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A CLASS OF C-TOTALLY REAL SUBMANIFOLDS OF SASAKIAN SPACE FORMS

FILIP DEFEVER, ION MIHAI and LEOPOLD VERSTRAELEN

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Abstract

Recently, Chen defined an invariant δ_M of a Riemannian manifold M. Sharp inequalities for this Riemannian invariant were obtained for submanifolds in real, complex and Sasakian space forms, in terms of their mean curvature. In the present paper, we investigate certain C-totally real submanifolds of a Sasakian space form $\tilde{M}^{2m+1}(c)$ satisfying Chen's equality.

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1. Introduction

We consider C-totally real submanifolds M^n of a Sasakian space form $\tilde{M}^{2m+1}(c)$; let H denote the mean curvature vector field of M^n in $\tilde{M}^{2m+1}(c)$. Precise definitions of the concepts used are given in Sections 2 and 3.

In [7] a general best possible inequality was obtained between the main intrinsic invariants of the submanifold M^n on one side, namely its sectional curvature function K and its scalar curvature function τ, and its main extrinsic invariant on the other side, namely its mean curvature function $|H|$.

More precisely, in the Sasakian case, Chen's inequality, relating K, τ and H, reads:

$$\text{(1)} \qquad \inf K \geq \tau - \frac{n^2(n-2)}{2(n-1)}|H|^2 - \frac{(n+1)(n-2)(c+3)}{8}.$$

The first author is Postdoctoral Researcher N.F.W.O., Belgium; the second author is Research Fellow of the Research Council of the K.U. Leuven.

[7] also classifies the C-totally real submanifolds M^n of $\tilde{M}^{2n+1}(c)$ with constant scalar curvature for which Chen's inequality becomes an equality.

In [6] a similar inequality for δ_M was established for totally real submanifolds of a complex space form; [6] and [4] contain also a classification of certain such submanifolds satisfying the equality.

In the present paper, we enlarge the investigation of [7] to the class of C-totally real submanifolds having nonconstant scalar curvature. Following [5], we consider C-totally real submanifolds M^n in $\tilde{M}^{2n+1}(c)$, satisfying Chen's equality, under some additional integrability condition. This extra condition then appropriately singles out, and conversely characterizes in some sense, a specific class of C-totally real submanifolds of $\tilde{M}^{2n+1}(c)$ capturing the particular example with nonconstant scalar curvature, that fell outside the range of the classification result of [7]. This condition is stated in terms of some distribution, introduced in this context in [2]. More precisely, we prove the following theorem.

THEOREM 1. *Let $\tilde{M}^{2n+1}(c)$ be a Sasakian space form and M^n an n-dimensional ($n > 2$) C-totally real submanifold with nonconstant scalar curvature such that the subspaces*

$$\mathscr{D}(p) = \{X \in T_pM^n; h(X, Y) = 0, \forall Y \in T_pM^n\}, \ p \in M^n,$$

define a differentiable subbundle and its complementary orthogonal subbundle $\mathscr{D}^\perp$ is involutive. Then M^n satisfies

$$\delta_M = \tau - \inf K = \frac{(n-2)(n+1)(c+3)}{8},$$

if and only if M^n is locally congruent to an immersion

$$\psi : (0, \tfrac{1}{2}\pi) \times_{\cos t} M^2 \times_{\sin t} S^{n-3} \to S^{2n+1}, \qquad \psi(t, p, q) = (\cos t)p + (\sin t)q,$$

where M^2 is a C-totally real minimal surface of S^5.

We remark that the example of a C-totally real submanifold with nonconstant scalar curvature satisfying Chen's equality given in [7] is included as a particular case of the above theorem, for $n = 3$.

2. C-totally real submanifolds of a Sasakian space form

Let $\tilde{M}^{2m+1}$ be an odd dimensional Riemannian manifold of class C^∞ with Riemannian metric tensor field g.

Let ϕ be a (1,1)-tensor field, ξ a vector field, and η a 1-form on $\tilde{M}^{2m+1}$, such that

$$\phi^2 X = -X + \eta(X)\xi, \qquad \phi\xi = 0, \qquad \eta(\phi X) = 0, \qquad \eta(\xi) = 1,$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \qquad \eta(X) = g(X, \xi).$$

If, in addition, $d\eta(X, Y) = g(\phi X, Y)$ for all vector fields X, Y on $\tilde{M}^{2m+1}$, then $\tilde{M}^{2m+1}$ is said to have a *contact metric structure* (ϕ, ξ, η, g), and $\tilde{M}^{2m+1}$ is called a *contact metric manifold.*

If moreover the structure is normal, that is if $[\phi X, \phi Y] + \phi^2[X, Y] - \phi[X, \phi Y] - \phi[\phi X, Y] = -2d\eta(X, Y)\xi$, then the contact metric structure is called a *Sasakian structure* (*normal contact metric structure*) and $\tilde{M}^{2m+1}$ is called a *Sasakian manifold.* For more details and background, see the standard references [1, 10].

A plane section σ in $T_p\tilde{M}^{2m+1}$ of a Sasakian manifold $\tilde{M}^{2m+1}$ is called a *ϕ-section* if it is spanned by X and ϕX, where X is a unit tangent vector field orthogonal to ξ. The sectional curvature $\tilde{K}(\sigma)$ with respect to a ϕ-section σ is called a *ϕ-sectional curvature.* If a Sasakian manifold $\tilde{M}^{2m+1}$ has constant ϕ-sectional curvature c, $\tilde{M}^{2m+1}$ is called a *Sasakian space form* and is denoted by $\tilde{M}^{2m+1}(c)$.

The curvature tensor $\tilde{R}$ of a Sasakian space form $\tilde{M}^{2m+1}(c)$ is given by [1]:

$$\begin{aligned}\tilde{R}(X, Y)Z = {} & \frac{c+3}{4}\big(g(Y, Z)X - g(X, Z)Y\big) \\ & + \frac{c-1}{4}\big(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi \\ & + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\big),\end{aligned}$$

for any tangent vector fields X, Y, Z to $\tilde{M}^{2m+1}(c)$.

An n-dimensional submanifold M^n of a Sasakian space form $\tilde{M}^{2m+1}(c)$ is called a *C-totally real submanifold* of $\tilde{M}^{2m+1}(c)$ if ξ is a normal vector field on M^n. A direct consequence of this definition is that $\phi(TM^n) \subset T^{\perp}M^n$, which means that M^n is an anti-invariant submanifold of $\tilde{M}^{2m+1}(c)$, (hence their name of 'contact'-totally real submanifolds); see for example [9].

The Gauss equation implies that

$$\begin{aligned}R(X, Y, Z, W) = {} & \frac{1}{4}(c+3)\big(g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\big) \\ & + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)),\end{aligned} \tag{2}$$

for all vector fields X, Y, Z, W tangent to M^n, where h denotes the second fundamental form and R the curvature tensor of M^n.

It is easily seen that

$$2\tau = n^2|H|^2 - \|h\|^2 + \frac{n(n-1)(c+3)}{4}. \tag{3}$$

3. Chen's inequality

Let M^n be an n-dimensional Riemannian manifold. Denote by $K(\pi)$ the sectional curvature of the plane section $\pi \subset T_pM^n$, $p \in M^n$. For any orthonormal basis $\{e_1, \ldots, e_n\}$ of the tangent space T_pM^n, the *scalar curvature* τ at p is defined by

$$\tau = \sum_{1\le i<j\le n} K(e_i \wedge e_j).$$

For each point $p \in M^n$, we put

$$(\inf K)(p) = \inf\{K(\pi);\ \pi \subset T_pM^n, \dim \pi = 2\}.$$

The function $\inf K$ is a well-defined function on M^n. Let δ_M denote the difference between the scalar curvature and $\inf K$, that is

$$\delta_M(p) = \tau(p) - (\inf K)(p);$$

δ_M is a well-defined Riemannian invariant, which is trivial when $n = 2$. The invariant δ_M was introduced by Chen in [2], where he gave a sharp inequality for δ_M for submanifolds in real space forms and also obtained a classification of the minimal submanifolds satisfying the equality-case (see also [3]).

We now state the inequality of Chen for the situation where the ambient space is a Sasakian space form [7].

THEOREM 2. *Let M^n be an n-dimensional ($n > 2$) C-totally real submanifold of a $(2m+1)$-dimensional Sasakian space form $\tilde{M}^{2m+1}(c)$. Then*

$$\delta_M \le \frac{n-2}{2}\left(\frac{n^2}{n-1}|H|^2 + \frac{1}{4}(n+1)(c+3)\right). \tag{4}$$

Moreover, the equality holds at a point $p \in M^n$ if and only if there exist a tangent basis $\{e_1, \ldots, e_n\} \subset T_pM^n$ and a normal basis $\{e_{n+1}, \ldots, e_{2m}, \xi\} \subset T_p^\perp M^n$ such that the shape operators take the following forms

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & \ldots & 0 \\ 0 & b & 0 & \ldots & 0 \\ 0 & 0 & & \mu I_{n-2} & \end{pmatrix}, \quad a + b = \mu,$$

$$A_r = \begin{pmatrix} h^r_{11} & h^r_{12} & 0 & \ldots & 0 \\ h^r_{12} & -h^r_{11} & 0 & \ldots & 0 \\ 0 & 0 & & 0_{n-2} & \end{pmatrix}, \quad r \in \{n+2, \ldots, 2m\},$$

and $A_\xi = 0$.

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4. Submanifolds with maximal dimension

We recall the following results, which we will need in the proof of Theorem 1.

PROPOSITION 3. *Let M^n, $(n > 2)$, be a C-totally real submanifold of a Sasakian space form $M^{2m+1}(c)$ which satisfies Chen's equality* (4). *Then for all X tangent to M^n, ϕX is perpendicular to H.*

From now on, we restrict our attention to the totally real submanifolds M^n of a Sasakian space form $\tilde{M}^{2m+1}(c)$ with lowest possible codimension or equivalently with maximal dimension, that is, we assume that $m = n$.

In this case, under the assumptions of Proposition 3, it follows that the mean curvature vector field H is in the direction of ξ along M^n. Hence, we have the following corollary.

COROLLARY 4. *Every C-totally real submanifold M^n $(n > 2)$, of a Sasakian space form $M^{2n+1}(c)$ which satisfies Chen's equality is minimal.*

For a proof of these, as well as of the following Proposition 5, we refer to [7].

PROPOSITION 5. *Let M^n be an n-dimensional $(n > 2)$ minimal C-totally real submanifold of a $(2n+1)$-dimensional Sasakian space form $\tilde{M}^{2n+1}(c)$. Then*

$$\delta_M \leq \frac{(n-2)(n+1)(c+3)}{8},$$

and the equality holds at a point $p \in M^n$ if and only if there exists a tangent basis $\{e_1, \dots, e_n\} \subset T_pM^n$ such that

$$h(e_1, e_1) = \lambda\phi e_1,\ h(e_1, e_2) = -\lambda\phi e_2,\ h(e_2, e_2) = -\lambda\phi e_1,\ h(e_i, e_j) = 0,\ i, j > 2,$$

where $\lambda \geq 0$ is given by

$$4\lambda^2 = \frac{n(n-1)(c+3)}{4} - 2\tau. \tag{5}$$

Next, we prove Theorem 1. Before doing so, we remark that the conditions under which this theorem is stated, can be formulated in a slightly more explicit form. Indeed, let M^n be a minimal C-totally real submanifold of a Sasakian space form $\tilde{M}^{2n+1}(c)$. For each $p \in M^n$, we put

$$\mathscr{D}(p) = \{X \in T_pM^n;\ h(X, Y) = 0, \forall Y \in T_pM^n\}.$$

The geometric meaning of $\mathscr{D}$ is clear, namely $\mathscr{D}$ is the kernel of the second fundamental form h. In [2], it was shown that if $\dim \mathscr{D}(p)$ is constant, then it is completely integrable and its dimension is either n or $n-2$.

In view of this last result, we can restate Theorem 1 in the following equivalent form, which is better suited for technical application.

THEOREM 2. *Let $\tilde{M}^{2n+1}(c)$ be a Sasakian space form and M^n an n-dimensional $(n > 2)$ C-totally real submanifold with nonconstant scalar curvature such that:*

(i) $\delta_M = \frac{1}{8}(n-2)(n+1)(c+3)$;

(ii) *the distributions $\mathscr{D}$ and $\mathscr{D}^{\perp}$ are both completely integrable.*

Then M^n is, up to a homothety, locally congruent to an immersion

$$\psi : (0, \frac{\pi}{2}) \times_{\cos t} M^2 \times_{\sin t} S^{n-3} \to S^{2n+1}, \qquad \psi(t, p, q) = (\cos t)p + (\sin t)q,$$

where M^2 is a C-totally real minimal surface of S^5.

PROOF. By Corollary 4, we know that M^n is actually a minimal submanifold. Hence, by Proposition 5, there exists at every point $p \in M^n$ an orthonormal basis $\{e_1, \dots, e_n\} \subset T_pM^n$ such that

$$h(e_1, e_1) = \lambda\phi e_1, \; h(e_1, e_2) = -\lambda\phi e_2, \; h(e_2, e_2) = -\lambda\phi e_1, \; h(e_i, e_j) = 0, \; i, j > 2,$$

with $\lambda \neq 0$. We remark that in contrast to the situation studied in [7], λ need not be a constant. Following the line of proof of the Lemmas 4.2 and 4.3 of [6], we can extend $\{e_1, \dots, e_n\}$ to vector fields $\{E_1, \dots, E_n\}$, which satisfy the above relations on a neighborhood of the point $p \in M^n$.

We observe that M^n cannot be totally geodesic. Indeed, if M^n were totally geodesic, (2) shows that in this case M^n should in fact be a real space form. This would however imply that its scalar curvature is constant, which is excluded by assumption. Since we know from [2] that $\dim \mathscr{D} = n$ only occurs when M^n is totally geodesic, we conclude that the dimension of $\mathscr{D}$ is $n-2$.

Hence, we may assume that locally, $\mathscr{D}^{\perp}$ is spanned by $\{E_1, E_2\}$. So, there exists $E_3 \in \mathscr{D}$ (unique up to sign) such that

$$\nabla_{E_1} E_1 = bE_2 + aE_3,$$

where a, b are C^∞-functions with $a \neq 0$.

We must have $c \neq -3$. Otherwise, $\tilde{M}^{2n+1}$ is locally Euclidean and from the Gauss equation it follows that $R(E_1, E_3, E_1, E_3) = 0$. This implies that M^n is totally geodesic, which leads to a contradiction, as already remarked above. By a homothety, we can arrange that the ambient space has normalized $c = 1$.

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We denote $\gamma_{ij}^k = g(\nabla_{E_i} E_j, E_k)$. Using the Codazzi equation, we find

$$\gamma_{ij}^1 = \gamma_{ij}^2 = 0, \quad \gamma_{11}^i = \gamma_{22}^i, \quad \gamma_{12}^i = \gamma_{21}^i = 0, \quad \gamma_{i1}^2 = -\tfrac{1}{3}\gamma_{12}^i; \quad i, j \geq 3.$$

Indeed, let $i, j \geq 3$. The equation $(\tilde{\nabla}_{E_1} h)(E_i, E_j) = (\tilde{\nabla}_{E_i} h)(E_1, E_j)$ is equivalent to $h(E_1, \nabla_{E_i} E_j) = 0$, which implies $\gamma_{ij}^1 = 0$. Analogously, $\gamma_{ij}^2 = 0$.

Similarly, $(\tilde{\nabla}_{E_2} h)(E_1, E_1) = (\tilde{\nabla}_{E_1} h)(E_2, E_1)$ leads to $\gamma_{12}^i = -\gamma_{21}^i$. Since moreover $\mathscr{D}^\perp$ is involutive, we also have that $\gamma_{12}^i = \gamma_{21}^i$. Therefore $\gamma_{12}^i = 0$. Finally, from $(\tilde{\nabla}_{E_i} h)(E_1, E_1) = (\tilde{\nabla}_{E_1} h)(E_i, E_1)$ it follows that $E_i(\lambda) = \lambda\gamma_{11}^i$, which together with $(\tilde{\nabla}_{E_i} h)(E_2, E_2) = (\tilde{\nabla}_{E_2} h)(E_i, E_2)$ implies $\gamma_{22}^i = \gamma_{11}^i$.

In a similar way, we obtain

$$E_1(\lambda) = -3\lambda\gamma_{22}^1, \qquad E_2(\lambda) = 3\lambda\gamma_{11}^2. \tag{6}$$

Denoting $d = \gamma_{21}^2$, by the above relations we may write

$$\nabla_{E_2} E_1 = dE_2, \quad \nabla_{E_2} E_2 = -dE_1 + aE_3, \quad \nabla_{E_i} E_1 = \nabla_{E_i} E_2 = 0.$$

Here, we clearly see that the assumption for the scalar curvature τ to be nonconstant is essential for the present proof. Indeed with τ constant, (2) shows that λ would also be constant in this case. However, the above would then imply that $a = b = d = 0$. But with $a = b = d = 0$, the following final part of the proof is no longer applicable.

In order to finish the proof, it suffices to check Hiepko's condition from [8].

We denote by $\mathscr{T}_1 = \mathrm{span}\{E_3\}$ and $\mathscr{T}_2 = \mathrm{span}\{E_4, \ldots, E_n\}$. So, it is sufficient to prove that:

(a) $\mathscr{T}_1$ is totally geodesic;
(b) $\mathscr{T}_2$ is spherical and $\mathscr{D}$ is totally geodesic;
(c) $\mathscr{D}^\perp$ is spherical and $\mathscr{T}_1 \oplus \mathscr{D}^\perp$ is totally geodesic in M^n.

Indeed, we have $0 = R(E_1, E_3, E_i, E_1) = -a\gamma_{33}^i$, $\forall i \geq 4$, which implies $\gamma_{33}^i = 0$. So, we have that $\mathscr{T}_1$ is totally geodesic, thus proving (a).

Next, we prove (b). For $i, j \geq 4$

$$\delta_{ij} = R(E_i, E_1, E_j, E_1) = g(\nabla_{E_i}(bE_2 + aE_3) + \nabla_{\nabla_{E_1} E_i} E_1, E_j) = a\gamma_{i3}^j.$$

Then $\nabla_{E_i} E_j = -(1/a)\delta_{ij} E_3 + Y$, $Y \in \mathscr{T}_2$. $\mathscr{T}_2$ is spherical if and only if a is constant. But $E_i(a) = R(E_i, E_1, E_3, E_1) = 0$.

For (c), obviously $\mathscr{T}_1 \oplus \mathscr{D}^\perp$ is totally geodesic. It remains to show that $\mathscr{D}^\perp$ is spherical. Let $p, q \in \{1, 2\}$; then $\nabla_{E_p} E_q = a\delta_{pq} E_3 + Z$, $Z \in \mathscr{D}^\perp$. It follows that $\mathscr{D}^\perp$ is totally umbilical and its mean curvature vector aE_3 is parallel. Thus $\mathscr{D}^\perp$ is spherical.

Using Hiepko's result [8], it follows that M^n is locally isometric to a warped product manifold

$$M^n = M_0 \times_{\rho_1} M_1 \times_{\rho_2} M_2.$$

Recall that by a result of [2], $\mathscr{D}$ is integrable and has dimension $n-2$. Since now both $\mathscr{T}_1$ and $\mathscr{D}$ are totally geodesic in S^{2n+1} and $\mathscr{T}_2$ is totally umbilical in S^{2n+1}, $\dim M_0 = 1$ and $\dim M_2 = n-3$. So, in fact, M_2 is locally a totally geodesic sphere of dimension $n-3$: $M_2 = S^{n-3}$. Then, by counting dimensions, we see that M_1 being spherical is lying in S^5; since M^n is minimal, M_1 is minimal in S^5 too. The warping functions can be determined from the equations (6), but we do not need explicit calculations. As the decomposition of S^{2n+1} into a warped product whose first factor is 1-dimensional is unique up to isometries (see [5]), following a similar argument as in [4], we can assume that

$$\rho_1 = \cos t, \quad \rho_2 = \sin t.$$

Therefore, we obtain that M^n is indeed immersed as desired. □

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Zuivere en Toegepaste Differentiaalmeetkunde
Departement Wiskunde
K. U. Leuven
Celestijnenlaan 200 B
B - 3001 Leuven
Belgium
e-mail: filip.defever@wis.kuleuven.ac.be

Faculty of Mathematics
University of Bucharest
Str. Academiei 14
70109 Bucharest
Romania
e-mail: imihai@math.math.unibuc.ro

Departement Wiskunde
K. U. Leuven
Celestijnenlaan 200 B
B - 3001 Leuven
Belgium,
and
Group of Exact Sciences
K.U. Brussel
Vrijheidslaan 17
B - 1081 Brussel
Belgium
e-mail: leopold.verstraelen@wis.kuleuven.ac.be