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JOINT SPECTRA FOR COMMUTING OPERATORS

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0. Introduction

The theory of joint spectra for commuting operators in a Hilbert space has recently been studied by several authors (Vasilescu [11, 12], Curto [4, 5], and Cho–Takaguchi [2, 3]). In this paper we will use the definition by Taylor [10] of the joint spectrum to show that the joint spectrum is determined by the action of certain "Laplacians" (cf. Curto [4, 5]) of a chain-complex of Hilbert spaces.

In particular, if $A_1,\dots,A_n$ is a doubly commuting set of bounded linear operators, then these Laplacians are all determined by the one single operator $D=A_1^*A_1+\cdots+A_n^*A_n$.

The paper is organized as follows. In Section 1 we briefly review the definition of joint spectrum. In Section 2 we discuss the role of the Laplacians in the chain-complex of Hilbert spaces described in Section 1. In Section 3 we look at the special case of doubly commuting operators and relate the spectrum of D above to the joint spectrum of $A_1,\dots,A_n$. In Section 4 we study the classification of points in the joint spectrum, particularly for the case of two commuting operators. In Section 5 we discuss an example and conjecture of Dash [7]. Finally in Section 6 the connection with the work of Vasilescu [11] is studied.

1. Joint spectrum

The concept of joint spectrum for commuting operators was introduced by Arens–Calderon [1]. Subsequently several definitions have been given notably by Dash [6] and Taylor [10]. We will review the definition by Taylor. It is known that in certain cases the Taylor spectrum and the Dash spectrum coincide (cf. [2]).

Let H be a complex Hilbert space and $A_1,\dots,A_N$ bounded commuting linear operators in H. Let $e_1,\dots,e_N$ be N indeterminates and construct the exterior algebra E^N with $e_1,\dots,e_N$ as generators. The elements of degree $p\geqq 0$ in E^N is the linear hull E_p^N of all elements of the form

$$e_{i_1}\wedge\cdots\wedge e_{i_p}\quad (1\leqq i_1<\cdots<i_p\leqq N).$$

The space $E_p^N(H)$ is defined as $H\otimes E_p^N$, the linear hull of all the elements of the form

$$x\,e_{i_1}\wedge\cdots\wedge e_{i_p}\quad (1\leqq i_1<\cdots<i_p\leqq N).$$

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$E_p^N(H)$ is canonically identified with a direct sum of $\binom{N}{p}$ copies of H and thus it is itself a Hilbert space. We now define the maps $\delta_p: E_p^N(H) \to E_{p+1}^N(H)$ for $p=0,1,\ldots,N-1$ (where $E_0^N(H)=E_N^N(H)=H$) by

$$\delta_p(xe_{i_1}\wedge\cdots\wedge e_{i_p})=\sum_{k=1}^{N} A_k x e_k \wedge e_{i_1}\wedge\cdots\wedge e_{i_p} \tag{1}$$

and extended by linearity. With these maps we can construct the following sequence

$$0\to E_0^N(H) \xrightarrow{\delta_0} E_1^N(H) \xrightarrow{\delta_1}\cdots\xrightarrow{\delta_{N-2}} E_{N-1}^N(H) \xrightarrow{\delta_{N-1}} E_N^N(H)\to 0. \tag{2}$$

Using the fact that the A's commute it is easily seen that (2) is a complex, i.e. that $\operatorname{im}\delta_p \subseteq \ker\delta_{p+1}$, for all p. This is the Koszul-complex. Furthermore, all the maps δ_p are bounded linear maps.

Definition 1. The N-tuple $A=(A_1,\ldots,A_N)$ is called *non-singular* if the complex (2) is exact, that is if

$$\operatorname{im}\delta_p=\ker\delta_{p+1}, \quad p=0,1,\ldots,N-1,$$

otherwise it is called *singular*.

Definition 2. The complex N-tuple $\lambda=(\lambda_1,\ldots,\lambda_N)$ is said to be in the *joint spectrum* of $A=(A_1,\ldots,A_N)$, denoted by $\sigma(A)=\sigma(A_1,\ldots,A_N)$, if $A-\lambda I=(A_1-\lambda_1 I,\ldots,A_N-\lambda_N I)$ is singular.

Example 1. In the case of a single operator A this reduces to the usual definition of spectrum. The Koszul-complex for this case looks like

$$0\to H \xrightarrow{A-\lambda I} H\to 0$$

and $A-\lambda I$ is non-singular if and only if $\ker(A-\lambda I)=\{0\}$ and $\operatorname{im}(A-\lambda I)=H$.

We can also define a dual complex by using the maps $\delta_p^*: E_{p+1}^N(H)\to E_p^N(H)$, defined by

$$\delta_p^*(x e_{i_1}\wedge\cdots\wedge e_{i_{p+1}})=\sum_{k=1}^{p+1}(-1)^{k-1}A_{i_k} x e_{i_1}\wedge\cdots\wedge \hat{e}_{i_k}\wedge\cdots\wedge e_{i_{p+1}} \tag{3}$$

and extended by linearity. The sign $\hat{e}_s$ denotes omission of the factor e_s. The dual complex is

$$0\leftarrow E_0^N(H) \xleftarrow{\delta_0^*} E_1^N(H) \xleftarrow{\delta_1^*}\cdots\xleftarrow{\delta_{N-2}^*} E_{N-1}^N(H) \xleftarrow{\delta_{N-1}^*} E_N^N(H)\leftarrow 0. \tag{4}$$

It is a simple exercise to show that (2) is exact if and only if (4) is exact. This follows from the facts that $\operatorname{im}\delta_p$ is closed if and only if $\operatorname{im}\delta_p^*$ is closed and that $\ker\delta_p=(\operatorname{im}\delta_p^*)^\perp$ ($\perp$ denoting orthogonal complement) (cf. Kato [8], Theorem 5.13, p. 234).

2. The Laplacians of a complex

In this section we will give a necessary and sufficient condition for a complex and its dual to be simultaneously exact at a particular point of the complex.

Consider the complex of Hilbert spaces

$$\cdots \to H \xrightarrow{A} H' \xrightarrow{B} H'' \to \cdots \tag{C}$$

and its dual

$$\cdots \leftarrow H \xleftarrow{A^*} H' \xleftarrow{B^*} H'' \leftarrow. \tag{C*}$$

Here A and B are closed densely defined maps between the respective Hilbert spaces. This means that AA^* and B^*B are densely defined selfadjoint operators on H' (cf. [**8**], Theorem 3.24, p. 275). We assume that the "Laplacian" $D = AA^* + B^*B$ is also a closed densely defined operator. D is easily seen to be symmetric and bounded from below by 0. Hence D always has a self-adjoint extension $\bar{D}_0$, the Friedrichs extension, with the same lower bound as D.

Example 2. Let us consider the case of two bounded linear operators A_1 and A_2 such that A_1 commutes with A_2 and A_2^*. Then also A_1^* commutes with A_2 and A_2^*. The Koszul-complex can be written

$$0 \to H \xrightarrow{\delta_0} H \oplus H \xrightarrow{\delta_1} H \to 0$$

where $\delta_0 x = A_1 x \oplus A_2 x$, $\delta_1(x_1 \oplus x_2) = A_1 x_2 - A_2 x_1$ and the dual complex

$$0 \leftarrow H \xleftarrow{\delta_0^*} H \oplus H \xleftarrow{\delta_1^*} H \leftarrow 0$$

with the maps

$$\delta_1^* x = (-A_2^* x) \oplus A_1^* x, \ \delta_0^*(x_1 \oplus x_2) = A_1^* x_1 + A_2^* x_2.$$

The Laplacians of the complex are

$$D_0 = \delta_0^* \delta_0 = A_1^* A_1 + A_2^* A_2$$

$$D_1 = \delta_0 \delta_0^* + \delta_1^* \delta_1 = (A_1 A_1^* + A_2^* A_2) \oplus (A_1^* A_1 + A_2 A_2^*)$$

$$D_2 = \delta_1 \delta_1^* = A_1 A_1^* + A_2 A_2^*.$$

Note that if A_1 and A_2 are normal commuting operators and we put $D = A_1^* A_1 + A_2^* A_2$, then

$$D_0 = D, \quad D_1 = D \oplus D, \quad D_2 = D.$$

Returning now to the complexes (C) and (C^*) we state the following.

Theorem 1 (cf. [5], Prop. 3.1, p. 395). *(C) and (C^*) are both exact at H', (i.e. $\operatorname{im} A = \ker B$, $\operatorname{im} B^* = \ker A^*$) if and only if D is selfadjoint and boundedly invertible on H'.*

For the proof we need the following two facts about closed densely defined linear operators $T: H_1 \to H_2$.

Lemma 1 (cf. [8], Theorem 5.13, p. 234). *$\operatorname{im} T$ is closed if and only if $\operatorname{im} T^*$ is closed. In this case we have*

$$\operatorname{im} T = (\ker T^*)^\perp, \ \operatorname{im} T^* = (\ker T)^\perp.$$

Lemma 2 (cf. [8], Theorem 5.2, p. 231), *T has closed range if and only if there is a constant $C > 0$, such that*

$$\|Tx\|_2 \geqq C\|x\|_1 \quad \textit{for all} \quad x \in D(T) \cap (\ker T)^\perp. \tag{5}$$

Proof of Theorem 1. First we note that since (C) and (C^*) are complexes we always have

$$\operatorname{im} A \subseteq \ker B, \ \operatorname{im} B^* \subseteq \ker A^*. \tag{6}$$

Furthermore $\ker B$ and $\ker A^*$ are closed since the maps B and A^* are closed.

Assume now that D is selfadjoint and $\operatorname{im} D = H'$. Then $\ker D = (\operatorname{im} D)^\perp = \{0\}$. Hence D^{-1} is a closed map defined on all of H and consequently D^{-1} is bounded. Now

$$H' = \operatorname{im} D \subseteq \operatorname{im} A + \operatorname{im} B^* \subseteq \operatorname{im} A + \overline{\operatorname{im} B^*} = \operatorname{im} A + (\ker B)^\perp \subseteq H'.$$

By (6) it follows that $\operatorname{im} A = \ker B$ and hence (C) is exact at H'. Similarly

$$H' = \operatorname{im} D \subseteq \operatorname{im} A + \operatorname{im} B^* \subseteq \overline{\operatorname{im} A} + \operatorname{im} B^* = (\ker A^*)^\perp + \operatorname{im} B^* \subseteq H'$$

showing that $\operatorname{im} B^* = \ker A^*$. Thus (C^*) is also exact at H'.

Conversely, assume that both (C) and (C^*) are exact at H'. Then

$$\operatorname{im} A = \ker B \Rightarrow \operatorname{im} A \quad \text{closed} \Rightarrow \operatorname{im} A^* \quad \text{closed by Lemma 1}$$

$$\operatorname{im} B^* = \ker A^* \Rightarrow \operatorname{im} B^* \quad \text{closed} \Rightarrow \operatorname{im} B \quad \text{closed by Lemma 1.}$$

Hence, there are constants $C_1, C_2 > 0$ such that

$$\|A^*x\| \geqq C_1\|x\| \quad x \in D(A^*) \cap (\ker A^*)^\perp$$

$$\|By\| \geqq C_2\|y\| \quad y \in D(B) \cap (\ker B)^\perp.$$

But any $u \in D(D)$ can be decomposed as $u = u_1 + u_2$, where $u_1 \in D(D) \cap (\ker B)$ and $u_2 \in D(D) \cap (\ker B)^{\perp}$. But then

$$u_1 \in D(D) \cap \ker B = D(D) \cap \operatorname{im} A = D(D) \cap (\ker A^*)^{\perp}$$

$$u_2 \in D(D) \cap (\ker B)^{\perp} = D(D) \cap (\operatorname{im} A)^{\perp} = D(D) \cap \ker A^*.$$

Hence $Du = AA^*u + B^*Bu = AA^*u_1 + B^*Bu_2$ and

$$\begin{aligned}(Du, u) &= (AA^*u_1, u) + (B^*Bu_2, u) \\ &= \|A_1^*u_1\|^2 + \|Bu_2\|^2 \\ &\geqq C_1\|u_1\|^2 + C_2\|u_2\|^2 \\ &\geqq C\{\|u_1\|^2 + \|u_2\|^2\} \quad C = \min(C_1, C_2) > 0 \\ &= C\|u\|^2 \qquad \text{for all} \quad u \in D(D).\end{aligned}$$

This shows that $\operatorname{im} D$ is closed and also that D is bounded from below by $C > 0$. Hence also $\bar{D}$ is bounded from below by $C > 0$. But then $\ker \bar{D} = \{0\}$ and $\operatorname{im} \bar{D} = H'$. But $\operatorname{im} D$ is dense in $\operatorname{Im} \bar{D}$ and closed. Thus we must have $\operatorname{im} D = H'$ and it follows that D is already selfadjoint. One sees easily that D^{-1} is bounded.

The question of the selfadjointness of D can often be settled by the following.

Theorem 2 (cf. [9], p. 88). *Let $G_1, \ldots, G_p$ be a finite set of selfadjoint operators such that $G_1, \ldots, G_p$ commute pairwise (i.e. their spectral families commute). In addition, suppose that $G_i \geqq 0$, $i = 1, \ldots, p$. Then $G = G_1 + \cdots + G_p$ is selfadjoint.*

3. Doubly commuting systems

Let $A_1, \ldots, A_N$ be an N-tuple of bounded linear operators on H.

Definition 3. $A = (A_1, \ldots, A_N)$ is called *doubly commuting* if $A_iA_j = A_jA_i$ and $A_iA_j^* = A_j^*A_i$ for all $i, j = 1, \ldots, N$. A is called *weakly doubly commuting* if, for all $i = 1, \ldots, N$

$$A_iA_j = A_jA_i \quad \text{and} \quad A_iA_j^* = A_j^*A_i \quad \text{for} \quad j \neq i.$$

In particular, in a doubly commuting system all the operators $A_1, \ldots, A_N$ are normal. The significance of these systems derive from the fact that all the Laplacians of the Koszul-complex are defined in terms of one single operator $D = A_1^*A_1 + \cdots + A_N^*A_N$. The dual complex also generates exactly the same Laplacians so that the complex is in some sense "self-dual".

Let $A = (A_1, \ldots, A_N)$ be a doubly commuting system. We now want to relate the spectral subspaces of H defined by the resolution of the identity belonging to D, to the operators $A_1, \ldots, A_N$.

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First we notice that if μ is an eigenvalue of D and $E(\mu)$ is the corresponding eigenspace, then from the commutativity

$$(D-\mu)A_ix = A_i(D-\mu)x = 0,$$

$$(D-\mu)A_i^*x = A_i^*(D-\mu)x = 0,$$

for all $x \in E(\mu)$ and $i = 1, \ldots, N$. Hence $E(\mu)$ is invariant under all A_i and A_i^*. But then, if $x \in E(\mu)$, $y \in E(\mu)^\perp$,

$$(x, A_iy) = (A_i^*x, y) = 0$$

$$(x, A_i^*y) = (A_ix, y) = 0$$

so that all the operators $A_1, \ldots, A_N$, $A_1^*, \ldots, A_N^*$ are reduced simultaneously by $E(\mu)$.

It is now easy to show that the discrete subspace H_d of H corresponding to the spectral resolution of D is invariant under all of A_i and A_i^*. Hence the continuous subspace H_c is also invariant and all the operators $A_1, \ldots, A_N, \ldots, A_N^*$ are simultaneously reduced by H_d and H_c. It is obvious that there are no joint eigenvalues of $A_1, \ldots, A_N$ nor of $A_1^*, \ldots, A_N^*$ in H_c.

Let us now specialize further and assume that $A_1, \ldots, A_N$ is a commuting set of selfadjoint bounded operators. Clearly, to every joint eigenvalue $(\lambda_1, \ldots, \lambda_N)$ to $(A_1, \ldots, A_N)$ there is an eigenvalue $\mu = \lambda_1^2 + \cdots + \lambda_N^2$ to D. Conversely, if μ is an eigenvalue to D and $E(\mu)$ the eigenspace, then $A_1, \ldots, A_N$ are commuting selfadjoint operators on $E(\mu)$ and hence they can be simultaneously diagonalized. In particular, if $\dim E(\mu) < \infty$, then there is a basis of $E(\mu)$ consisting of joint eigenvectors to $A_1, \ldots, A_N$. This leads us to the following.

Theorem 3. *Suppose $A_1, \ldots, A_N$ are bounded commuting selfadjoint operators such that $D = A_1^2 + \cdots + A_N^2$ is compact. Then*

(i) *if* $\dim \ker D < \infty$ *there is a complete system of joint eigenvectors of $A_1, \ldots, A_N$ in H.*

(ii) *if* $\dim \ker D = \infty$ *there is a complete system of joint eigenvectors to $A_1, \ldots, A_N$ in* $(\ker D)^\perp$.

Proof. D compact $\Rightarrow H_c = \{0\}$ if $\dim \ker D < \infty$, otherwise $H_c = \ker D$. Every non zero eigenvalue has finite multiplicity which means that there is a basis in the eigenspace consisting of joint eigenvectors. From this the theorem follows easily.

Remark. If μ is an eigenvalue of infinite multiplicity to D there need not be any joint eigenvectors to $A_1, \ldots, A_N$ in the eigenspace as is shown by the following example.

Example 3. Let $E(\lambda)$ be a continuous spectral family. Define $A_1 = \int \cos \lambda \, dE(\lambda)$, $A_2 = \int \sin \lambda \, dE(\lambda)$. Then A_1 and A_2 are bounded selfadjoint commuting operators with continuous spectra only. But $D = A_1^2 + A_2^2$ is the operator

$$D = \int (\cos^2 \lambda + \sin^2 \lambda) \, \mathrm{d}E(\lambda) = I,$$

which implies that D has eigenvalue 1 with infinite multiplicity.

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In order to clarify the situation when D or D^{-1} is compact we prove

Theorem 4. *Let $D=A_1^*A_1+\cdots+A_N^*A_N$. Then*

(i) *D is compact if and only if all the A_i are compact.*

(ii) *If at least one of the A_i has compact inverse then D has compact inverse.*

Proof. (i) Assume all A_i compact. Then trivially D is compact. Conversely, if D is compact, let x_n be a sequence such that $x_n \to 0$ weakly. Then $Dx_n \to 0$ strongly. But then

$$\sum_{i=1}^{N} \|A_i x_n\|^2 = (Dx_n, x_n) \to 0.$$

Hence all $A_i x_n \to 0$ strongly as $n \to \infty$ which implies that all A_i compact.

(ii) Suppose A_1 has compact inverse. Then also A_1^* has compact inverse. Furthermore

$$(Du, u) = \sum_{i=1}^{N} \|A_i u\|^2 \geqq \|A_1 u\|^2 \geqq c\|u\|^2.$$

Hence D has a bounded inverse. If we now write

$$D = A_1^* A_1 \left(I + (A_1^* A_1)^{-1} \sum_{i=2}^{N} A_i^* A_i \right) = A_i^* A_i (I + C),$$

where C is a positive operator. But then $(I+C)^{-1}$ exists and

$$D^{-1} = (I+C)^{-1} A_1^{-1} A_1^{*-1},$$

which is compact.

4. Classification of the spectrum

$A=(A_1,\ldots,A_N)$ is an N-tuple of bounded commuting linear operators on H.
The following definitions are given by Dash [6].

Definition 4. $\lambda=(\lambda_1,\ldots,\lambda_N)\in\mathbb{C}^N$ is in the *joint approximate point spectrum* $\sigma_\pi(A)$ if there is a sequence of unit vectors $x_n \in H$ such that

$$\|(A_i - \lambda_i I)x_n\| \to 0 \quad \text{as} \quad n \to \infty, \quad i = 1, \ldots, N.$$

Definition 5. $\lambda=(\lambda_1,\ldots,\lambda_N)\in\mathbb{C}^N$ is in the *joint approximate compression spectrum* $\sigma_\rho(A)$, if there is a sequence of unit vectors $x_n \in H$ such that

$$\|(A_i - \lambda_i I)^* x_n\| \to 0 \quad \text{as} \quad n \to \infty, \quad i = 1, \ldots, N.$$

Let us furthermore introduce the *joint point spectrum*

$$\sigma_p(A)=\{\lambda\in\mathbb{C}^N;\text{ there is a non-zero } x\in H, \text{ such that}$$
$$(A_i-\lambda_i I)x=0,\quad i=1,\dots,N\}$$

and the *joint compression spectrum*

$$\sigma_{\rho p}(A)=\{\lambda\in\mathbb{C}^N;\text{ there is a non-zero } x\in H, \text{ such that}$$
$$(A_i-\lambda_i I)^*x=0,\ i=1,\dots,N\}.$$

If we now construct the Koszul-complex for the operators $A_i-\lambda_i I$, $i=1,\dots,N$ we find easily that the Laplacians D_0 and D_N are given by

$$D_0=\sum_{i=1}^{N}(A_i-\lambda_i I)^*(A_i-\lambda_i I)$$

and

$$D_N=\sum_{i=1}^{N}(A_i-\lambda_i I)(A_i-\lambda_i I)^*$$

respectively. From this it is follows easily that

$$\sigma_p(A)\subseteq\sigma_\pi(A)=\{\lambda\in\mathbb{C}^N; D_0\text{ not boundedly invertible}\}$$

and

$$\sigma_{\rho p}(A)\subseteq\sigma_\rho(A)=\{\lambda\in\mathbb{C}^N; D_N\text{ not boundedly invertible}\}.$$

We now propose to classify the spectrum according to which Laplacians are boundedly invertible.

Definition 6. The *joint discrete spectrum* $\sigma_d(A)$ is defined as

$$\sigma_d(A)=\bigcup_{k=0}^{N}\sigma_d^k(A),$$

where

$$\sigma_d^k(A)=\{\lambda\in\mathbb{C}^N;\ker D_k\neq\{0\}\}.$$

Note that the sets $\sigma_d^k(A)$ need not be disjoint.

Definition 7. The *joint continuous spectrum* $\sigma_c(A)$ is defined as

$$\sigma_c(A)=\sigma(A)\backslash\sigma_d(A).$$

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We decompose this in the following (not necessarily disjoint) subsets

$$\sigma_c(A) = \bigcup_{k=0}^{N} \sigma_c^k(A),$$

where

$$\sigma_c^k(A) = \{\lambda \in \sigma_c(A); D_k \text{ has an unbounded inverse}\}.$$

We see easily that if $A_1, \ldots, A_N$ is a doubly commuting system then

$$\sigma_d(A) = \sigma_d^0(A), \quad \sigma_c(A) = \sigma_c^0(A).$$

This follows since all of the Laplacians can be expressed as direct sums of D_0. Hence if $\lambda \notin \sigma_d^0(A) \cup \sigma_c^0(A), D_0$ is boundedly invertible and then all D_j are boundedly invertible. Hence the complex is exact.

Example 4. Let us look at the case of a single operator. The Koszul-complex is

$$0 \to H \underset{\delta_0^*}{\overset{\delta_0}{\rightleftarrows}} H \to 0$$

where

$$\delta_0 x = (A - \lambda I)x, \quad \delta_0^* x = (A - \lambda I)^* x.$$

The Laplacians are

$$D_0 = (A - \lambda I)^*(A - \lambda I)$$

$$D_1 = (A - \lambda I)(A - \lambda I)^*.$$

Hence

$$\sigma_d^0 = \{\lambda \in \mathbb{C}; \ker(A - \lambda I) \neq \{0\}\} = \text{point spectrum of } A$$

$$\sigma_d^1 = \{\lambda \in \mathbb{C}; \ker(A - \lambda I)^* \neq \{0\}\} = \text{compression spectrum of } A$$

$$\sigma_c^0 = \{\lambda \in \mathbb{C}; \text{im}(A - \lambda I) \text{ dense in } H \text{ but not closed}\} = \text{approximate point spectrum.}$$

Also $\sigma_c^1 = \sigma_c^0$ since $\text{im}\,\delta_0$ is not closed if and only if $\text{im}\,\delta_0^*$ is not closed. This is the usual decomposition of the spectrum for a single operator.

Example 5. Two commuting operators A_1 and A_2. The corresponding Koszul-complex is (cf. Example 2)

$$0 \to H \underset{\delta_0^*}{\overset{\delta_0}{\rightleftarrows}} H \oplus H \underset{\delta_1^*}{\overset{\delta_1}{\rightleftarrows}} H \to 0$$

$\sigma_d^0=\{\lambda; ker\,\delta_0\neq\{0\}\}=$joint point spectrum

$\sigma_d^1=\{\lambda;\ker D_1=\ker\delta_0^*\cap\ker\delta_1\neq\{0\}\}$

$\sigma_d^2=\{\lambda;\ker\delta_1^*\neq\{0\}\}=$joint compression spectrum

$\sigma_c^0=\{\lambda;\operatorname{im}\delta_0^*$ not closed$\}$. But then $\operatorname{im}\delta_0$ is not closed which implies that $\sigma_c^0\subseteq\sigma_c^1$. Similarly $\sigma_c^2\subseteq\sigma_c^1$.

Conversely, if $\lambda\in\sigma_c^1$ then either $\operatorname{im}\delta_0$ or $\operatorname{im}\delta_1^*$ is not closed and hence $\lambda\in\sigma_c^0$ or $\lambda\in\sigma_c^2$. Hence

$$\sigma_c^1=\sigma_c^0\cup\sigma_c^2$$

and we get the following decomposition of the joint spectrum

$$\sigma(A)=\sigma_d^0\cup\sigma_d^1\cup\sigma_d^2\cup\sigma_c^0\cup\sigma_c^2.$$

This contains the results of lemma p. 867 in **[3]**.

Example 6. In case the underlying Hilbert space is finite-dimensional we clearly have

$$\sigma_c=\emptyset.$$

Also for the case of two operators in a finite-dimensional Euclidean space we have the relation

$$\sigma(A_1,A_2)=\sigma_d^1(A_1,A_2)\quad(\text{which implies that }\sigma_d^0\cup\sigma_d^2\subseteq\sigma_d^1).$$

This can be seen as follows: $(\lambda_1,\lambda_2)\notin\sigma_d^1$ if and only if $\ker D_1=\{0\}$, equivalently $\ker\delta_0^*\cap\ker\delta_1=\{0\}$. Hence $\ker D_1=\{0\}$ if and only if the system

$$(A_1-\lambda_1I)^*x_1+(A_2-\lambda_2I)^*x_2=0\quad(\text{i.e. }\delta_0^*(x_1\oplus x_2)=0)$$

$$-(A_2-\lambda_2I)x_1+(A_1-\lambda_1I)x_2=0\quad(\text{i.e. }\delta_1(x_1\oplus x_2)=0)$$

has only the zero solution. But then the operator

$$\alpha(A)=\begin{pmatrix}(A_1-\lambda_1I)^* & (A_2-\lambda_2I)^*\\ -(A_2-\lambda_2I) & (A_1-\lambda_1I)\end{pmatrix}$$

is invertible on $H\oplus H$. Hence by Theorem 1.1 in Vasilescu **[11]**, $A-\lambda I$ is non-singular, that is $\lambda\notin\sigma(A)$.

We will here also give a proof of Theorem 3.1 in **[11]** based on the methods developed here.

Let $A_1:H_1\to H_1$ and $A_2:H_2\to H_2$ be bounded linear operators with spectra $\sigma(A_1)$ and $\sigma(A_2)$ respectively. In $H=H_1\otimes H_2$ we can construct two commuting operators

$$B_1=A_1\otimes I_2 \quad \text{and} \quad B_2=I_1\otimes A_2.$$

Let $\sigma(B)$ denote the joint spectrum of B_1 and B_2 in H.

Theorem 5. $\sigma(B)=\sigma(A_1)\times\sigma(A_2)$.

Proof. We have the complex

$$0\to H \xrightarrow{\delta_0} H\oplus H \xrightarrow{\delta_1} H\to 0,$$

where $\delta_0 x=B_1x\oplus B_2x$, $\delta_1(x_1\oplus x_2)=B_1x_2-B_2x_1$.

Note that it is enough to consider the point $(0,0)$ since

$$B_1-\lambda_1 I=(A_1-\lambda_1 I_1)\otimes I_2 \quad \text{and} \quad B_2-\lambda_2 I=I_1\otimes(A_2-\lambda_2 I_2).$$

The Laplacians for the complex are calculated to be

$$D_0=(A_1^*A_1)\otimes I_2+I_1\otimes(A_2^*A_2)$$

$$D_1=[(A_1A_1^*)\otimes I_2+I_1\otimes(A_2^*A_2^*)]\oplus[(A_1^*A_1)\otimes I_2+I_1\otimes(A_2A_2^*)]$$

$$D_2=(A_1A_1^*)\otimes I_2+I_1\otimes(A_2A_2^*).$$

Assume now that $(0,0)\notin\sigma(A_1)\cup\sigma(A_2)$. Then at least one of A_1 and A_2, say A_1, is non-singular, so $\ker A_1=\{0\}$ and $\operatorname{im} A_1=H$. But then $\operatorname{im} A_1$ is closed and consequently $\operatorname{im} A_1^*$ is closed and we have

$$\operatorname{im} A_1^*=(\ker A_1)^\perp=H_1, \quad \ker A_1^*=(\operatorname{im} A_1)^\perp=\{0\}.$$

Hence A_1^* is also non-singular. Now D_0, being the sum of two positive operators, one of which is invertible, is boundedly invertible. Using the remark above about A_1^* we can use the same kind of argument to show that D_1 and D_2 are boundedly invertible. Hence $(0,0)\notin\sigma(B)$.

Conversely, if $(0,0)\notin\sigma(B)$, then D_0 is boundedly invertible. Hence, if $w=u\otimes v$

$$(D_0w,w)=\|A_1u\|^2\|v\|^2+\|u\|^2\|A_2v\|^2\geqq C\|u\|^2\|v\|^2. \tag{*}$$

But then at least one of A_1 and A_2 must be boundedly invertible, for suppose A_2 is not. Then $0\notin\sigma(A_2)$ and either

(i) 0 is an eigenvalue of A_2 which implies that there is a v such that $\|v\|=1$, $A_2v=0$. (*) then gives

$$\|A_1u\|^2\geqq C\|u\|^2 \text{ which implies that } 0\notin\sigma(A_1)$$

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or

(ii) 0 is in the approximate point spectrum of A_2 which implies that there is a sequence v_n such that $\|v_n\|=1$, $A_2v_n \to 0$. But then (*) gives

$$\|A_1u\|^2 \geqq (C-\|A_2v_n\|^2)\|u\|^2 \geqq \frac{C}{2}\|u\|^2$$

if n is large enough, implying $0 \notin \sigma(A_1)$

or

(iii) 0 is in the compression spectrum of A_2 which implies that there is a v such that $\|v\|=1$, $A_2^*v=0$. But then from (*) on using the Laplacian D_2 instead we have

$$\|A_1^*u\|^2 \geqq C\|u\|^2.$$

Hence $0 \notin \sigma(A_1^*)$ and so $0 \notin \sigma(A_1)$. Consequently if A_2 is not boundedly invertible, then A_1 must be and it follows that $(0,0) \notin \sigma(A_1) \times \sigma(A_2)$. From this follows that $\sigma(B)=\sigma(A_1)\times\sigma(A_2)$.

5. An example by Dash

In this section we will show that it is possible for a point (λ_1, λ_2) to be in σ_d^1 but not in $\sigma_d^0 \cup \sigma_d^2$. The example is given by Dash in [7] in order to disprove a certain conjecture.

Let $H=\oplus_{n=1}^{\infty} l^2$ so that each element $X \in H$ is a sequence of elements $X_n \in l^2$, $n=1,2,\ldots$. If each X_n is given by $X_n=(x_{n1}, x_{n2}, \ldots)$ then the norm in H is

$$\|X\|^2=\sum_{n=1}^{\infty}\|X_n\|^2=\sum_{n,k=1}^{\infty}|x_{nk}|^2.$$

Define the operators A_1 and A_2 in H by the matrices

$$A_1=\begin{pmatrix} 0 & I & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & I & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & & & \\ \cdots & \cdots & \cdots & & \ddots & \\ \cdots & \cdots & \cdots & & & \end{pmatrix} \text{ and } A_2=\begin{pmatrix} V & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & V & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & V & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & & \ddots & \\ \cdots & \cdots & \cdots & & & \end{pmatrix}$$

where I is the identity operator in l^2, 0 the zero operator and V the unilateral shift. Clearly A_1 and A_2 commute. Also A_1 and A_2^* commute so that A_1 and A_2 form a weakly doubly commuting system.

We find for the Laplacians of the corresponding complex

$$(D_0f, f)=\|A_1f\|^2+\|A_2f\|^2 \geqq \|A_2f\|^2$$

$$=\sum_{i=1}^{\infty}\|Vf_i\|^2=\sum_{i=1}^{\infty}\|f_i\|^2=\|f\|^2.$$

This shows that D_0 is boundedly invertible and so $(0,0) \notin \sigma_d^0$. Also

$$(D_2 f, f) = \|A_1^* f\|^2 + \|A_2^* f\|^2 \geqq \|A_1^* f\|^2 = \|f\|^2$$

which implies that D_2 is boundedly invertible and consequently

$$(0,0) \notin \sigma_d^2.$$

However $\ker D_1 \neq \{0\}$ as the following shows:

$D_1 : H \oplus H \to H \oplus H$. Let $f = X \oplus Y \in \ker D_1$. Then

$$A_1 Y - A_2 X = 0$$

$$A_1^* X + A_2^* Y = 0.$$

Choose $X = 0$ which implies that

$$A_1 Y = 0 \quad \text{and} \quad A_2^* Y = 0.$$

If $Y = (Y_1, Y_2, \ldots)$ then

$$A_1 Y = 0 \text{ if and only if } Y_2 = Y_3 = \cdots = 0 \text{ with } Y_1 \text{ arbitrary.}$$

Now $A_2^* Y = 0$ if and only if $V^* Y_1 = 0$ which is satisfied for $Y_1 = \{1, 0, 0, \ldots\} \in l^2$. Hence $(0,0) \in \sigma_d^1$.

In the same paper Dash makes the following conjecture

$$(0,0) \notin \sigma(A_1, A_2) \Leftrightarrow \text{there is an } \varepsilon > 0 \text{ such that}$$

(i) $\|A_1 f\|^2 + \|A_2 f\|^2 \geqq \varepsilon \|f\|^2$
(ii) $\|A_1^* f\|^2 + \|A_2^* f\|^2 \geqq \varepsilon \|f\|^2$
(iii) $\|A_1^* f\|^2 + \|A_2 f\|^2 \geqq \varepsilon \|f\|^2$
(iv) $\|A_1 f\|^2 + \|A_2^* f\|^2 \geqq \varepsilon \|f\|^2$.

If it is easily seen that (i) holds if any only if D_0 is boundedly invertible, and that (ii) holds if and only if D_2 is boundedly invertible.

The Laplacian D_1 is in general given by a matrix operator in $H \oplus H$

$$D_1 = \begin{pmatrix} A_1 A_1^* + A_2^* A_2 & A_1 A_2^* - A_2^* A_1 \\ A_2 A_1^* - A_1^* A_2 & A_1^* A_1 + A_2 A_2^* \end{pmatrix}.$$

In the case of a weakly doubly commuting system the off-diagonal operators are both zero and we get

$$D_1=\begin{pmatrix} A_1A_1^*+A_2^*A_2 & 0 \\ 0 & A_1^*A_1+A_2A_2^* \end{pmatrix}.$$

For this Laplacian (iii) and (iv) hold if and only if D_1 is boundedly invertible.

Hence Dash's conjecture holds if A_1, A_2 are weakly doubly commuting. In the general case, however, D_1 has the following structure

$$D_1=\begin{pmatrix} P & Q \\ Q^* & R \end{pmatrix}$$

where $P=A_1A_1^*+A_2^*A_2$, $R=A_1^*A_1+A_2A_2^*$ are positive operators and $Q=A_1A_2^*-A_2^*A_1$. If $u=f\oplus g$ is a general vector in $H\oplus H$, D_1 is boundedly invertible if and only if there is an $\varepsilon>0$ such that $(D_1u,u)\geqq\varepsilon\|u\|^2$ or equivalently

(v) $(Pf,f)+2\mathrm{Re}(f,Qg)+(Rg,g)\geqq\varepsilon\{\|f\|^2+\|g\|^2\}$.

If $g=0$ this is inequality (iii) and if $f=0$ it is inequality (iv). Hence (iii)–(iv) are necessary for $(0,0)\notin\sigma(A_1,A_2)$. However, without further conditions on the operator Q it is not known whether conditions (iii)–(iv) imply (v).

6. Connections with [11]

In [11] Vasilescu proves the following theorem.

Theorem. $(0,0)\notin\sigma(A_1,A_2)\Leftrightarrow$ *the operator*

$$\alpha(A)=\begin{pmatrix} A_1^* & A_2^* \\ -A_2 & A_1 \end{pmatrix}$$

is boundedly invertible on $H\oplus H$.

We give another proof. A simple calculation shows that

$$\alpha(A)^*=\begin{pmatrix} A_1 & -A_2^* \\ A_2 & A_1^* \end{pmatrix}$$

and that

$$\alpha(A)^*\alpha(A)=D_1 \tag{7}$$

$$\alpha(A)\alpha(A^*)=\begin{pmatrix} D_0 & 0 \\ 0 & D_2 \end{pmatrix} \tag{8}$$

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where D_0, D_1, D_2 are the Laplacians of the complex for A_1 and A_2. Hence

$$(D_1 u, u) = \|\alpha(A)u\|^2 \quad \text{and}$$

$$(D_0 f, f) + (D_2 g, g) = \|\alpha(A)^*(f \oplus g)\|^2.$$

From this follows that if $\alpha(A)$ boundedly invertible and consequently also $\alpha(A)^*$. Then all of D_0, D_1 and D_2 are boundedly invertible which implies that $(0,0) \notin \sigma(A_1, A_2)$.

Conversely $(0,0) \notin \sigma(A_1, A_2)$ implies that all three Laplacians have bounded inverses. Hence

$$\|\alpha(A)u\|^2 \geqq C\|u\|^2$$

which implies that $\ker \alpha(A) = \{0\}$ and $\operatorname{im} \alpha(A)$ is closed. Also $\ker \alpha(A)^* = \{0\}$ and hence

$$\operatorname{im} \alpha(A) = (\ker \alpha(A)^*)^\perp = H.$$

Consequently $\alpha(A)$ is boundedly invertible.

From (7) and (8) follows also an interwining property of the Laplacians. By evaluating $\alpha(A)\alpha(A)^*\alpha(A)$ in two different ways according to (7) and (8) we find

$$\alpha(A)D_1 = \begin{pmatrix} D_0 & 0 \\ 0 & D_2 \end{pmatrix} \alpha(A). \tag{9}$$

We can also easily show the equivalence of our approach with that of Vasilescu in [12]. He defines the operators δ_A and δ_A^* on $\oplus_{k=0}^N E_k^N(H)$ by

$$\delta_A = \begin{pmatrix} 0 & 0 & \dots & \dots \\ \delta_0 & 0 & & \dots \\ 0 & \delta_1 & & \dots \\ \dots & 0 & & 0 \\ \dots & \dots & \delta_{N-1} & 0 \end{pmatrix}, \quad \delta_A^* = \begin{pmatrix} 0 & \delta_0^* & \dots & \dots & \dots \\ 0 & 0 & \delta_1^* & & \dots \\ \dots & \dots & & & \delta_{N-1}^* \\ 0 & \dots & & 0 & 0 \end{pmatrix} \tag{10}$$

and $\delta = \delta_A + \delta_A^*$. Clearly δ is a selfadjoint operator. The main theorem in [12] is that the complex is exact if and only if δ is boundedly invertible.

We notice that $\delta_A^2 = \delta_A^{*2} = 0$ because of the properties $\delta_{i+1}\delta_i = 0$ and $\delta_i^*\delta_{i+1}^* = 0$. Hence $\delta^2 = \delta_A\delta_A^* + \delta_A^*\delta_A$, which has the same structure as a Laplacian. A simple computation shows that

$$\delta^2 = \operatorname{diag}(D_0, \dots, D_N) \quad \text{(diagonal matrix)} \tag{11}$$

where $D_0, \dots, D_N$ are the Laplacians of the complex. From this it follows immediately that δ is boundedly invertible if and only if all the Laplacians $D_0, \dots, D_N$ are boundedly invertible.

The intertwining property (9) follows in this case by evaluating δ^3 in two ways as

$$\delta \operatorname{diag}(D_0, \dots, D_N) = \operatorname{diag}(D_0, \dots, D_N)\delta.$$

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