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Uniqueness and nonuniqueness in mean boundary value problems

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We give some sufficient conditions to guarantee the uniqueness of certain mean boundary value problems for a circle. Also we show that, in general, we cannot expect uniqueness of the problem for an arc unless the function is analytic in a neighborhood of the unit circle or some shifted means of the function are also known.

Let U denote the open unit disc and T the unit circle. For a function f continuous on an arc $\left\{e^{i2\pi t},\ t_1 \le t \le t_2\right\}$ of T, we consider the arithmetic means,

$$s_n\left(f;\ t_1,\ t_2\right) = \frac{1}{n} \sum_{k=1}^{n} f\left(e^{i2\pi k(t_2-t_1)/n+i2\pi t_1}\right) ,$$

$$\tilde{s}_n\left(f;\ t_1,\ t_2\right) = \frac{f\left(e^{i2\pi t_1}\right)+f\left(e^{i2\pi t_2}\right)}{2n} + \frac{1}{n} \sum_{k=1}^{n-1} f\left(e^{i2\pi k(t_2-t_1)/n+i2\pi t_1}\right) ,$$

$n = 1, 2, \ldots$, and the limit,

$$\begin{aligned} s_\infty\left(f;\ t_1,\ t_2\right) &= \lim_{n\to\infty} s_n\left(f;\ t_1,\ t_2\right) \\ &= \lim_{n\to\infty} \tilde{s}_n\left(f;\ t_1,\ t_2\right) . \end{aligned}$$

Then $s_n(f;\ 0,\ 1) = \tilde{s}_n(f;\ 0,\ 1)$ for all n. As usual, let H^p be the Hardy spaces and A be the space of functions holomorphic in U and

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continuous on $\overline{U}$. We have the following results.

THEOREM 1. *Let* $f(z) = \sum a_n z^n$ *be in* A *and satisfy* $s_n(f;\ 0,\ 1) = 0$ *for all* n . *Then* f *is the zero function if one of the following conditions is satisfied:*

(i) $a_n = O\left(\frac{1}{n^{1+\varepsilon}}\right)$ *for some* $\varepsilon > 0$ *;*

(ii) $\sum_{k=N}^{\infty} |a_k| = O\left(\frac{1}{N}\right)$ *;*

(iii) there exists a prime $q > 0$ *such that* $a_k = 0$ *if* $k \neq q^m$ *for some integer* m *;*

(iv) f' *belongs to* H^p *for some* p *,* $1 < p \leq \infty$.

By similar methods we can conclude that f is determined by the means

$$\frac{1}{n} \sum_{k=1}^{n} f(z_{n,k})$$

with $z_{n,k} = \rho\left(e^{i2\pi k/n}\right)$ for any diffeomorphism ρ of T satisfying $\overline{\rho\left(e^{i\theta}\right)} = \rho\left(e^{-i\theta}\right)$.

We remark that there exist polynomials p_m with $s_n\left(p_m;\ 0,\ 1\right) = \delta_{m,n}$ (*cf.* [*1*]) and that if $\sum |s_n - s_\infty| n^{\varepsilon}$ converges for some $\varepsilon > 0$ and one of the above four conditions holds, we can use results in [*1*] to obtain an explicit formula to recapture f from its means $s_n(f;\ 0,\ 1)$. However, for a proper subarc $K = \left\{e^{i2\pi t} : t_1 \leq t \leq t_2,\ 0 < t_2 - t_1 < 1\right\}$, we can construct a nonzero function f in A , infinitely differentiable relative to $\overline{U}$, holomorphic in a neighborhood of K , and $s_n\left(f;\ t_1,\ t_2\right) = \tilde{s}_n\left(f;\ t_1,\ t_2\right) = 0$ for all $n = 1,\ 2,\ \ldots$. On the other hand, we have uniqueness for an arc if certain extra conditions are satisfied, namely,

THEOREM 2. *Let* f *be in* A *such that either* $\tilde{s}_n(f;\ 0,\ \delta) = 0$ *or*

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$s_n(f;\ 0,\ \delta) = 0$ *for all* $n = 1, 2, \ldots$. *Then* f *is the zero function if one of the following conditions holds:*

(i) f *is holomorphic in a neighborhood of the closed unit disk;*

(ii) $f' \in H^p$ *for* $1 < p \le \infty$ *, and*

$$t_n(f;\ 0,\ \delta) = \frac{f(e^{i2\pi\delta})}{2n} + \frac{1}{n}\sum_{k=1}^{n-1} f\left(e^{i2\pi\delta(2k-1)/(2n-1)}\right) = 0 ,$$

for $n = 1, 2, \ldots$.

We note that combining Theorem 2 and the results of [1] and [2], we can obtain an explicit formula to recapture a $C^{1+\varepsilon}$ function f if its means s_n, t_n are known on an arc, namely,

$$f(z) = \lim_{\lambda\to\infty} \lambda h_\lambda(z) \int_0^\delta \frac{\overline{h_\lambda}(e^{i2\pi t})g(e^{i2\pi t})}{1-ze^{-i2\pi t}}\,dt$$

with

$$g(e^{i2\pi t}) = \sum_{m=1}^{\infty} [\tilde{s}_{2m}(f;\ 0,\ \delta)-s_\infty(f;\ 0,\ \delta)]p_{2m}(e^{i2\pi t/\delta})$$
$$+ \sum_{m=1}^{\infty}\left[\frac{2m}{2m-1}\, t_m(f;\ 0,\ \delta)-s_\infty(f;\ 0,\ \delta)\right]p_{2m-1}(e^{i2\pi t/\delta}) + s_\infty(f;\ 0,\ \delta)$$

and

$$h_\lambda(z) = \exp\left\{\frac{-\log(1+\lambda)}{2}\int_0^\delta \frac{1+ze^{-i2\pi t}}{1-ze^{-2\pi t}}\,dt\right\} .$$

Here

$$p_n(z) = \sum_{k|n} \mu\left(\frac{n}{k}\right)z^k$$

as in [1]. If f is holomorphic in a neighborhood of $\overline{U}$ the formula for g is a little simpler. The details of the proofs and related results will appear elsewhere.

References

[1] Chin-Hung Ching and Charles K. Chui, "Representation of a function in terms of its mean boundary values", *Bull. Austral. Math. Soc.* **7** (1972), 425-427.

[2] D.J. Patil, "Representation of H^p functions", *Bull. Amer. Math. Soc.* **78** (1972), 617-620.

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