^2 = \sqrt{3}\,v.$$

Pari easily calculates that a minimum polynomial for the fundamental unit v of N is given by $v^3 - v^2 - v - 1 = 0$, thus completing our eta evaluation.

2 Evaluation of Eta for $d = -56$

At first sight it would appear that the technique of the previous section, where an L-series is expressed in terms of a quotient of zeta functions, can only be successful when the class number of the quadratic order under consideration is 3. In this section we see that similar techniques can however be applied in more general situations.

In particular, in this section we consider the case with fundamental discriminant $d = -56$, where the class number is 4. In fact, there are four reduced binary quadratic forms for this discriminant, $(1, 0, 14)$, $(2, 0, 7)$ and $(3, \pm 2, 5)$.

Again using the techniques of [2] we have that

$$\sum{}' \frac{1}{(x^2+47y^2)^s} + \sum{}' \frac{i^k}{(3x^2+2xy+5y^2)^s} + \sum{}' \frac{i^{2k}}{(2x^2+7y^2)^s} + \sum{}' \frac{i^{3k}}{(3x^2-2xy+5y^2)^s} = 2L(s,\chi) \tag{12}$$

where the L-series and its character χ belong to the extension H/K where H is the Hilbert class field of $K = \mathbb{Q}(\sqrt{-56})$.

The Hilbert class field H in this case is a cyclic extension of degree four of K. If we let the Galois group of H/K be generated by σ of order 4, then there is also an intermediate field L which is the fixed field of σ^2. This field L also plays an important part in what follows.

If χ is a generator of the character group of $G = \mathrm{Gal}(H/K)$, we have the relations

$$\zeta_H(s) = \zeta_K(s)L(s,\chi)L(s,\chi^2)L(s,\chi^3) \quad \text{and} \quad \zeta_L(s) = \zeta_K(s)L(s,\chi^2)$$

between the various primitive L-series of the extensions H/K and L/K respectively. We note that $L(s,\chi^2)$ is the same L-series in both of these expressions.

Note also that $L(s,\chi) = L(s,\overline{\chi}) = L(s,\chi^3)$. It is now clear that $L(s,\chi^2)$ can be calculated from the second of these relations, then allowing $L(s,\chi)$ to be calculated from the first.

If we let $\xi(F)$ be as per the previous section, we have that

$$L(1,\chi^2) = 2\pi\,\frac{\xi(L)}{\xi(K)} \quad \text{and} \quad L(1,\chi) = 2\pi\sqrt{\frac{\xi(H)}{\xi(L)}}. \tag{13}$$

The fields K, L, and H have class numbers 4, 2, and 1 and discriminants -56, 56^2, and 56^4, respectively. All the fields contain 2 roots of unity. Applying the formula (10), with these values, then allows us to obtain expressions for the L-series in terms of the regulators of the fields H and L.

We can apply the Kronecker limit formula to the left-hand side of (12) and thus it is possible to derive expressions for the logarithms of quotients of the various eta values in terms of the regulators of L and H in a similar manner to the previous section.

This method however leads to some undesirable quantities appearing in the final solution. In particular, the regulator of the field H is not the logarithm of a single fundamental unit. In order to improve the situation, we make use of Pari to manipulate the units involved. The result turns out to be much neater, but of course relies on the additional information we are able to gain numerically.

We let $\alpha = \sqrt{-1+2\sqrt{2}}$ and $\beta = \sqrt{-1-2\sqrt{2}}$. We note that $\alpha\beta = \sqrt{-7}$. The Hilbert class field of K is in fact $H = \mathbb{Q}(\sqrt{-14},\alpha) = \mathbb{Q}(\sqrt{-14},\beta) = \mathbb{Q}(\alpha,\beta)$.

The Galois group of $H/\mathbb{Q}$ is dihedral of order 8. It is generated by σ of order 4 whose action is characterised by

$$\sigma(\alpha) = \beta,\ \sigma(\beta) = -\alpha,$$

and by complex conjugation denoted τ.

With these definitions we were able to find with Pari, the following set of units of H

$$\epsilon_0 = (1+\alpha+\alpha\sqrt{2})/2, \qquad \epsilon_1 = (1+\beta-\beta\sqrt{2})/2,$$
$$\epsilon_2 = (1-\alpha-\alpha\sqrt{2})/2, \qquad \epsilon_3 = (1-\beta+\beta\sqrt{2})/2,$$

any three of which constitute a system of fundamental units of H.

They are all roots of

$$\lambda^4 - 2\lambda^3 - \lambda^2 + 2\lambda - 1 = 0,$$

and they have the property that $\sigma(\epsilon_n) = \epsilon_{n+1}$ and $\tau(\epsilon_n) = \epsilon_{-n}$ with subscripts modulo 4.

Now we let l_i be the logarithm of the absolute value of the unit ϵ_i. In this notation, $l_0 + l_1 + l_2 + l_3 = 0$ and since ϵ_1 and ϵ_3 are non-real complex conjugates, we have that $l_1 = l_3 = -\frac{1}{2}(l_0 + l_2)$. Also the regulator of L turns out to be $R_L = 2(l_0 + l_2)$ since the fundamental unit of L is given by $1 + \sqrt{2} = |\epsilon_0\epsilon_2|$.

The regulator of H can now be written as the following determinant

$$\begin{vmatrix} 2l_0 & -(l_0+l_2) & 2l_2 \\ -(l_0+l_2) & 2l_0 & -(l_0+l_2) \\ 2l_2 & -(l_0+l_2) & 2l_0 \end{vmatrix} = 4(l_0+l_2)(l_0-l_2)^2.$$

Using the analytic class number formula we have the following values

$$\xi(K) = \frac{1}{\sqrt{14}}, \quad \xi(L) = \frac{(l_0+l_2)}{2^2\cdot 7}, \quad \xi(H) = \frac{(l_0+l_2)(l_0-l_2)^2}{2^5\cdot 7^2}.$$

Now we can retrieve the values of the L-series as per the relations (13),

$$L(1,\chi^2) = \frac{\pi(l_0+l_2)}{\sqrt{14}} \quad \text{and} \quad L(1,\chi) = \frac{\pi(l_0-l_2)}{\sqrt{14}}.$$

Finally, by applying the Kronecker limit formula to the Epstein zeta functions of (12), we have

$$\log|\eta(\sqrt{-14})| + \log\left|2^{-\frac{1}{4}}\eta\left(\frac{\sqrt{-14}}{2}\right)\right| - 2\log\left|3^{-\frac{1}{4}}\eta\left(\frac{1+\sqrt{-14}}{3}\right)\right| = -\frac{1}{2}(l_0+l_2)$$

and

$$\log|\eta(\sqrt{-14})| - \log\left|2^{-\frac{1}{4}}\eta\left(\frac{\sqrt{-14}}{2}\right)\right| = -\frac{1}{2}(l_0-l_2).$$

It is now clear that by adding or subtracting these two equations, one has expressions for the quotients of eta values. Combining this with the information provided by the Chowla-Selberg formula (1) then leads to complete evaluations of all the eta values involved.

This computation raises the possibility that, in some cases at least, a more direct method may exist which relates eta quotients to some canonical choice of fundamental units.

3 Evaluation of Eta for $d = -47$

In this section we will provide an evaluation of an eta quotient at a point in a quadratic number field, of discriminant $d = -47$, which has class number 5. We will not provide eta evaluations for all roots of reduced forms of this discriminant but will provide solely an evaluation of the quotient

$$a = \left|\frac{\eta((1+\sqrt{-47})/4)}{\eta((1+\sqrt{-47})/2)}\right|.$$

Further evaluations for the discriminant $d = -47$ have been completed and are reported in the thesis [8] of the second author. This simple, but nevertheless interesting, eta quotient is evaluated here primarily to motivate the material of the next section.

We begin by noticing that the eta quotient above can be expressed in terms of one of the Weber functions (see [16, §34])

$$\mathfrak{f}(\tau) = e^{-\frac{\pi i}{24}} \frac{\eta\left(\frac{\tau+1}{2}\right)}{\eta(\tau)}, \quad \mathfrak{f}_1(\tau) = \frac{\eta\left(\frac{\tau}{2}\right)}{\eta(\tau)}, \quad \mathfrak{f}_2(\tau) = \sqrt{2}\,\frac{\eta(2\tau)}{\eta(\tau)}.$$

In fact,

$$a = \left|\mathfrak{f}_1\left((1+\sqrt{-47})/2\right)\right| = \left|\mathfrak{f}_1\left((-1+\sqrt{-47})/2\right)\right|, \tag{14}$$

the latter equality following from (3).

By applying the modular substitution $\tau \to \tau + 1$ to $\mathfrak{f}$, we find by the modular substitution rules for the Weber functions [16, §34] that

$$a = \left|\mathfrak{f}\left((1+\sqrt{-47})/2\right)\right|. \tag{15}$$

Now we can use the elegant identity [16, §34]

$$\mathfrak{f}(\tau)\,\mathfrak{f}_1(\tau)\,\mathfrak{f}_2(\tau) = \sqrt{2},$$

which holds for all τ in the complex upper half plane. From (14) and (15) we have

$$a^2 \left|\mathfrak{f}_2\left((1+\sqrt{-47})/2\right)\right| = \sqrt{2}.$$

Now we can use the identity

$$\mathfrak{f}_1(\tau) = \frac{\sqrt{2}}{\mathfrak{f}_2(\tau/2)}$$

from which we obtain

$$a^2 = \left|\mathfrak{f}_1\left(1+\sqrt{-47}\right)\right| = \left|\mathfrak{f}\left(\sqrt{-47}\right)\right|,$$

making use of a final modular substitution $\tau \to \tau + 1$ for the function $\mathfrak{f}_1$.

The value of $\mathfrak{f}(\sqrt{-47})$ is given by Weber, [16, §131 (3)]. He states that

$$\mathfrak{f}(\sqrt{-47}) = \sqrt{2}\,x, \text{ where } x^5 - x^3 - 2x^2 - 2x - 1 = 0. \tag{16}$$

This evaluation of Weber's is of course sufficient to complete our evaluation of the eta quotient a. However it is instructive to know something about how Weber completed his evaluation of $\mathfrak{f}(\sqrt{-47})$. He does this by making use of his *modular equations of irrational type*.

In particular, for this case, he calculates the modular equation of *degree*[1] 47 which relates the functions

$$u = \mathfrak{f}(\tau), \qquad u_1 = \mathfrak{f}_1(\tau), \qquad u_2 = \mathfrak{f}_2(\tau),$$
$$v = \mathfrak{f}(47\tau), \quad v_1 = \mathfrak{f}_1(47\tau), \quad v_2 = \mathfrak{f}_2(47\tau).$$

This modular equation is given in [16, §75]. Letting

$$2A = uv + u_1v_1 + u_2v_2,$$
$$B = 2/(u_1v_1) + 2/(u_2v_2) + 2/(uv),$$

then the modular equation in question is listed as

$$A^2 - A - B = 2. \tag{17}$$

Now Weber noted [16, §31] that if one sets $\tau = -1/\sqrt{-47}$ then $47\tau = \sqrt{-47}$. But one can apply the modular transformation laws for the Weber functions to $\mathfrak{f}(-1/\sqrt{-47})$, and similarly for the other two Weber functions. The result is

$$2A = \mathfrak{f}(\sqrt{-47})^2 + 2\mathfrak{f}_1(\sqrt{-47})\mathfrak{f}_2(\sqrt{-47}),$$
$$B = 2/\mathfrak{f}(\sqrt{-47})^2 + 4/\mathfrak{f}_1(\sqrt{-47})\mathfrak{f}_2(\sqrt{-47}).$$

Now making use of the identity (3), this can be written entirely in terms of $\mathfrak{f}(\sqrt{-47})$, giving the result we mentioned above.

Note that there is a typographical error in the relation above (1) of §131 in Weber, which ought to read $B = 4x + (-1)^{(m+1)/8}/x^2$.

This modular equation method forms the starting point for a powerful eta evaluation technique. In particular Weber's modular equations of irrational type and Schläfli type can be used to obtain further eta evaluation where the class number of the underlying quadratic field K is 5.

It seems unlikely that a method akin to that of the first few sections of this paper can be used to evaluate such eta quotients. The main obstacle is that when the class number is prime, there are no intermediate fields between K and its Hilbert class field from which we can glean information. The earlier techniques will only allow us to

[1] A modular equation relating some functions $f_i(\tau)$ and $f_i(n\tau)$ is said to be of *degree n*

evaluate a product of all the primitive L-series involved, by taking a quotient of zeta function residues. However those methods do not provide a way of splitting up this product of L-series.

In the next section we investigate the logical successor to the Weber function modular equation technique, namely the use of modular equations for generalized Weber functions.

4 Evaluations Via Modular Equations of Weber Type Functions

In this section we will provide an eta evaluation at a point in the quadratic order of discriminant $D = -2^2 \cdot 71$, which has class number 7.

We will make use of a modular equation for the *level three Weber functions*, introduced in [9]. These functions are defined as follows

$$\mathfrak{g}_\infty(\tau) = \sqrt{3}\,\frac{\eta(3\tau)}{\eta(\tau)}, \quad \mathfrak{g}_0(\tau) = \frac{\eta(\frac{\tau}{3})}{\eta(\tau)}, \quad \mathfrak{g}_1(\tau) = \zeta_{24}\,\frac{\eta(\frac{\tau+1}{3})}{\eta(\tau)}, \quad \mathfrak{g}_2(\tau) = \frac{\eta(\frac{\tau+2}{3})}{\eta(\tau)}.$$

The following theorem, which also appears in [8], describes the action of modular transformations on these functions

Theorem 4.1

$$\begin{pmatrix} \mathfrak{g}_\infty \\ \mathfrak{g}_0 \\ \mathfrak{g}_1 \\ \mathfrak{g}_2 \end{pmatrix} \circ T = \begin{pmatrix} \zeta_{12}\,\mathfrak{g}_\infty \\ \zeta_{12}^{-1}\,\mathfrak{g}_1 \\ \mathfrak{g}_2 \\ \mathfrak{g}_0 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} \mathfrak{g}_\infty \\ \mathfrak{g}_0 \\ \mathfrak{g}_1 \\ \mathfrak{g}_2 \end{pmatrix} \circ S = \begin{pmatrix} \mathfrak{g}_0 \\ \mathfrak{g}_\infty \\ \mathfrak{g}_2 \\ \mathfrak{g}_1 \end{pmatrix} \tag{18}$$

where T stands for the transformation $\tau \to \tau + 1$ and S for $\tau \to -1/\tau$.

In order to obtain our eta evaluation we will need a modular equation of degree 71 for the function $\mathfrak{g}_0$. The technique for constructing such a modular equation is as follows. We abbreviate $u = \mathfrak{g}_0(\tau)$ and $v = \mathfrak{g}_0(71\tau)$ and then define

$$A = (uv) + 3/(uv) \quad \text{and} \quad B = (u/v)^6 + (v/u)^6.$$

We are then guaranteed that if a polynomial $F(X, Y) \in \mathbb{Z}[X, Y]$ exists such that the q-series of $F(A, B)$ vanishes up to and including the constant term, then $F(A, B) = 0$ is a modular equation (an identity for all τ in the complex upper half plane). We find such a polynomial $F(X, Y)$ relatively easily with the help of Pari (see the program and its output in the appendices at the end of this paper).

Once we have a modular equation, we substitute in a particular value of τ as was the case for the Weber function modular equations of the previous section. As it is, the modular equation can be thought of as a polynomial relationship between $u = \mathfrak{g}_0(\tau)$ and $v = \mathfrak{g}_0(71\tau)$. We choose the specific value $\tau = 2 - 1/\sqrt{-71}$, then $71\tau = 142 + \sqrt{-71}$.

From the transformation laws (18), we easily see that these values become

$$u = \mathfrak{g}_0(2 - 1/\sqrt{-71}) = \zeta_{12}^{-1}\,\mathfrak{g}_1(\sqrt{-71})$$
$$v = \mathfrak{g}_0(142 + \sqrt{-71}) = \mathfrak{g}_1(\sqrt{-71}).$$

Thus both of the values can be expressed in terms of $\mathfrak{g}_1(\sqrt{-71})$ which itself can be expressed in terms of $x = \eta((\sqrt{-71}+1)/3)/\eta(\sqrt{-71})$. In fact, in terms of the value x, A and B can be written as follows

$$A = x^2 + 3/x^2 \quad \text{and} \quad B = -2.$$

Making these substitutions in the modular equation given in the appendix, we obtain a polynomial equation solely in x. Pari is able to factorize this polynomial, and finally $y = x^2$ is found to have minimum polynomial over $\mathbb{Q}$

$$\begin{aligned}(19)\quad y^{14} - 48y^{13} + 806y^{12} - 5751y^{11} + 21673y^{10} - 45772y^9 + 54296y^8 \\ - 57799y^7 + 162888y^6 - 411948y^5 + 585171y^4 - 465831y^3 \\ + 195858y^2 - 34992y + 2187 = 0.\end{aligned}$$

A Pari Code for Computing the Modular Equation

```
etaq = prod(n=1,300,1-q^n+O(q^300))
etaq71 = subst(etaq,q,q^71) + O(q^300)
g0q = etaq/subst(etaq,q,q^3)
g0q71 = etaq71/subst(etaq71,q,q^3)
a = vector(36);b = vector(7)
a[1] = 1;b[1] = 1
a[2] = q^(-6)*g0q*g0q71 + 3/(q^(-6)*g0q*g0q71)
b[2] = q^(-35)*(g0q71/g0q)^6 + q^(35)*(g0q/g0q71)^6
for(i=3,36,a[i]=a[i-1]*a[2])
for(i=3,7,b[i]=b[i-1]*b[2])
n = a[36]-b[7]
m = A^35-B^6
l(z) = for(i=0,6,if((z-i*35)%6==0,return(B^i*A^((z-i*35)/6))))
larr(x) = for(p=0,6,if((x-p*35)%6==0,return(b[p+1]*a[(x-p*35)/6+1])))
for(j=0,209,if(polcoeff(n,j-209,q)<>0,m=m-polcoeff(n,j-209,q)*l(209-j);
    n=n-polcoeff(n,j-209,q)*larr(209-j)))
s = subst(m,A,x^2+3/x^2)
t = subst(s,B,-2)
factor(t*x^70)}
```

This program operates as follows. First, the q-series of $\eta(\tau)$ and $\eta(71\tau)$ are calculated but leaving out temporarily the $q^{1/24}$ that normally sits at the front of the eta function q-series.

Next the q-series of $\mathfrak{g}_0(\tau)$ and $\mathfrak{g}_0(71\tau)$ are calculated but with q raised to a power of 3 to avoid fractional powers of q (this does not affect the resulting modular equation).

Note that by making use of the q-series calculated for the eta function, etc., the q-series for $\mathfrak{g}_0$ is also given temporarily without its leading factor, which is a fractional power of q.

The next few lines set up arrays containing powers of the functions A and B. This time the leading factors of A and B are no longer fractional powers of q and are reinserted manually. The q-series, however, are still given with q raised to the third power to eliminate fractional powers of q elsewhere.

Next two partial versions of the same modular equation are given by setting them equal to $A^{35} - B^6$. One of these partial modular equations, m, is given symbolically in terms of the symbols A and B, the other, n, is set equal to the q-series of this same expression.

Each step of the program from this point successively adds terms to the modular equation which are chosen to remove the leading power of q in n. The symbolic version m is updated at each iteration to keep track of which terms have been added. The function `l(z)` in the program calculates, in symbols, the appropriate combination of A and B which ought to be subtracted from the partial modular equation to make the leading q^z disappear. The function `larr(z)` does the same thing but returns a q-series which can actually be subtracted from n.

Eventually all the negative powers of q are removed from n and according to the theory it automatically vanishes identically. Then the symbolic version of the modular equation m contains the full modular equation.

Note that this program can be adapted to calculate modular equations of other degrees. However when one does that it is often necessary to adjust the number of terms one uses in the q-series for eta. This is done by changing the two occurrences of 300 which appear in the first line of the program.

B The Modular Equation of Degree 71

The above program to compute the modular equation of degree 71 returns the following

$$
\begin{aligned}
&- B^6 + \big(45227A^5 + 18133258A^4 + 1004418327A^3 + 15824008560A^2 + 88322894388A \\
&\qquad\qquad + 157324900428\big)\, B^5 \\
&+ \big(66669A^{11} - 185850168A^{10} + 31705014713A^9 - 1129499268613A^8 + 16963947184109A^7 \\
&\quad - 125649510945386A^6 + 638487275388813A^5 - 1552554751084017A^4 \\
&\quad + 3962029486572324A^3 - 2145200065176420A^2 + 4235574300492312A \\
&\quad + 334371371330052\big)\, B^4 \\
&+ \big(18247A^{17} + 34960116A^{16} + 1985710534A^{15} + 63662864828A^{14} - 1619835597267A^{13} \\
&\quad - 28418996389238A^{12} + 394965895741207A^{11} - 187791277402124A^{10} \\
&\quad - 10179088007868766A^9 + 20019911390817886A^8 + 111511305467794059A^7 \\
&\quad - 333469916522532868A^6 - 509811465099875238A^5 + 2627214146579699208A^4 \\
&\quad - 1206872909586515880A^3 - 5931857326644330336A^2 + 9235831515856639776A
\end{aligned}
$$

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$$
\begin{aligned}
&\quad - 4008745359795714912\big)\, B^3 \\
&+ \big(\, 1775A^{23} + 1790904A^{22} + 408865712A^{21} + 12871929238A^{20} \\
&\quad - 60399761912A^{19} - 1759341605969A^{18} + 16697289179911A^{17} \\
&\quad - 18710026180642A^{16} - 402705493601961A^{15} + 2155011792551217A^{14} \\
&\quad - 1103475784401276A^{13} - 31980555336397002A^{12} + 131177720091704128A^{11} \\
&\quad + 11926052958698609A^{10} - 1517574002674915623A^{9} + 3403840142322671190A^{8} \\
&\quad + 4496662009244846610A^{7} - 2805295557073054 4145A^{6} + 25255225780844178636A^{5} \\
&\quad + 57256834475348917704A^{4} - 144559132807457045664A^{3} + 106543495553558624928A^{2} \\
&\quad - 8454658425463893312A - 14745527697401357040\big)\, B^2 \\
&+ \big(\, 71A^{29} + 5822A^{28} - 956796A^{27} + 26642750A^{26} + 78210405A^{25} \\
&\quad - 14164206628A^{24} + 241690913057A^{23} - 1525521821072A^{22} \\
&\quad - 2004401003479A^{21} + 82943327780330A^{20} - 375841940625707A^{19} \\
&\quad - 681048887773444A^{18} + 11773412555795589A^{17} - 28491477688680246A^{16} \\
&\quad - 108997374517525685A^{15} + 738373388780098900A^{14} - 638805705893682584A^{13} \\
&\quad - 6375317111152758400A^{12} + 20850266961115278615A^{11} + 5500865969368792926A^{10} \\
&\quad - 153692286902281334970A^{9} + 263060657744880166722A^{8} + 271389775577407043076A^{7} \\
&\quad - 1528637479450636943052A^{6} + 1619928576395104645512A^{5} + 1453763796022806072192A^{4} \\
&\quad - 5525852731622636301648A^{3} + 6065527992813100255104A^{2} - 3117067970664803783616A \\
&\quad + 630826576037767118016\big)\, B \\
&+ \big(\, A^{35} - 142A^{34} + 9054A^{33} - 340445A^{32} + 8308916A^{31} \\
&\quad - 135427601A^{30} + 1428113239A^{29} - 8058972505A^{28} \\
&\quad - 7952479568A^{27} + 528259830655A^{26} - 3735834343906A^{25} \\
&\quad + 3638855548174A^{24} + 108008357830670A^{23} - 659276078110314A^{22} \\
&\quad + 1917442737404 28A^{21} + 1408275574734 9906A^{20} - 55378046862581733A^{19} \\
&\quad - 59237478478334524A^{18} + 1024424013442984916A^{17} - 2087336003407726453A^{16} \\
&\quad - 6669829903375533626A^{15} + 37907373346470307583A^{14} - 24919854119837821203A^{13} \\
&\quad - 251421122783419924585A^{12} + 680182144509383585028A^{11} + 243840770951310089451A^{10} \\
&\quad - 4043235668975293503168A^{9} + 5873249504452222862532A^{8} + 5951872483525684883592A^{7} \\
&\quad - 28656447529979605048164A^{6} + 28021183251863944427712A^{5} \\
&\quad + 21201112775376895278960A^{4} - 79633122661012442270400A^{3} + 85227106422343349504448A^{2} \\
&\quad - 43816610901692558132352A + 9199757182384381477824\big) \\
&= 0
\end{aligned}
$$

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