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ON THE CENTER OF QUASI-CENTRAL BANACH ALGEBRAS WITH BOUNDED APPROXIMATE IDENTITY

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1. Introduction. Let A be a quasi-central complex Banach algebra with a bounded approximate identity and Prim A the structure space of A. In [**15**], we have shown that every central double centralizer T on A can be represented as a bounded continuous complex-valued function Φ_T on Prim A such that $Tx + P = \Phi_T(P)(x + P)$ for all $x \in A$ and $P \in$ Prim A when the center $Z(A)$ of A is completely regular. Here $x + P$ for $P \in$ Prim A denotes the canonical image of x in A/P. In particular, in the case of quasi-central C^*-algebras, this result is equivalent to the Dixmier's representation theorem of central double centralizers on C^*-algebras (see [**3**, Section 2] and [**9**, Theorem 5]).

In this paper, it is shown that if $Z(A)$ is completely regular then the space Prim A is locally quasi-compact and for each element z of $Z(A)$, Φ_{Lz} vanishes at infinity, where Lz for $z \in Z(A)$ is the central double centralizer on A defined by $Lz(x) = zx$ for all $x \in A$. Furthermore the following stronger result can be proved. If $Z(A)$ is completely regular then for each central double centralizer T on A such that μT belongs to the kernel of $Z(D(A))$-hull $\tau(Z(A))$, Φ_T vanishes at infinity. Here, as can be observed in Section 2, μ denotes the canonical isomorphism of the central double centralizer algebra onto the ideal center $Z(D(A))$ of A and τ denotes the canonical isomorphism of A into the dual space of $A^\# = \{f \cdot a : a \in A, f \in A^*\}$. Also, $Z(D(A))$-hull $\tau(Z(A))$ denotes the hull of $\tau(Z(A))$ in the structure space of $Z(D(A))$. Conversely, it is proved that if A is semi-simple and $Z(D(A))$ has a Hausdorff structure space then for each central double centralizer T on A such that Φ_T vanishes at infinity, μT belongs to the kernel of $Z(D(A))$-hull $\tau(Z(A))$.

C. Delaroche [**7**] has established that the center of an arbitrary quasi-central C^*-algebra A is canonically *-isomorphic with the algebra of all continuous complex-valued functions on Prim A which vanish at infinity. Moreover, R. J. Archbold [**1**] has given another proof of the above Delaroche's theorem. In his proof, the classical Dini's theorem is used. Our main theorems imply immediately the Delaroche's theorem from [**5**]. However, our proof of Delaroche's theorem is quite different from that given in [**7**] and [**1**].

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In order to get our main theorems, we need to obtain a compactification $K(\text{Prim } A)$ of Prim A such that there is a simple relation between the bounded continuous complex-valued functions on Prim A and that on $K(\text{Prim } A)$. If A_1 is the Banach algebra obtained from A adjointing by an identity element in the standard manner, then there is no simple relation between the bounded continuous complex-valued functions on Prim A and that on $K(\text{Prim } A)$ as was observed in [**5**] and [**9**]. However, we will show that this difficulty can be circumvented by considering an extension $U(A)$ of A different from A_1. In [**9**], this algebra $U(A)$ can be seen as the algebraic sum of A and its ideal center in the enveloping von Neumann algebra of A when A is a C^*-algebra.

Finally, we consider a Banach algebra which has a quasi-compact structure space as an application of the main theorems.

2. Notations and preliminaries. Let A be a complex Banach algebra with a bounded approximate identity $\{e_\alpha\}$ and A^* the dual space of A. We set

$$A^\# = \{f \in A^*:\lim_\alpha \|f \cdot e_\alpha - f\| = 0\}.$$

Here $f \cdot a$ for $a \in A$ and $f \in A^*$ denotes an element of A^* defined by $f \cdot a(x) = f(ax)$ for all $x \in A$. Actually, $A^\#$ is a closed subspace of A^* and $A^\# = \{f \cdot a: f \in A^*,\ a \in A\}$. Hence the dual space $(A^\#)^*$ of $A^\#$ becomes a Banach algebra under the restriction to $A^\#$ of the Arens product on the second dual space A^{**} of A (cf. [**4**, **6**, **15**]). Therefore there exists a norm reducing isomorphism τ of A into $(A^\#)^*$. Set

$$D(A) = \{F \in (A^\#)^*: F \cdot \tau(A) + \tau(A) \cdot F \subset \tau(A)\},$$

where the operation $\cdot$ denotes the restriction to $A^\#$ of the Arens product on A^{**}. We then see that $D(A)$ is a Banach subalgebra of $(A^\#)^*$ and $\tau(A)$ is a closed two-sided ideal of $D(A)$. The center $Z(D(A))$ of $D(A)$ is called the ideal center of A (see [**15**, Definition 2.1]). In fact, $Z(D(A))$ becomes the ideal center of A in the sense of [**9**] when A is a C^*-algebra.

Now let $M(A)$ be the double centralizer algebra of A and $Z(M(A))$ its center. An element T of $Z(M(A))$ is called the central double centralizer on A and may be identified with a bounded linear operator T on A such that $(Tx)y = x(Ty)$ for all $x, y \in A$. Let μ be the map of $Z(M(A))$ into $(A^\#)^*$ defined by

$$\mu T = \text{weak}^*\text{-}\lim_\alpha \tau(Te_\alpha)$$

for all $T \in Z(M(A))$. This map is called the Davenport's representation of $Z(M(A))$ and it is a continuous algebraic isomorphism of $Z(M(A))$ onto $Z(D(A))$ (see [**6**, Theorem 2.8] and [**15**, Lemma 2.2]).

For every algebra A, we will denote by $Z(A)$ the center of A and denote by Prim A the structure space of A, that is the set of all primitive ideals of A, with the hull-kernel topology. Also when B is a subset of A we denote by A-hull B, let us say the hull of B in Prim A, the subspace of Prim A consisting of all primitive ideals of A which contain B. However we will frequently write hull B in place of A-hull B for the hull of B in Prim A without confusion. An algebra A is said to be quasi-central provided hull $Z(A) = \emptyset$ (cf. [**15**, Definition 3.5]). Next for every topological space Ω, we denote by $C^b(\Omega)$ the Banach algebra of all bounded continuous complex-valued functions on Ω, with the supremum norm and denote by $C_0(\Omega)$ the Banach subalgebra of $C^b(\Omega)$ consisting of all functions f of $C^b(\Omega)$ such that for arbitrary $\epsilon > 0$, the set $\{\omega \in \Omega : |f(\omega)| \geqq \epsilon\}$ is quasi-compact (i.e., it satisfies the Borel–Lebesque axiom without necessarily being Hausdorff). An arbitrary element of $C_0(\Omega)$ is said to vanish at infinity. Let A be a complex Banach algebra with a bounded approximate identity. For every $P \in$ Prim A, there exists a unique element P' of Prim $D(A)$ such that $P' \cap \tau(A) = \tau(P)$ from [**12**, Theorem 2.6.6]. Furthermore for every $T \in Z(M(A))$ and $P \in$ Prim A, there exists a unique complex number $\Phi_T(P)$ such that $\mu T + P' = \Phi_T(P)(J + P')$, where J is an identity element of $(A^\#)^*$ (and hence belongs to $D(A)$). Also Φ_T is a bounded complex-valued function on Prim A for each $T \in Z(M(A))$ (cf. [**15**, Section 3]). Let $R(A)$ be the radical of A, that is the intersection of all primitive ideals of A, and $ZM_R(A)$ the closed ideal of $Z(M(A))$ consisting of all $T \in Z(M(A))$ such that $T(A) \subset R(A)$. In [**15**], we have shown the following two theorems.

THEOREM A. *If A is a complex Banach algebra with a bounded approximate identity such that* Prim $Z(D(A))$ *is Hausdorff, then the map $T \rightarrow \Phi_T$ is a continuous homomorphism of $Z(M(A))$ into C^b*(Prim A) *such that $Tx + P = \Phi_T(P)(x + P)$ for all $x \in A$ and $P \in$* Prim A, *the kernel of the homomorphism being equal to $ZM_R(A)$.*

THEOREM B. *If A is a quasi-central complex Banach algebra with a bounded approximate identity such that $Z(A)$ is completely regular, then the map $T \rightarrow \Phi_T$ is a continuous homomorphism of $Z(M(A))$ into C^b*(Prim A) *such that $Tx + P = \Phi_T(P)(x + P)$ for all $x \in A$ and $P \in$* Prim A.

In the next section, we first state that Prim A is locally quasi-compact under the conditions given in Theorem B. We will next give a characterization of an element T of $Z(M(A))$ such that $\Phi_T \in C_0$(Prim A) under some conditions.

3. Main theorems. We assume throughout this section that A is a quasi-central complex Banach algebra with a bounded approximate identity. We will know from Lemma 4.4 part (i) that $\tau(Z(A))$ is a closed ideal of $Z(D(A))$ and so denote by $\widetilde{\tau(Z(A))}$ the kernel of $Z(D(A))$-hull

$\tau(Z(A))$, that is

$$\widetilde{\tau(Z(A))} = \cap \{M \in \text{Prim } Z(D(A)) : \tau(Z(A)) \subset M\}.$$

Also for each element z of $Z(A)$, let Lz be an element of $Z(M(A))$ defined by $Lz(x) = zx$ for all $x \in A$. Our main theorems are the following.

THEOREM 3.1. *If the center $Z(A)$ of A is completely regular, then the space* Prim A *is locally quasi-compact and Φ_{Lz} belongs to $C_0(\text{Prim } A)$ for each element z of $Z(A)$.*

THEOREM 3.2. *If $Z(A)$ is completely regular, then Φ_T belongs to $C_0(\text{Prim } A)$ for each element T of $\mu^{-1}(\tau(Z(A)))$.*

THEOREM 3.3. *If A is semi-simple and if the ideal center $Z(D(A))$ of A has a Hausdorff structure space, then an arbitrary element T of $Z(M(A))$ such that $\Phi_T \in C_0(\text{Prim } A)$ belongs to $\mu^{-1}(\tau(Z(A)))$.*

Theorem 3.2 and 3.3 imply immediately that an element T of $Z(M(A))$ belongs to $\mu^{-1}(\widetilde{\tau(Z(A))})$ if and only if Φ_T belongs to $C_0(\text{Prim } A)$ provided A is a semi-simple and the ideal center of A has a Hausdorff structure space.

Now in order to prove these theorems, we have to prepare some lemmas. Those lemmas will be stated in the next section. In the remainder of this paper, we denote by χ_M the non-zero homomorphism of A onto the complex field induced by $M \in \text{Prim } A$ and denote by Hom A the carrier space of A, with A-topology whenever A is a complex commutative Banach algebra.

4. Lemmas. The first lemma can be observed in the proof of [**15**, Theorem 3.6] and hence we will omit the proof.

LEMMA 4.1. *Let A be a quasi-central complex Banach algebra with a bounded approximate identity such that $Z(A)$ is completely regular. Then*

(4.1.1) $\quad \chi_{P' \cap Z(D(A))}\tau | Z(A) = \chi_{P \cap Z(A)}$

and

(4.1.2) $\quad \Phi_T(P) = \chi_{P' \cap Z(D(A))}(\mu T)$

for all $P \in$ Prim A and $T \in Z(M(A))$.

The following result is a basic lemma for Theorem 3.1 and 3.2, and it follows immediately from [**12**, Theorem 3.6.15].

LEMMA 4.2. *Let A be a complex commutative Banach algebra and K a closed subset of* Prim A. *Let u be an element of A such that $|\chi_P(u)| \geqq \delta > 0$ for all $P \in K$. Then* ker K *is modular and there exists an element e of A such that $\chi_P(e) = 1$ for all $P \in K$.*

The following lemma will be used only to show Theorem 3.2.

LEMMA 4.3. (i) *Let A be an algebra, I a two-sided ideal of A and $\tilde{I}$ the kernel of hull I. Then $\tilde{I}/I$ is a radical algebra.*

(ii) *Let A be a commutative Banach algebra and I a closed ideal of A. If I is completely regular, then so is $\tilde{I}$.*

Proof. Suppose first that $\tilde{I}/I$ is not radical. Then there exists a primitive ideal P of $\tilde{I}/I$ from [**12**, Theorem 2.3.2 part (i)] and hence there exists a primitive ideal Q of $\tilde{I}$ such that $I \subset Q$ and $Q/I = P$. Furthermore, there exists a primitive ideal R of A such that $R \cap \tilde{I} = Q$. Then R belongs to hull I and so it contains $\tilde{I}$. It follows that $\tilde{I} = R \cap \tilde{I} = Q$. But this is impossible since Q is a proper ideal of $\tilde{I}$ and completes the proof of part (i).

Suppose next that A is a commutative Banach algebra and I is a closed ideal of A which is completely regular. By part (i), we have $\tilde{I}$-hull $I = \emptyset$. Hence there is a homeomorphism (which we denote by ϕ) of Prim $\tilde{I}$ onto Prim I from [**12**, Theorem 2.6.6]. Then Prim $\tilde{I}$ is Hausdorff from the complete regularity of I. Now let M be a fixed element of Prim $\tilde{I}$. Then $\phi(M)$ has a neighbourhood V such that ker V is modular from the complete regularity of I. Therefore $\bar{V}$ is quasi-compact and hence so is $\overline{\phi^{-1}(V)}$, where the bar denotes the hull-kernel closure. Thus ker $\overline{\phi^{-1}(V)}$ is modular from [**12**, Theorem 3.6.7]. Note also that

$$\ker \overline{\phi^{-1}(V)} \subset \ker \phi^{-1}(V).$$

Then $\phi^{-1}(V)$ is a neighbourhood of M such that ker $\phi^{-1}(V)$ is modular. Noting that since I and $\tilde{I}$ are commutative Banach algebras their structure spaces and strong structure spaces coincide, part (ii) follows.

We now define $U(A)$ to be the set

$$U(A) = \tau(A) + Z(D(A)).$$

Then $U(A)$ is a subalgebra of $D(A)$ since $\tau(A)$ is a two-sided ideal of $D(A)$. If A is a C^*-algebra, then $U(A)$ is automatically complete. However, $U(A)$ is not generally complete. This non-completeness puts us to trouble but it could be circumvented by showing that the center of the difference algebra $U(A)/R$ modulo $R \in$ Prim $U(A)$ reduces to the complex field. We may further reduce Theorem 3.3 by considering the algebra $U(A)$ and its structure space.

LEMMA 4.4. *Let A be a complex Banach algebra with a bounded approximate identity. Then*

(i) $\tau(Z(A)) = Z(D(A)) \cap \tau(A)$,

(ii) $Z(\boldsymbol{U}(A)) = Z(D(A))$.

Proof. We first show (i). It is obvious that

$$\tau(Z(A)) \supset Z(D(A)) \cap \tau(A).$$

Let $z \in Z(A)$ and $F \in D(A)$. Note that $\tau(A)$ is weak*-dense in $(A^{\#})^*$ as was seen in the proof of [**15**, Lemma 2.2]. Then choose a net $\{x_\lambda\}$ in A such that

$$F = \text{weak*-}\lim_\lambda \tau(x_\lambda).$$

By [**6**, Lemma 2.6], we have

$$\begin{aligned} F \cdot \tau(z) &= \text{weak*-}\lim_\lambda \tau(x_\lambda) \cdot \tau(z) \\ &= \text{weak*-}\lim_\lambda \tau(z) \cdot \tau(x_\lambda) \\ &= \tau(z) \cdot F. \end{aligned}$$

It follows that $\tau(Z(A)) \subset Z(D(A))$ and hence

$$\tau(Z(A)) = Z(D(A)) \cap \tau(A).$$

We next show (ii). It is clear that $Z(D(A)) \subset Z(U(A))$. To show the reverse inclusion relation, let $F \in Z(U(A))$. So we can write $F = \tau(a) + Z$, where $a \in A$ and $Z \in Z(D(A))$. For each $x \in A$, we have

$$\tau(xa) = \tau(x) \cdot F - \tau(x) \cdot Z = F \cdot \tau(x) - Z \cdot \tau(x) = \tau(ax),$$

so that $xa = ax$ since τ is one-to-one. Thus $a \in Z(A)$ and hence $\tau(a) \in Z(D(A))$ from (i). It follows that $F \in Z(D(A))$. In other words, $Z(U(A)) \subset Z(D(A))$ and (ii) is proved.

Remark. The above lemma is an extension of [**9**, Theorem 8] and it also implies that $\tau(Z(A))$ is a closed two-sided ideal of $Z(D(A))$. If A has an identity element, then $D(A) = U(A) = \tau(A)$.

LEMMA 4.5. *Let A be a complex Banach algebra with a bounded approximate identity. Then there exists a homeomorphism Ψ of $U(A)$-hull $\tau(A)$ onto $Z(D(A))$-hull $\tau(Z(A))$ such that*

(4.5.1) $\quad \tau(A) + \Psi(R) = R,$

(4.5.2) $\quad \Psi(R) = R \cap Z(D(A))$

for all $R \in U(A)$-hull $\tau(A)$.

Proof. By [**12**, Theorem 2.6.6], the map $\Psi_1 : R \to R/\tau(A)$ is a homeomorphism of $U(A)$-hull $\tau(A)$ onto Prim $U(A)/\tau(A)$ and the map $\Psi_2 : Q \to Q/\tau(Z(A))$ is also a homeomorphism of $Z(D(A))$-hull $\tau(Z(A))$ onto Prim $Z(D(A))/\tau(Z(A))$ (since $\tau(Z(A))$ is a two-sided ideal of $Z(D(A))$. Moreover, the map $\sigma : Z + \tau(Z(A)) \to Z + \tau(A)$ is an algebraic isomorphism of $Z(D(A))/\tau(Z(A))$ onto $U(A)/\tau(A)$ from Lemma 4.4 part (i). Therefore, the map $\Psi_3 : M \to \sigma(M)$ is a homeomorphism of Prim $Z(D(A))/\tau(Z(A))$ onto Prim $U(A)/\tau(A)$. Set

$$\Psi = \Psi_2^{-1} \cdot \Psi_3^{-1} \cdot \Psi_1.$$

We thus obtain the homeomorphism Ψ of $U(A)$-hull $\tau(A)$ onto $Z(D(A))$-hull $\tau(Z(A))$. Let R be an arbitrary element of $U(A)$-hull $\tau(A)$. We first show that Ψ satisfies (4.5.1). In fact, if $F \in R$ then

$$F + \tau(A) \in \Psi_1(R) = \Psi_3 \cdot \Psi_2 \cdot \Psi(R).$$

Also,

$$\Psi_2 \cdot \Psi(R) = \Psi(R)/\tau(Z(A))$$

and hence there exists an element Z of $\Psi(R)$ such that

$$F + \tau(A) = \sigma(Z + \tau(Z(A))).$$

This shows that $F - Z \in \tau(A)$. In other words, $R \subset \tau(A) + \Psi(R)$. Now let $Z \in \Psi(R)$. Then

$$Z + \tau(Z(A)) \in \Psi_2 \cdot \Psi(R)$$

and so

$$\sigma(Z + \tau(Z(A))) \in \Psi_1(R)$$

since $\Psi_1 = \Psi_3 \cdot \Psi_2 \cdot \Psi$. Hence there exists an element F' of R such that $Z - F' \in \tau(A)$. Since $\tau(A) \subset R$, it follows that $Z \in R$ and hence $\Psi(R) \subset R$. Therefore $\tau(A) + \Psi(\mathrm{R}) \subset R$ and so Ψ satisfies (4.5.1).

We next show that Ψ satisfies (4.5.2). It follows immediately from (4.5.1) that

$$\Psi(R) \subset R \cap Z(D(A)).$$

To show $R \cap Z(D(A)) \subset \Psi(R)$, let $F \in R \cap Z(D(A))$. Then there exist $a \in A$ and $Z \in \Psi(R)$ such that $F = \tau(a) + Z$ from (4.5.1). Therefore, $\tau(a) = F - Z \in Z(D(A))$ and hence

$$\tau(a) \in Z(D(A)) \cap \tau(A) = \tau(Z(A)) \subset \Psi(R)$$

from Lemma 4.4 part (i) and the construction of Ψ. It follows that

$$F \in \Psi(R) + \Psi(R) = \Psi(R)$$

and so

$$R \cap Z(D(A)) \subset \Psi(R).$$

Thus it is shown that Ψ satisfies (4.5.2).

LEMMA 4.6. *Let A be a complex normed algebra and M a maximal modular left ideal of A. Let P be the quotient $M:A$ of M in A, that is the two-sided ideal of A consisting of all $a \in A$ such that $aA \subset M$. If P does not contain the center $Z(A)$ of A, and if M is closed, then $Z(A/P)$ reduces to the complex field and $P \cap Z(A)$ belongs to* Prim $Z(A)$.

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Proof. Note that the difference space A/M is a normed space under the infimum norm

$$\|x + M\| = \inf \{\|x + m\| : m \in M\}$$

since M is closed. Moreover, by considering a homomorphism ρ of A into the algebra of all bounded linear operators on A/M defined by $\rho(a)(x + M) = ax + M$ for all $a, x \in A$, we can regard A/M as a left A-module. Denote by $\mathrm{Hom}_A(A/M)$ the A-endomorphism ring on A/M. Now choose an element u of A such that $u \notin M$. Then $\rho(A)(u + M) = A/M$ since ρ is a strictly irreducible representation of A on A/M. We first show that $\mathrm{Hom}_A(A/M)$ is a complex division algebra. In fact, let $T \in \mathrm{Hom}_A(A/M)$, $x \in A$ and λ a complex number. Then there exists an element a of A such that $\rho(a)(u + M) = x + M$. We therefore have

$$\begin{aligned} T(\lambda(x + M)) &= T\rho(\lambda a)(u + M) \\ &= \rho(\lambda a)T(u + M) \\ &= \lambda T\rho(a)(u + M) \\ &= \lambda T(x + M). \end{aligned}$$

Hence T is linear. In other words, $\mathrm{Hom}_A(A/M)$ is a complex algebra. Also, since ρ is strictly irreducible, A/M is an A-simple module, so that $\mathrm{Hom}_A(A/M)$ is a division ring from Schur's lemma.

We next show that $\mathrm{Hom}_A(A/M)$ is normed. Actually, set

$$|x + M| = \inf \{\|\rho(a)\| : x + M = \rho(a)(u + M)\}$$

for each $x \in A$. Here $\|\rho(a)\|$ denotes the operator norm of $\rho(a)$. It is easy to see that $|x + M|$ is a norm of A/M. We now have

$$\begin{aligned} (4.6.1)\quad |\rho(a)(x + M)| & \\ &= \inf \{\|\rho(b)\| : \rho(b)(u + M) = \rho(a)(x + M)\} \\ &\leqq \inf \{\|\rho(a)\rho(c)\| : \rho(c)(u + M) = x + M\} \\ &\leqq \|\rho(a)\|\, |x + M| \end{aligned}$$

for all $a, x \in A$. We can show that

$$|T(x + M)| \leqq |T(u + M)|\, |x + M|$$

for each $T \in \mathrm{Hom}_A(A/M)$ and $x \in A$. In fact, if a is an element of A such that $x + M = \rho(a)(u + M)$, then

$$\begin{aligned} |T(x + M)| &= |T\rho(a)(u + M)| \\ &= |\rho(a)T(u + M)| \\ &\leqq \|\rho(a)\|\, |T(u + M)| \text{ (from 4.6.1).} \end{aligned}$$

Taking the infimum over all such a, we get the desired inequality. We thus see that T is bounded with respect to $|x + M|$, so that $\mathrm{Hom}_A(A/M)$

can be normed. These observations imply that $\mathrm{Hom}_A(A/M)$ reduces to the complex field from the Mazur–Gelfand theorem on normed division algebras. Note that

$$\rho(Z(A)) \neq \{0\} \quad \text{and} \quad \rho(Z(A)) \subset Z(\rho(A)) \subset \mathrm{Hom}_A(A/M).$$

Hence $Z(\rho(A))$ is isomorphic with the complex field. Since $\rho(A)$ is isomorphic with A/P, if follows that $Z(A/P)$ reduces to the complex field.

Finally, we observe that $P \cap Z(A) \in \mathrm{Prim}\, Z(A)$. The map

$$z + P \cap Z(A) \rightarrow z + P$$

is an algebraic isomorphism of $Z(A)/P \cap Z(A)$ into $Z(A/P)$. Since $Z(A)/P \cap Z(A) \neq \{0\}$ and Z(A/P) is isomorphic with the complex field, it follows that $P \cap Z(A)$ is a maximal modular ideal of $Z(A)$. In other words, $P \cap Z(A) \in \mathrm{Prim}\, Z(A)$.

Remark. In the proof of the above lemma, we have referred to the proof of [**12**, Lemma 2.4.4]. Also, we can by the same method show that if A is a real algebra then $Z(A/P)$ is isomorphic to the real or complex field. Moreover, these results contain [**12**, Corollary 2.4.5 and 3.1.2].

LEMMA 4.7. *Let A be a normed algebra, R a right ideal of A and L a left ideal of A. If R has a bounded left approximate identity $\{u_\alpha : \alpha \in \Lambda\}$, then $\overline{R \cap L} = \bar{R} \cap \bar{L}$, where the bar denotes the norm-closure in A.*

Proof. It is trivial that $\overline{R \cap L} \subset \bar{R} \cap \bar{L}$. To show the reverse inclusion let $x \in \bar{R} \cap \bar{L}$. Choose a sequence $\{a_n\}$ in R such that $\lim_n \|a_n - x\| = 0$ and a sequence $\{b_n\}$ in L such that $\lim_n \|b_n - x\| = 0$. For any positive number ϵ, there exists an integer $n(\epsilon)$ and an element $\alpha(\epsilon)$ of Λ such that

$$\|a_{n(\epsilon)} - x\| < \min\{\epsilon/4, \epsilon/4d\},$$
$$\|b_{n(\epsilon)} - x\| < \epsilon/4d$$

and

$$\|u_{\alpha(\epsilon)}a_{n(\epsilon)} - a_{n(\epsilon)}\| < \epsilon/4,$$

where d denotes the bound on $\{u_\alpha : \alpha \in \Lambda\}$. We then have

$$\begin{aligned}
&\|u_{\alpha(\epsilon)}b_{n(\epsilon)} - x\| \\
&\quad \leqq \|u_{\alpha(\epsilon)}b_{n(\epsilon)} - u_{\alpha(\epsilon)}x\| + \|u_{\alpha(\epsilon)}x - u_{\alpha(\epsilon)}a_{n(\epsilon)}\| \\
&\quad + \|u_{\alpha(\epsilon)}a_{n(\epsilon)} - a_{n(\epsilon)}\| + \|a_{n(\epsilon)} - x\| \\
&\quad < d \times (\epsilon/4d) + d \times (\epsilon/4d) + \epsilon/4 + \ /4 \\
&\quad = \epsilon.
\end{aligned}$$

We thus obtain that

$$\lim_{\epsilon \downarrow 0} \|u_{\alpha(\epsilon)}b_{n(\epsilon)} - x\| = 0.$$

Since each $u_{\alpha(\epsilon)}b_{n(\epsilon)}$ $(\epsilon > 0)$ is an element of $R \cap L$, it follows that $x \in \overline{R \cap L}$, so that the reverse inclusion relation is proved.

The following result covers the non-completeness of the algebra $U(A)$.

LEMMA 4.8. *Let A be a complex Banach algebra with a bounded approximate identity $\{e_\alpha : \alpha \in \Lambda\}$ and R an arbitrary element of* Prim $U(A)$. *Then the center $Z(U(A)/R)$ reduces to the complex field and $R \cap Z(D(A))$ belongs to* Prim $Z(D(A))$.

Proof. Case 1. $R \in U(A)$-hull $\tau(A)$. By Lemma 4.5, $R \cap Z(D(A))$ belongs to Prim $Z(D(A))$. Hence $Z(D(A))/R \cap Z(D(A))$ reduces to the complex field. Moreover, since $\tau(A) \subset R$, the map

$$Z + R \cap Z(D(A)) \rightarrow Z + R$$

is an algebraic isomorphism of $Z(D(A))/R \cap Z(D(A))$ onto $U(A)/R$. In other words, $Z(U(A))/R = U(A)/R$ reduces to the complex field.

Case 2. $R \notin U(A)$-hull $\tau(A)$. Choose a maximal modular left ideal M of $U(A)$ such that $R = M : U(A)$. Since R does not contain $\tau(A)$, $M \cap \tau(A)$ is a proper left ideal of $\tau(A)$. Furthermore the ideal $M \cap \tau(A)$ is modular. In fact, the maximality of M and $M \cap \tau(A) \neq \tau(A)$ imply that

$$(4.8.1) \quad \tau(A) + M = U(A).$$

Choose $a \in A$ and $m \in M$ such that $J = \tau(a) + m$ (J is the identity element of $D(A)$ and hence $U(A)$). Then

$$\tau(A) \cdot (1 - \tau(a)) = \tau(A) \cdot m \in \tau(A) \cap M,$$

so that $M \cap \tau(A)$ is modular. We now assert that $M \cap \tau(A)$ is maximal. To see this, let L be any proper left ideal of $\tau(A)$ which contains $M \cap \tau(A)$. Since $\tau(A)$ is a Banach algebra and L is modular, the norm-closure $\bar{L}$ of L is also a proper left ideal of $\tau(A)$. Furthermore $M \cdot \bar{L} \subset \bar{L}$. Indeed, let $m \in M$ and $a \in A$ with $\tau(a) \in \bar{L}$. Choose a sequence $\{\tau(a_n)\}$ in L such that

$$\lim_n \|\tau(a_n) - \tau(a)\| = 0.$$

For any positive number ϵ, there are an integer $n(\epsilon)$ and an element $\alpha(\epsilon) \in \Lambda$ such that

$$\|\tau(a_{n(\epsilon)}) - \tau(a)\| < \epsilon/2\|m\|$$

and

$$\|\tau(e_{\alpha(\epsilon)}) \cdot \tau(a_{n(\epsilon)}) - \tau(a_{n(\epsilon)})\| < \epsilon/2\|m\|.$$

We then have

$$\begin{aligned}\|m\cdot\tau(e_{\alpha(\epsilon)})\cdot\tau(a_{n(\epsilon)}) - m\cdot\tau(a)\| \\ &\leqq \|m\cdot\tau(e_{\alpha(\epsilon)})\cdot\tau(a_{n(\epsilon)}) - m\cdot\tau(a_{n(\epsilon)})\| \\ &+ \|m\cdot\tau(a_{n(\epsilon)}) - m\cdot\tau(a)\| \\ &\leqq \|m\|(\epsilon/2\|m\| + \epsilon/2\|m\|) \\ &= \epsilon.\end{aligned}$$

Hence

$$\lim_{\epsilon\downarrow 0}\|m\cdot\tau(e_{\alpha(\epsilon)})\cdot\tau(a_{n(\epsilon)}) - m\cdot\tau(a)\| = 0.$$

Since each $m\cdot\tau(e_{\alpha(\epsilon)})\cdot\tau(a_{n(\epsilon)})$ ($\epsilon > 0$) is an element of L, it follows that $m\cdot\tau(a)\in\bar{L}$. In other words, $M\cdot\bar{L}\subset\bar{L}$. Now set

$$M' = M + \bar{L}.$$

We then have

$$\begin{aligned}U(A)\cdot M' &\subset U(A)\cdot M + U(A)\cdot\bar{L} \\ &\subset M + \tau(A)\cdot\bar{L} + M\cdot\bar{L} \text{ (from 4.8.1)} \\ &\subset M + \bar{L} + \bar{L} \\ &= M'.\end{aligned}$$

Thus M' is a left ideal of $U(A)$ which contains M and so M must be either equal to $U(A)$ or M. Assume that $M' = U(A)$. Choose $m_0\in M$ and $\tau(a_0)\in\bar{L}$ such that $J = m_0 + \tau(a_0)$. Then

$$\begin{aligned}\tau(A) &\subset \tau(A)\cdot m_0 + \tau(A)\cdot\tau(a_0) \\ &\subset \tau(A)\cap M + \bar{L} \\ &\subset L + \bar{L} = \bar{L},\end{aligned}$$

so that $\tau(A) = \bar{L}$. This contradicts that $\bar{L}$ is proper. We thus obtain that $M' = M$. This implies the following:

$$\bar{L}\subset M'\cap\tau(A) = M\cap\tau(A)\subset L\subset\bar{L},$$

so that $M\cap\tau(A) = L$. Therefore the left ideal $M\cap\tau(A)$ is maximal. We now next show that M is closed. Suppose, on the contrary, that $M\neq\bar{M}$ (the norm-closure of M). The maximality of M implies that $\bar{M} = U(A)$. Notice that the maximality and the modularity of $M\cap\tau(A)$ imply that $M\cap\tau(A)$ is closed since $\tau(A)$ is a Banach algebra. We then have, from Lemma 4.7, that

$$\tau(A) = \tau(A)\cap\bar{M} = \overline{\tau(A)\cap M} = \tau(A)\cap M.$$

However this is impossible since M does not contain $\tau(A)$, and hence M is closed. It follows from Lemma 4.6 and 4.4 part (ii) that $Z(U(A)/R)$ reduces to the complex field and $R\cap Z(D(A))$ belongs to Prim $Z(D(A))$.

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LEMMA 4.9. *Let A, B be two algebras and η a homomorphism of B into A. If $\eta^{-1}(P) \in$ Prim B for each $P \in$ Prim A, then the map*: $P \to \eta^{-1}(P)$ *is continuous on* Prim A.

Proof. Set $T_\eta(P) = \eta^{-1}(P)$ for each $P \in$ Prim A. Let K be a closed subset of Prim B and P_0 a limit point of $T_\eta^{-1}(K)$. Then we have

$$P_0 \supset \ker T_\eta^{-1}(K) = \cap \{P \in \text{Prim } A : \eta^{-1}(P) \in K\},$$

and hence

$$T_\eta(P_0) \supset \cap \{\eta^{-1}(P) : \eta^{-1}(P) \in K\} \supset \ker K.$$

Therefore $T_\eta(P_0)$ belongs to the hull-kernel closure of K and hence K. Thus $T_\eta^{-1}(K)$ is also closed. In other words, the map: $P \to \eta^{-1}(P)$ is continuous on Prim A.

LEMMA 4.10. *Let A be a complex Banach algebra with a bounded approximate identity. Then $R \to R \cap Z(D(A))$ is a continuous map of* Prim $U(A)$ *into* Prim $Z(D(A))$.

Proof. Let i be the inclusion map of $Z(D(A))$ into $U(A)$. Note that for each element R of Prim $U(A)$, $i^{-1}(R) = R \cap Z(D(A))$ and hence $i^{-1}(R)$ belongs to Prim $Z(D(A))$ from Lemma 4.8. Therefore the desired result is easily obtained from Lemma 4.9.

5. The construction of Φ_T^U and its application. We assume throughout this section that A is a complex Banach algebra with a bounded approximate identity. Here we will construct a bounded complex-valued function Φ_T^U on Prim A for each $T \in Z(M(A))$, which is similar to Φ_T. Also as an application of this construction, we see that Prim $U(A)$ is a desired compactification K(Prim A) of Prim A.

Let T be a central double centralizer on A and R a primitive ideal of $U(A)$. Then $\mu T \in Z(D(A))$ from [**15**, Lemma 2.2], so that $\mu T + R$ belongs to the center of $U(A)/\text{R}$. Hence there exists a unique complex number $\Phi_T^U(R)$ such that

$$\mu T + R = \Phi_T^U(R)(J + R)$$

from Lemma 4.8. Moreover,

$$\begin{aligned} |\Phi_T^U(R)| &\leqq \|\Phi_T^U(R)(J + R)\| = \|\mu T + R\| \\ &\leqq \|\mu T\| \leqq \|\mu\| \, \|T\|. \end{aligned}$$

We thus obtain a bounded complex-valued function Φ_T^U on Prim $U(A)$ for each $T \in Z(M(A))$. Also by [**12**, Theorem 2.6.6], there exists a unique homeomorphism ϕ of Prim A into Prim $U(A)$ such that $\phi(P) \cap \tau(A) = \tau(P)$ for all $P \in$ Prim A.

THEOREM 5.1. *If the ideal center $Z(D(A))$ of A has a Hausdorff structure space, then the map $T \to \Phi_T{}^U$ is a continuous homomorphism of $Z(M(A))$ into $C(\text{Prim } U(A)) = C^b(\text{Prim } U(A))$ such that $\Phi_T = \Phi_T{}^U \cdot \phi$ for all $T \in Z(M(A))$.*

Proof. We will refer to the proof of Theorem A. The map $T \to \Phi_T{}^U$ is a continuous homomorphism of $Z(M(A))$ into the Banach algebra of all bounded complex-valued functions on Prim $U(A)$ from the construction of $\Phi_T{}^U$. We first show that for each $T \in \dot{Z}(M(A))$, $\Phi_T{}^U$ is continuous on Prim $U(A)$. By Lemma 4.10, $R \to R \cap Z(D(A))$ is a continuous map of Prim $U(A)$ into Prim $Z(D(A))$. Also since Prim $Z(D(A))$ is Hausdorff, the algebra $Z(D(A))$ is completely regular. It follows from [**12**, Theorem 3.7.1] that the map $Q \to \chi_Q$ is a homeomorphism of Prim $Z(D(A))$ onto the carrier space Hom $Z(D(A))$ of $Z(D(A))$. We thus observe that the map

$$R \to \chi_{R \cap Z(D(A))}(Z)$$

is a continuous complex-valued function on Prim $U(A)$ for each $Z \in Z(D(A))$. Let $T \in Z(M(A))$ and $R \in$ Prim $U(A)$. Since

$$\mu T + R = \Phi_T{}^U(R)(J + R),$$

we have that

$$\mu T - \Phi_T{}^U(R)J \in R \cap Z(D(A)).$$

It follows that

$$\Phi_T{}^U(R) = \chi_{R \cap Z(D(A))}(\mu T).$$

Then $\Phi_T{}^U$ is continuous on Prim $U(A)$ from the above argument.

We next show that $\Phi_T = \Phi_T{}^U \cdot \phi$ for all $T \in Z(M(A))$. To see this, let $T \in Z(M(A))$ and $P \in$ Prim A. Choose an element x_0 of A with $x_0 + P \neq 0$. Then

$$\Phi_T(P)(x_0 + P) = Tx_0 + P$$

from Theorem A. On the other hand, $\Phi_T{}^U(\phi(P))J - \mu T$ belongs to $\phi(P)$, and hence we have

$$\begin{aligned}\tau(\Phi_T{}^U(\phi(P))x_0 - Tx_0) &= (\Phi_T{}^U(\phi(P))J - \mu T) \cdot \tau(x_0)\\ &\in \phi(P) \cap \tau(A) = \tau(P).\end{aligned}$$

Then $\Phi_T{}^U(\phi(P))x_0 - Tx_0 \in P$ since τ is one-to-one. Therefore we have that

$$\Phi_T{}^U(\phi(P))(x_0 + P) = Tx_0 + P = \Phi_T(P)(x_0 + P).$$

It follows that $\Phi_T{}^U \cdot \phi(P) = \Phi_T(P)$ and the proof is complete.

The following result is a Urysohn's lemma for the structure space of an arbitrary algebra and it is used to show the theorem below.

LEMMA 5.2. *Let B be an arbitrary algebra. Let K_0 be a closed subset of* Prim B *and K_1 a closed subset of* Prim B *disjoint from K_0 such that* ker K_1 *is modular. Then there exists an element u of* ker K_0 *such that $u + P = e + P$ for all $P \in K_1$, where e is an identity for B modulo* ker K_1.

Proof. Note that $\ker K_0 + \ker K_1 = B$. In fact, if $\ker K_0 + \ker K_1 \neq B$, then there exists a maximal modular two-sided ideal M of A such that $\ker K_0 + \ker K_1 \subset M$ since $\ker K_0 + \ker K_1$ is modular. However M belongs to Prim B, and hence $M \in K_0 \cap K_1$. This contradicts that $K_0 \cap K_1 = \emptyset$. Thus there exist $x_0 \in \ker K_0$ and $x_1 \in \ker K_1$ such that $x_0 + x_1 = e$. Setting $u = x_0$, we see that u is the desired element.

Remark. There is the Urysohn's lemma for the strong structure space of an arbitrary algebra (see [**12**, Lemma 2.6.9]). Also there are the Urysohn's lemmas for normed algebras and C^*-algebras (see [**11**, Lemma 1] and [**13**, Theorem]).

THEOREM 5.3. *If A is quasi-central and semi-simple, and if $Z(D(A))$ has a Hausdorff structure space, then $\phi($*Prim *$A)$ is dense in* Prim $U(A)$.

Proof. Suppose, on the contrary, that $\phi(\text{Prim } A)$ is not dense in Prim $U(A)$. Choose an element R_0 of Prim $U(A)$ which does not belong to $\overline{\phi(\text{Prim } A)}$. Here the bar denotes the hull-kernel closure in Prim $U(A)$. Recall that $R \to R \cap Z(D(A))$ is a continuous map, let us say θ, of Prim $U(A)$ into Prim $Z(D(A))$ from Lemma 4.10. Since $U(A)$ possesses the identity element J, the space Prim $U(A)$ is quasi-compact and hence so is $\overline{\phi(\text{Prim } A)}$. Now we show that $\theta(R_0)$ does not belong to $\theta(\overline{\phi(\text{Prim } A)})$. Indeed, assume that there exists an element R_1 of $\overline{\phi(\text{Prim } A)}$ such that $\theta(R_0) = \theta(R_1)$. If R_1 belongs to $\phi(\text{Prim } A)$, then there exists $P_1 \in \text{Prim } A$ such that $R_1 = \phi(P_1)$. Since R_0 does not belong to $\phi(\text{Prim } A)$, $\tau(A) \subset R_0$ and hence $\tau(Z(A)) \subset R_0$. Note also that

$$R_0 \cap Z(D(A)) = \theta(R_0) = \theta(R_1) = \phi(P_1) \cap Z(D(A)).$$

We therefore have

$$\begin{aligned} \tau(Z(A)) &= R_0 \cap \tau(Z(A)) \\ &= R_0 \cap Z(D(A)) \cap \tau(Z(A)) \\ &\qquad\qquad\qquad\qquad \text{(from Lemma 4.4 part (i))} \\ &= \phi(P_1) \cap Z(D(A)) \cap \tau(Z(A)) \\ &= \tau(P_1) \cap \tau(Z(A)). \end{aligned}$$

It follows that $\tau(Z(A)) \subset \tau(P_1)$ and so $Z(A) \subset P_1$ since τ is one-to-one. But this is impossible since A is quasi-central. On the other hand, if R_1 does not belong to $\phi(\text{Prim } A)$, then $R_1 \in U(A)$-hull $\tau(A)$. Notice also that $R_0 \in U(A)$-hull $\tau(A)$. Hence, we have from (4.5.2) that

$$\begin{aligned}\Psi(R_0) &= R_0 \cap Z(D(A)) = \theta(R_0) = \theta(R_1)\\ &= R_1 \cap Z(D(A)) = \Psi(R_1).\end{aligned}$$

Here Ψ denotes the homeomorphism of $U(A)$-hull $\tau(A)$ onto $Z(D(A))$-hull $\tau(Z(A))$ given in Lemma 4.5. By the above equality, we have $R_0 = R_1$. However this is also impossible since

$$R_0 \notin \overline{\phi(\text{Prim } A)} \quad \text{and} \quad R_1 \in \overline{\phi(\text{Prim } A)}.$$

Then these observations imply that $\theta(R_0)$ does not belong to $\theta(\overline{\phi(\text{Prim } A)})$. Now since Prim $Z(D(A))$ is Hausdorff, the quasi-compact subset $\theta(\overline{\phi(\text{Prim } A)})$ of Prim $Z(D(A))$ is closed. It follows from Lemma 5.2 that there exists an element Z_0 of $Z(D(A))$ such that

$$Z_0 + \theta(R_0) = J + \theta(R_0)$$

and

$$Z_0 \in Q \text{ for all } Q \in \theta(\overline{\phi(\text{Prim } A)}).$$

Set $T_0 = \mu^{-1}(Z_0)$, so that $T_0 \in Z(M(A))$. Since

$$Z_0 - J \in \theta(R_0) = R_0 \cap Z(D(A)) \subset R_0,$$

it follows that

$$\begin{aligned}|\Phi_T{}^U(R_0)|\, \|J + R_0\| &= \|\mu T_0 + R_0\| = \|Z_0 + R_0\|\\ &= \|J + R_0\|,\end{aligned}$$

so that $|\Phi_{T_0}{}^U(R_0)| = 1$. On the other hand, for each $P \in$ Prim A,

$$\theta(\phi(P)) \in \theta(\overline{\phi(\text{Prim } A)})$$

and hence

$$Z_0 \in \theta(\phi(P)) = \phi(P) \cap Z(D(A)) \subset \phi(P)$$

from the construction of Z_0. We then have, from Theorem 5.1,

$$\begin{aligned}|\Phi_{T_0}(P)| &\leq \|\Phi_{T_0}(P)(J + \phi(P))\|\\ &= \|\Phi_{T_0}{}^U(\phi(P))(J + \phi(P))\|\\ &= \|\mu T_0 + \phi(P)\| = \|Z_0 + \phi(P)\| = 0\end{aligned}$$

for all $P \in$ Prim A. In other words, T_0 belongs to the kernel of the homomorphism $T \to \Phi_T$, which coincides with $ZM_R(A)$ from Theorem A. However since A is semi-simple, $ZM_R(A) = \{0\}$. It follows that $T_0 = 0$ and so $\Phi_{T_0}{}^U = 0$ since the map $T \to \Phi_T{}^U$ is homomorphic from Theorem 5.1. This contradicts that $\Phi_{T_0}{}^U(R_0) \neq 0$ as was observed in the above argument and the theorem is proved.

6. The proofs of the main theorems.

6.1. *Proof of Theorem* 3.1. Let $\{e_\alpha\}$ denote the bounded approximate identity of A and let z be an arbitrary element of $Z(A)$. We then have

$$(6.1.1)\quad \mu(Lz) = \text{weak*-}\lim_{\alpha} \tau(Lz(e_\alpha)) = \text{weak*-}\lim_{\alpha} \tau(ze_\alpha) = \tau(z).$$

Therefore it follows from Lemma 4.1 that

$$(6.1.2)\quad \Phi_{Lz}(P) = \chi_{P' \cap Z(D(A))}(\mu(Lz)) = \chi_{P' \cap Z(D(A))}(\tau z) = \chi_{P \cap Z(A)}(z)$$

for all $P \in \operatorname{Prim} A$. Now let ϵ be an arbitrary positive number. Recall that the map $T \to \Phi_T$ is a continuous homomorphism of $Z(M(A))$ into $C^b(\operatorname{Prim} A)$ from Theorem B. Set

$$W(z; \epsilon) = \{P \in \operatorname{Prim} A : |\Phi_{Lz}(P)| \geqq \epsilon\}.$$

We first show that the set $W(z; \epsilon)$ is quasi-compact. Notice that $W(z; \epsilon)$ is closed. Then we have only to show that $\ker W(z; \epsilon)$ is modular from [**12**, Theorem 2.6.4]. The map $\Psi_1 : P \to P \cap Z(A)$ is a continuous map of $\operatorname{Prim} A$ into $\operatorname{Prim} Z(A)$ from [**12**, Theorem 2.7.5] and the map $\Psi_2 : M \to \chi_M(z)$ is a continuous complex-valued function on $\operatorname{Prim} Z(A)$ from the complete regularity of $Z(A)$ and [**12**, Theorem 3.7.1]. Therefore we have that $\Phi_{Lz} = \Psi_2 \cdot \Psi_1$ from (6.1.2). Set

$$K = \{M \in \operatorname{Prim} Z(A) : |\Psi_2(M)| \geqq \epsilon\}.$$

Hence $W(z; \epsilon) = \{P \in \operatorname{Prim} A : \Psi_1(P) \in K\}$. Since K is closed in $\operatorname{Prim} Z(A)$, it follows from Lemma 4.2 that there exists an element z' of $Z(A)$ such that $\chi_M(z') = 1$ for all $M \in K$. Let P be an element of $W(z; \epsilon)$. Since $\Psi_1(P) \in K$, it follows from (6.1.2) that

$$\Phi_{Lz'}(P) = \chi_{P \cap Z(A)}(z') = \chi_{\Psi_1(P)}(z') = 1.$$

Therefore for each $x \in A$, we have

$$z'x + P = Lz'(x) + P = \Phi_{Lz'}(P)(x + P) = x + P$$

for all $P \in W(z; \epsilon)$. Then we conclude immediately that $\ker W(z; \epsilon)$ is modular. These observations imply that Φ_{Lz} belongs to $C_0(\operatorname{Prim} A)$ for all $z \in Z(A)$.

We next show that $\operatorname{Prim} A$ is locally quasi-compact. To show this, let P_0 be any fixed element of $\operatorname{Prim} A$. Then there exists an element z_0 of $Z(A)$

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such that $z_0 \notin P_0$ from the quasi-centrality of A. Set

$$\epsilon_0 = |\Phi_{Lz_0}(P_0)|.$$

By (6.1.2), ϵ_0 must be a non-zero number. Also set

$$V(P_0) = \{P \in \text{Prim } A : |\Phi_{Lz_0}(P)| > \epsilon_0/2\}.$$

Then $V(P_0)$ is an open subset of Prim A which contains P_0. Furthermore consider the set $W(z_0; \epsilon_0/2)$. This set is quasi-compact from the above argument. We also see that

$$\overline{V(P_0)} \subset W(z_0; \epsilon_0/2),$$

where the bar denotes the hull-kernel closure in Prim A. Therefore $V(P_0)$ is an open neighbourhood of P_0 such that $\overline{V(P_0)}$ is quasi-compact. In other words, Prim A is locally quasi-compact.

6.2. *Proof of Theorem* 3.2. We first construct a continuous map Γ of Prim A into Prim $\widetilde{\tau(Z(A))}$ such that

$$\Phi_T(P) = \chi_{\Gamma(P)}(\mu T)$$

for all $P \in$ Prim A and all $T \in \mu^{-1}(\tau(Z(A)))$. Recall that $P \to P'$ is a continuous map of Prim A into Prim $D(A)$. Moreover, $Q \to Q \cap Z(D(A))$ is also a continuous map of Prim $D(A)$ into Prim $Z(D(A))$ as was observed in the proof of [**15**, Theorem 3.2]. Also the map $M \to M \cap \widetilde{\tau(Z(A))}$ is a homeomorphism of $(Z(D(A))\text{-hull } \widetilde{\tau(Z(A))})^c$ onto Prim $\widetilde{\tau(Z(A))}$ from [**12**, Theorem 2.6.6], where c denotes the complement. Notice that $P' \cap Z(D(A))$ belongs to $(Z(D(A))\text{-hull } \widetilde{\tau(Z(A))})^c$ for each $P \in$ Prim A. Suppose, on the contrary, that

$$\widetilde{\tau(Z(A))} \subset (P_0)' \cap Z(D(A))$$

for some element P_0 of Prim A. By Lemma 4.4 part (i), we have

$$\begin{aligned} \tau(Z(A)) &= \widetilde{\tau(Z(A))} \cap \tau(A) \\ &\subset (P_0)' \cap \tau(A) \cap Z(D(A)) \\ &= \tau(P_0) \cap \tau(Z(A)), \end{aligned}$$

so that $\tau(Z(A)) \subset \tau(P_0)$ and hence $Z(A) \subset P_0$. This contradicts the quasi-centrality of A. Now set

$$\Gamma(P) = P' \cap Z(D(A)) \cap \widetilde{\tau(Z(A))} \quad (= P' \cap \widetilde{\tau(Z(A))})$$

for each $P \in$ Prim A. Then we see, from the above argument, that Γ is a continuous map of Prim A into Prim $\widetilde{\tau(Z(A))}$. Let P be an arbitrary element of Prim A and T an arbitrary element of $\mu^{-1}(\widetilde{\tau(Z(A))})$. Then there exists an element z_0 of $Z(A)$ such that $z_0 \notin P$ from the quasi-centrality of A. Note that the element

$$\chi_{P' \cap Z(D(A))}(\mu T)\tau(z_0) - (\mu T) \cdot \tau z_0$$

belongs to $\Gamma(P)$. Therefore we have

$$\chi_{P' \cap Z(D(A))}(\mu T)\chi_{\Gamma(P)}(\tau z_0) - \chi_{\Gamma(P)}(\mu T)\chi_{\Gamma(P)}(\tau z_0) = 0.$$

Since $z_0 \notin P$ implies that $\chi_{\Gamma(P)}(\tau z_0) \neq 0$, it follows that

$$\chi_{P' \cap Z(D(A))}(\mu T) = \chi_{\Gamma(P)}(\mu T).$$

On the other hand, we have

$$\Phi_T(P) = \chi_{P' \cap Z(D(A))}(\mu T)$$

from (4.1.2). These observations imply that $\Phi_T(P) = \chi_{\Gamma(P)}(\mu T)$ for all $P \in \operatorname{Prim} A$ and all $T \in \mu^{-1}(\widetilde{\tau(Z(A))})$.

We next show that Φ_T belongs to $C_0(\operatorname{Prim} A)$ for all $T \in \mu^{-1}(\widetilde{\tau(Z(A))})$. To see this, let T be an arbitrary element of $\mu^{-1}(\widetilde{\tau(Z(A))})$ and ϵ an arbitrary positive number. Set

$$K = \{P \in \operatorname{Prim} A : |\Phi_T(P)| \geqq \epsilon\}$$

and

$$D = \{Q \in \operatorname{Prim} \widetilde{\tau(Z(A))} : |\chi_Q(\mu T)| \geqq \epsilon\}.$$

Then $K = \Gamma^{-1}(D)$. Now $Z(D(A))$ is a commutative Banach algebra with identity and $\tau(Z(A))$ is a closed ideal of $Z(D(A))$. Also $\tau(Z(A))$ is completely regular from the assumption. It follows from Lemma 4.3 part (ii) that $\widetilde{\tau(Z(A))}$ is completely regular. Therefore the map $Q \to \chi_Q(\mu T)$ is a continuous complex-valued function on $\operatorname{Prim} \widetilde{\tau(Z(A))}$ and hence D is closed. Note also that $\widetilde{\tau(Z(A))}$ is a complex commutative Banach algebra. Then the difference algebra $\widetilde{\tau(Z(A))}/\ker D$ has an identity element $E' = E + \ker D$ from Lemma 4.2. In this case, we can choose an element z of $Z(A)$ such that $E' = \tau(z) + \ker D$. In fact, $\widetilde{\tau(Z(A))}/\tau(Z(A))$ is a radical algebra from Lemma 4.3 part (i). Also the map

$$\Delta : \tau(z) + \tau(Z(A)) \cap \ker D \to \tau(z) + \ker D$$

is an algebraic isomorphism of $\tau(Z(A))/\tau(Z(A)) \cap \ker D$ into $\widetilde{\tau(Z(A))}/\ker D$. Denote by Image Δ the image of Δ. Hence Image Δ is an ideal of $\widetilde{\tau(Z(A))}/\ker D$ and hence there exists the canonical homomorphism of $\widetilde{\tau(Z(A))}/\tau(Z(A))$ onto $(\widetilde{\tau(Z(A))}/\ker D)/\text{Image } \Delta$. Since $\widetilde{\tau(Z(A))}/\tau(Z(A))$ is radical, it follows that $(\widetilde{\tau(Z(A))}/\ker D)/\text{Image } \Delta$ is also radical and hence it does not possess an identity element. Thus E' must belong to Image Δ and so there exists an element z of $Z(A)$ such that $E' = \tau(z) + \ker D$. We then have

$$\chi_Q(\tau z) = \chi_{Q/\ker D}(E') = 1$$

for all $Q \in D$. Moreover, Lz belongs to $\mu^{-1}(\widetilde{\tau(Z(A))})$ from (6.1.1). It follows that

$$\begin{aligned}\Phi_{Lz}(P) &= \chi_{\Gamma(P)}(\mu(Lz)) \\ &= \chi_{\Gamma(P)}(\tau z) \ \text{(from (6.1.1))} \\ &= 1\end{aligned}$$

for all $P \in K$. Therefore we have

$$\begin{aligned}zx + P &= Lz(x) + P \\ &= \Phi_{Lz}(P)(x + P) \\ &= x + P\end{aligned}$$

for all $P \in K$ and all $x \in A$. This shows that ker K is modular. However K is closed in Prim A from the continuity of Γ (or Φ_T). It follows from [**12**, Theorem 2.6.4] that K must be quasi-compact and the theorem is proved.

6.3. *Proof of Theorem* 3.3. We will refer to the method in the proof of [**14**, Theorem 1]. Now we have to show that an arbitrary element T of $Z(M(A))$ with $\Phi_T \in C_0(\text{Prim } A)$ belongs to $\mu^{-1}(\widehat{\tau(Z(A))})$. Suppose, on the contrary, that there exists an element T_0 of $Z(M(A))$ such that $\Phi_{T_0} \in C_0(\text{Prim } A)$ but $\mu T_0 \notin \widehat{\tau(Z(A))}$. Then, by the definition of $\widehat{\tau(Z(A))}$, there exists a primitive ideal M_0 of $Z(D(A))$ such that $\tau(Z(A)) \subset M_0$ but $\mu T_0 \notin M_0$. Since M_0 belongs to $Z(D(A))$-hull $\tau(Z(A))$, it follows from Lemma 4.5 that there exists an element R_0 of $U(A)$-hull $\tau(A)$ such that

$$\tau(A) + M_0 = R_0 \quad \text{and} \quad M_0 = R_0 \cap Z(D(A)).$$

In this case, $\mu T_0 \notin R_0$. Actually, if $\mu T_0 \in R_0$ then there are $a_0 \in A$ and $Z_0 \in M_0$ such that $\mu T_0 = \tau(a_0) + Z_0$. By Lemma 4.4 part (i),

$$\tau(a_0) = \mu T_0 - Z_0 \in \tau(A) \cap Z(D(A)) = \tau(Z(A)) \subset M_0$$

and hence $\mu T_0 \in M_0$. But this is impossible since $\mu T_0 \notin M_0$. We put

$$\epsilon_0 = |\chi_{M_0}(\mu T_0)|.$$

Then $\epsilon_0 > 0$. Note that

$$\Phi_{T_0}{}^U(R_0)(J + R_0) = \mu T_0 + R_0$$

and so

$$\Phi_{T_0}{}^U(R_0)J - \mu T_0 \in R_0 \cap Z(D(A)) = M_0.$$

Therefore

$$\chi_{M_0}(\mu T_0) = \Phi_{T_0}{}^U(R_0)\chi_{M_0}(J) = \Phi_{T_0}{}^U(R_0),$$

so

$$|\Phi_{T_0}{}^U(R_0)| = \epsilon_0.$$

Let $\{U_\lambda(R_0):\lambda \in \Lambda\}$ be a fundamental system of neighbourhoods of R_0 in Prim $U(A)$, where Λ is a direct set and put

$$K = \{P \in \text{Prim } A : \Phi_{T_0}(P) = \Phi_{T_0}{}^U(R_0)\}.$$

We see that $\phi(K) \cap \overline{U_\lambda(R_0)} \neq \emptyset$ for all $\lambda \in \Lambda$, where ϕ is the homeomorphism of Prim A into Prim $U(A)$ given in Section 5 and the bar denotes the hull-kernel closure in Prim $U(A)$. Indeed suppose, on the contrary, that there exists $\lambda_0 \in \Lambda$ such that

$$\phi(K) \cap \overline{U_{\lambda_0}(R_0)} = \emptyset.$$

Since $\Phi_{T_0}{}^U$ is continuous on Prim $U(A)$ from Theorem 5.1, there exists, for an arbitrary $\epsilon > 0$, an element $\lambda(\epsilon)$ of Λ such that

$$(6.3.1) \quad |\Phi_{T_0}{}^U(R) - \Phi_{T_0}{}^U(R_0)| < \epsilon$$

for all $R \in U_{\lambda(\epsilon)}(R_0)$. We can assume that $U_{\lambda(\epsilon)}(R_0) \subset U_{\lambda_0}(R_0)$. Recall that $\phi(\text{Prim } A)$ is dense in Prim $U(A)$ from Theorem 5.3. Then we can find an element P_ϵ of Prim A such that $\phi(P_\epsilon) \in U_{\lambda(\epsilon)}(R_0)$. Let I_0 be the open interval $(0, \epsilon_0/2)$ on the real numbers and put

$$K_0 = \{P \in \text{Prim } A : |\Phi_{T_0}(P)| \geqq \epsilon_0/2\}.$$

Since Φ_{T_0} belongs to $C_0(\text{Prim } A)$, K_0 is a closed quasi-compact subset of Prim A. Moreover since $\Phi_{T_0} = \Phi_{T_0}{}^U \cdot \phi$ from Theorem 5.1, it follows from (6.3.1) that P_ϵ belongs to K_0 for all $\epsilon \in I_0$. Now choose a positive integer N such that $1/N \in I_0$. Then the sequence $\{P_{1/n}:n \geqq N\}$ is contained in K_0. Thus there exists a subnet $\{P_{1/nj}:j = 1, 2, \ldots\}$ which converges to some element P_0 of K_0 in the hull-kernel topology. We have, from (6.3.1), that

$$\begin{aligned} |\Phi_{T_0}(P_{1/nj}) - \Phi_{T_0}{}^U(R_0)| &= |\Phi_{T_0}{}^U(\phi(P_{1/nj})) - \Phi_{T_0}{}^U(R_0)| \\ &\leqq 1/n_j \end{aligned}$$

for all $j \in J$. Hence, after taking the limit with respect to j, we obtain that $\Phi_{T_0}(P_0) = \Phi_{T_0}{}^U(R_0)$ and hence $P_0 \in K$. On the other hand,

$$\phi(P_{1/nj}) \in U_{\lambda(1/nj)}(R_0) \subset U_{\lambda_0}(R_0)$$

for all $j \in J$ and hence

$$\phi(P_0) \in \overline{U_{\lambda_0}(R_0)}.$$

In other words,

$$\phi(P_0) \in \phi(K) \cap \overline{U_{\lambda_0}(R_0)}.$$

This is a contradiction and therefore $\phi(K) \cap \overline{U_\lambda(R_0)} \neq \emptyset$ for all $\lambda \in \Lambda$. Thus we can find an element Q_λ of K such that $\phi(Q_\lambda) \in \overline{U_\lambda(R_0)}$ for each $\lambda \in \Lambda$. Notice that K is a closed subset of Prim A which is contained in the quasi-compact subset K_0. Therefore we can assume without loss of generality that the net $\{Q_\lambda : \lambda \in \Lambda\}$ converges to some element Q_0 of K in the hull-kernel topology. Now let z be an arbitrary element of $Z(A)$ and $\delta > 0$ be chosen arbitrarily. By the continuity of $\Phi_{Lz}{}^U$, there exists $\lambda(\delta) \in \Lambda$ such that

$$(6.3.2) \quad |\Phi_{Lz}{}^U(R) - \Phi_{Lz}{}^U(R_0)| < \delta$$

for all $R \in U_{\lambda(\delta)}(R_0)$. On the other hand, we have

$$\begin{aligned}\Phi_{Lz}{}^U(R_0)(J + \mathrm{R}_0) &= \mu(Lz) + R_0 \\ &= \tau(z) + R_0 \text{ (from (6.1.1))}\end{aligned}$$

and therefore

$$\Phi_{Lz}{}^U(R_0)J - \tau(z) \in R_0 \cap Z(D(A)) = M_0.$$

Hence

$$\Phi_{Lz}{}^U(R_0) = \chi_{M_0}(\tau z).$$

Since $\tau(Z(A)) \subset M_0$, it follows that $\Phi_{Lz}{}^U(R_0) = 0$. We thus have, from (6.3.2) that

$$|\Phi_{Lz}{}^U(R)| \leqq \delta$$

for all $R \in \overline{U_{\lambda(\delta)}(R_0)}$. However since

$$\phi(Q_\lambda) \in \overline{U_\lambda(R_0)} \subset \overline{U_{\lambda(\delta)}(R_0)} \quad \text{for each } \lambda \geqq \lambda(\delta),$$

it follows that

$$|\Phi_{Lz}(Q_\lambda)| = |\Phi_{Lz}{}^U(\phi(Q_\lambda))| \leqq \delta$$

for all $\lambda \geqq \lambda(\delta)$. By taking the limit with respect to λ,

$$|\Phi_{Lz}(Q_0)| \leqq \delta.$$

We then conclude that $\Phi_{Lz}(Q_0) = 0$ since δ is arbitrary. Therefore

$$\begin{aligned}\chi_{Q_0 \cap Z(A)}(z) &= \Phi_{Lz}(Q_0) \text{ (from 6.1.2))} \\ &= 0\end{aligned}$$

for all $z \in Z(A)$. In other words, $Z(A) \subset Q_0$. This contradicts the quasi-centrality of A and the theorem is proved.

7. Application. In this section, we first show, from the main theorems, the following result established by C. Delaroche.

COROLLARY 7.1 [**7**, Proposition 1]. *Let A be an arbitrary quasi-central C^*-algebra. Then $Z(A)$ is isometrically $*$-isomorphic with C_0*(Prim A).

Proof. The map $T \to \Phi_T$ is an isometric $*$-isomorphism of $Z(M(A))$ onto C^b(Prim A) from [**15**, Corollary 3.4]. Then the map $z \to \Phi_{Lz}$ is also an isometric $*$-isomorphism of $Z(A)$ into C_0(Prim A) from Theorem

3.1. Note also that

$$\widetilde{\tau(Z(A))} = \tau(Z(A))$$

from [**8**, Théorème 2.9.7]. Hence, Theorem 3.3 and (6.1.1) imply that this map is surjective.

COROLLARY 7.2. *If A is a quasi-central C^*-algebra such that* Prim A *is quasi-compact, then A has an identity element.*

Proof. Since $C_0(\text{Prim } A) = C^b(\text{Prim } A)$, we can choose an element e of $Z(A)$ such that Φ_{Le} is the identity element of $C^b(\text{Prim } A)$ from Corollary 7.1. We then have

$$x + P = \Phi_{Le}(P)(x + P) = ex + P$$

for all $P \in \text{Prim } A$ and $x \in A$. Since A is semi-simple, it follows that $x = ex = xe$ for all $x \in A$ and the proof is complete.

We next give an extension of the above corollary. It is well known that when A is a strongly semi-simple, completely regular Banach algebra, A has an identity element if and only if the strong structure space of A is quasi-compact (cf. [**12**, Theorem 2.7.10]). As an application of Theorem 3.3, we have the following theorem which is similar to the above result and is also an extension of Corollary 7.2.

THEOREM 7.3. *Let A be a complex quasi-central Banach algebra with a bounded approximate identity $\{e_\alpha\}$. Suppose that* Prim $Z(D(A))$ *is Hausdorff and A is semi-simple. Then A has an identity element if and only if* Prim A *is quasi-compact.*

Proof. Let id be the identity map on A and so $\text{id} \in Z(M(A))$. If Prim A is quasi-compact, then Φ_{id} must belong to $C_0(\text{Prim } A)$ and hence $\text{id} \in \mu^{-1}(\widetilde{\tau(Z(A))})$ from Theorem 3.3. Also,

$$J = \text{weak*-}\lim_{\alpha} \tau(\text{id}(e_\alpha)) \quad (\text{from [6, Lemma 2.7]})$$

$$= \mu(\text{id}).$$

We thus obtain that $J \in \widetilde{\tau(Z(A))}$. In this case, $J \in \tau(Z(A))$. Indeed, if $J \notin \tau(Z(A))$, then

$$Z(D(A))\text{-hull } \tau(Z(A)) \neq \emptyset$$

and so $\widetilde{\tau(Z(A))}$ is a proper ideal of $Z(D(A))$. This contradicts that $J \in \widetilde{\tau(Z(A))}$. Therefore there is an element e of $Z(A)$ with $J = \tau(e)$. We can easily see that e is an identity element of A.

Conversely, if A has an identity element then Prim A is obviously quasi-compact from [**12**, Corollary 2.6.5].

Remark. Let Ω be a compact Hausdorff space and $C(\Omega) = C^b(\Omega)$. Also let R be the space of complex measurable functions f on the positive real numbers such that

$$||f|| = \int_0^\infty \exp(-te^t)|f(t)|dt < \infty,$$

and let

$$(f * g)(t) = \int_0^t f(s)g(t-s)ds.$$

Then R is a commutative radical Banach algebra with a bounded approximate identity (cf. [**10**, Section 4] or [**2**, p. 255]). Now define $A = C(\Omega) \oplus R$, where the algebra operations and norm in A are given in the standard manner (see [**12**, p. 169]). Then A is a commutative Banach algebra with a bounded approximate identity. The radical of A is equal to R. Moreover Prim A is homeomorphic with Ω, so that Prim A is a compact Hausdorff space. However A does not possess an identity element. Thus it seems that the semi-simplicity is needed in the above Theorem.

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