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THE TROPICAL j-INVARIANT

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Abstract

If $(Q, \mathcal{A})$ is a marked polygon with one interior point, then a general polynomial $f \in \mathbb{K}[x, y]$ with support $\mathcal{A}$ defines an elliptic curve C_f on the toric surface $X_{\mathcal{A}}$. If $\mathbb{K}$ has a non-archimedean valuation into $\mathbb{R}$ we can tropicalize C_f to get a tropical curve $\mathrm{Trop}(C_f)$. If in the Newton subdivision induced by f is a triangulation and the interior point occurs as the vertex of a triangle, then $\mathrm{Trop}(C_f)$ will be a graph of genus one and we show that the lattice length of the cycle of that graph is the negative of the valuation of the j-invariant of C_f.

1. *Introduction*

Previous work by Grisha Mikhalkin [**13**], by Michael Kerber and Hannah Markwig [**11**] and by Magnus Vigeland [**18**] shows that the length of the cycle of a tropical curve of genus one has properties which one classically attributes to the j-invariant of an elliptic curve without giving a direct link between these two numbers. In [**9**] we established such a direct link for plane cubics by showing that the tropicalization of the j-invariant is *in general* the negative of the cycle length. In the present paper we generalize this result to elliptic curves on other toric surfaces using the same methods. The main result as described in the abstract is Theorem 13.

In the case where the triangulation is *unimodular*, i.e. all the triangles have area $\frac{1}{2}$, this result was independently derived by David Speyer [**17**, Proposition 9.2] using Tate uniformization of elliptic curves. David Speyer's result is more general though in the sense that it applies to curves in arbitrary toric varieties.

This paper is organized as follows. In Section 2 we consider toric surfaces defined by a marked lattice polygon with one interior point, we recall the classification of these polygons and we consider the impact on the j-invariant for the corresponding elliptic curves. Section 3 recalls the notion of tropicalization and of plane tropical curves. We then introduce in Section 4 the notion of tropical j-invariant and give a formula to compute it. Section 5 shows that the tropical j-invariant is preserved by integral unimodular affine transformations. With this preparation we are able to state our main result in Section 6. Section 7 is then devoted to the reduction of the proof to considering only three marked polygons and Section 8 shows how these three cases can be dealt with using procedures from the SINGULAR library `jinvariant.lib` (see [**10**]) which is available via the URL

www.lms.ac.uk/jcm/12/lms2008-012/appendix-a/jinvariant.lib.

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Finally, in Section 9 we describe in more detail how computer algebra was used to prove the main result. The actual computations are done using `polymake` [4], TOPCOM [16] and SINGULAR [6]. The tropical curves in this paper and their Newton subdivisions were produced using the procedure `drawTropicalCurve` from the SINGULAR library `tropical.lib` (see [8]) which can be obtained via the URL

www.mathematik.uni-kl.de/~keilen/en/tropical.html.

The paper comes with an appendix consisting of four files. These are available online via the web page of the journal.

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2. *Toric surfaces*

Throughout this paper we consider mainly marked polygons $(Q, \mathcal{A})$ such that Q contains a single interior lattice point, where by a *marked polygon* we mean a convex lattice polygon Q in $\mathbb{R}^2$ together with a subset $\mathcal{A} \subseteq Q \cap \mathbb{Z}^2$ of the lattice points of Q containing at least the vertices of Q (cf. [5, Section 2.A]). Fixing a base field $\mathbb{K}$ such a polygon defines a polarized *toric surface*

$$X_{\mathcal{A}} \subset \mathbb{P}_{\mathbb{K}}^{|\mathcal{A}|-1}.$$

In the torus $(\mathbb{K}^*)^2 \subset X_{\mathcal{A}}$ the hyperplane section, say C_f, defined by the linear form $\sum_{(i,j)\in\mathcal{A}} a_{ij} \cdot z_{ij}$ is the vanishing locus of the Laurent polynomial

$$f = \sum_{(i,j)\in\mathcal{A}} a_{ij} \cdot x^i y^j$$

(cf. [5, Chapter 5]). Since the arithmetical genus of the hyperplane sections is the number of interior lattice points of Q (cf. [3, p. 91]), the general hyperplane section will be a smooth *elliptic curve*. The *j-invariant* of such a curve is an element of the base field which characterizes the curve up to isomorphism.

An *integral unimodular affine transformation* of $\mathbb{R}^2$ is an affine map

$$\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 : \alpha \mapsto A \cdot \alpha + \tau$$

with $\tau \in \mathbb{Z}^2$ and $A \in \mathrm{Gl}_2(\mathbb{Z})$ invertible over the integers. Such an integral unimodular affine transformation ϕ maps each face of Q to a face of the convex lattice polygon $\phi(Q)$ and preserves thereby the number of lattice points on each face. Moreover, ϕ induces an isomorphism of the polarized toric surfaces $X_{\mathcal{A}}$ and $X_{\phi(\mathcal{A})}$ (cf. [5, Proposition 5.1.2]). From the point of view of toric surfaces it therefore suffices to consider the marked polygon $(Q, \mathcal{A})$ only up to integral unimodular affine transformations, and if we suppose $\mathcal{A} = Q \cap \mathbb{Z}^2$ then there are precisely sixteen of them which we divide into two groups, Q_a, Q_b and Q_c respectively $Q_{ca}, \ldots, Q_{cm}$ (see Figure 1, cf. [15] or [14]). We fix the interior point at position $(1, 1)$.

The marked polygon $(Q_c, \mathcal{A}_c)$ corresponds to $\mathbb{P}^2_{\mathbb{K}}$ embedded into $\mathbb{P}^9_{\mathbb{K}}$ via the 3-uple Veronese embedding. The marked polygon $(Q_b, \mathcal{A}_b)$ corresponds to $\mathbb{P}^1_{\mathbb{K}} \times \mathbb{P}^1_{\mathbb{K}}$ embedded into $\mathbb{P}^8_{\mathbb{K}}$ via the $(2, 2)$-Segre embedding. The marked polygon $(Q_a, \mathcal{A}_a)$ describes the singular weighted projective plane $\mathbb{P}_{\mathbb{K}}(2, 1, 1)$ embedded into $\mathbb{P}^8$.

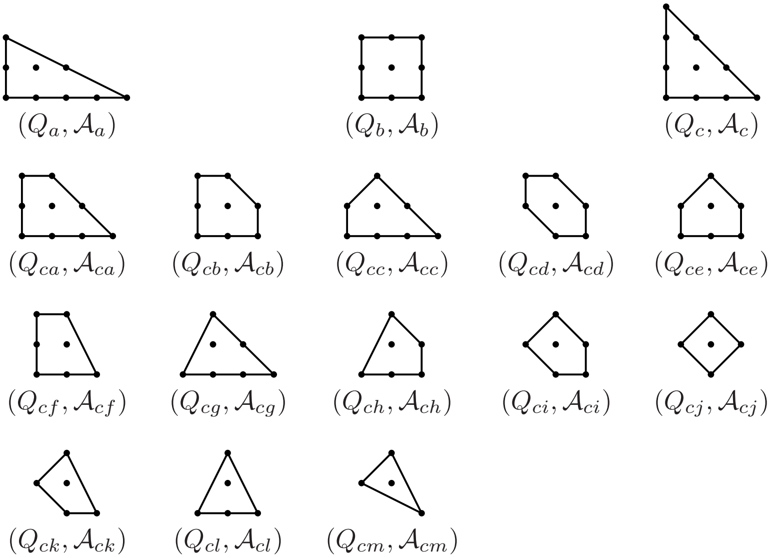


Figure 1: The 16 convex lattice polygons with one interior lattice point

If a marked polygon $(Q', \mathcal{A}')$ is derived from $(Q, \mathcal{A})$ by cutting off one lattice point (k, l), like $(Q_{ca}, \mathcal{A}_{ca})$ is derived from $(Q_c, \mathcal{A}_c)$, then the toric surface $X_{\mathcal{A}'}$ is a blow up of $X_{\mathcal{A}}$ in a single point. Moreover, in the torus $(\mathbb{K}^*)^2$ the hyperplane sections corresponding to

$$f = \sum_{(i,j)\in\mathcal{A}'} a_{ij} \cdot x^i y^j = \sum_{(i,j)\in\mathcal{A}} a_{ij} \cdot x^i y^j$$

with $a_{kl} = 0$ coincide. In particular, if they are both smooth their *j-invariant* coincides since two birationally equivalent smooth curves are already isomorphic (cf. [7, Section I.6]). Since the 13 polygons $Q_{ca}, \ldots, Q_{cm}$ in the second group in Figure 1 are all subpolygons of Q_c the corresponding toric surfaces are all obtained from the projective plane by a couple of blow ups. When we want to compute the j-invariant of the curve corresponding to some Laurent polynomial with support in one of these 13 polygons, we can instead consider the plane curve with support in $\mathcal{A}_c$ but with the appropriate coefficients being zero.

Once we are able to compute the j-invariant for polynomials with support $\mathcal{A}_a$, $\mathcal{A}_b$ and $\mathcal{A}_c$ we are therefore able to compute the j-invariant for every Laurent polynomial with support on the lattice points of a lattice polygon with only one interior point.

We still assume that $(Q, \mathcal{A})$ is a marked lattice polygon with only one interior lattice point as above. Moreover, we use the notation $\underline{a} = (a_{ij} \mid (i,j) \in \mathcal{A})$, and we suppose that

$$f = \sum_{(i,j)\in\mathcal{A}} a_{ij} \cdot x^i y^j,$$

then the j-invariant

$$j(C_f) = j(f) = \frac{A_{\mathcal{A}}}{B_{\mathcal{A}}}$$

of the curve C_f in $X_{\mathcal{A}}$ defined by f can be expressed as a quotient of two homogeneous polynomials $A_{\mathcal{A}}, B_{\mathcal{A}} \in \mathbb{Q}[\underline{a}]$ of degree 12. In the case of $\mathcal{A} = \mathcal{A}_c$ $A_{\mathcal{A}}$ has 1607 terms and $B_{\mathcal{A}}$ has 2040. In the case of $\mathcal{A} = \mathcal{A}_b$ $A_{\mathcal{A}}$ has 990 terms and $B_{\mathcal{A}}$ has 1010. And finally in the case $\mathcal{A} = \mathcal{A}_a$ $A_{\mathcal{A}}$ has 267 terms and $B_{\mathcal{A}}$ has 312. Every other case can be reduced to these three via some integral unimodular affine transformation and by setting some coefficients equal to zero. The reader interested in seeing or using the polynomials can consult the procedure `invariantsDB` in the Singular library jinvariant.lib (see [**10**]). The proof of our result relies heavily on the investigation of the combinatorics of these polynomials.

3. *Tropicalization*

In this section we want to pass from the algebraic to the tropical side. For this we specify a field $\mathbb{K}$ with a *non-archimedean valuation* $\mathrm{val} : \mathbb{K}^* \to \mathbb{R}$ as base field and we extend the valuation to $\mathbb{K}$ by $\mathrm{val}(0) = \infty$. We call $\mathrm{val}(k)$ also the *tropicalization* of k. In the examples that we consider $\mathbb{K}$ will always be the *field of Puiseux series*

$$\bigcup_{N=1}^{\infty} \mathrm{Quot}\left(\mathbb{C}\left[\left[t^{\frac{1}{N}}\right]\right]\right) = \left\{ \sum_{\nu=m}^{\infty} c_\nu \cdot t^{\frac{\nu}{N}} \;\middle|\; c_\nu \in \mathbb{C}, N \in \mathbb{Z}_{>0}, m \in \mathbb{Z} \right\}$$

and the valuation of a Puiseux series is its *order*.

If $f = \sum a_{ij} \cdot x^i y^j \in \mathbb{K}[x, y, x^{-1}, y^{-1}]$ is any Laurent polynomial, we call the set

$$\mathrm{supp}(f) = \{(i,j) \in \mathbb{Z}^2 \mid a_{ij} \neq 0\}$$

the *support* of f and the convex hull $\mathrm{N}(f)$ of $\mathrm{supp}(f)$ in $\mathbb{R}^2$ is called the *Newton polygon* of f. If $\mathrm{supp}(f) \subseteq \mathcal{A} \subseteq \mathrm{N}(f) \cap \mathbb{Z}^2$ then f defines a curve C_f in the toric surface $X_{\mathcal{A}}$ as described in Section 2 and we define the *tropicalization* of C_f as

$$\mathrm{Trop}\left(C_f\right) = \overline{\mathrm{val}\left(C_f \cap (\mathbb{K}^*)^2\right)} \subseteq \mathbb{R}^2,$$

i.e. the closure of $\mathrm{val}\left(C_f \cap (\mathbb{K}^*)^2\right)$ with respect to the Euclidean topology in $\mathbb{R}^2$. Here by abuse of notation

$$\mathrm{val} : (\mathbb{K}^*)^2 \longrightarrow \mathbb{Q}^2 : (k_1, k_2) \mapsto \big(\mathrm{val}(k_1), \mathrm{val}(k_2)\big)$$

denotes the Cartesian product of the above valuation map.

A better way to compute the tropicalization of C_f is as the tropical curve defined by the *tropicalization* of the polynomial f, i.e. the piecewise linear map

$$\mathrm{trop}(f) : \mathbb{R}^2 \longrightarrow \mathbb{R} : (x,y) \mapsto \min\{\mathrm{val}(a_{ij}) + i \cdot x + j \cdot y \mid (i,j) \in \mathrm{supp}(f)\}.$$

Given any *plane tropical Laurent polynomial*

$$F : \mathbb{R}^2 \longrightarrow \mathbb{R} : (x,y) \mapsto \min\{u_{ij} + i \cdot x + j \cdot y \mid (i,j) \in \mathcal{A}'\}$$

with *support* $\mathrm{supp}(F) = \mathcal{A}' \subset \mathbb{Z}^2$ finite and $u_{ij} \in \mathbb{R}$, we call the *locus* $\mathcal{C}_F$ *of non-differentiability* of F, i.e. the set of points $(x,y) \in \mathbb{R}^2$ where the minimum is attained at least twice, the *plane tropical curve* defined by F. The convex hull $\mathrm{N}(F)$ of $\mathrm{supp}(F)$ is again called the *Newton polygon* of F.

By *Kapranov's Theorem* (see [2, Theorem 2.1.1]), $\mathrm{Trop}(C_f)$ coincides with the plane tropical curve defined by the plane tropical polynomial $\mathrm{trop}(f)$. In particular, $\mathrm{Trop}(C_f)$ is a piece-wise linear graph with some unbounded edges.

The plane tropical Laurent polynomial F induces a *marked subdivision* (cf. [5, Definition 7.2.1]) of the marked polygon $\big(\mathrm{N}(F), \mathcal{A}\big)$ with $\mathrm{supp}(F) \subseteq \mathcal{A} \subseteq \mathrm{N}(F) \cap \mathbb{Z}^2$ in the following way: project the lower faces of the convex hull of

$$\{(i, j, u_{ij}) \mid (i, j) \in \mathrm{supp}(F)\}$$

into the xy-plane to subdivide of $\mathrm{N}(F)$ into smaller polygons and mark those lattice points for which (i, j, u_{ij}) is contained in a lower face.

This subdivision is *dual* to the tropical curve $\mathcal{C}_F$ in the following sense (see [12, Prop. 3.11]): Each marked polygon of the subdivision is dual to a vertex of $\mathcal{C}_F$, and each facet of a marked polygon is dual to an edge of $\mathcal{C}_F$. Moreover, if the facet, say e, has end points (x_1, y_1) and (x_2, y_2) then the vector

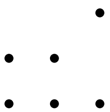
$$v(E) = (y_2 - y_1, x_1 - x_2)^t$$

points in the direction of the dual edge E in $\mathcal{C}_F$. We will call this vector $v(E)$ the *direction vector* of E . Note that it is only well defined up to sign. In particular, the edge E is orthogonal to its dual facet e. Finally, the edge E is unbounded if and only if its dual facet e is contained in a facet of $\mathrm{N}(F)$.

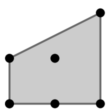
Example 1. Consider the polynomial

$$f = xy + t \cdot (y + x + x^2 + x^2y^2) + t^3$$

The following diagram shows the support of f and its marked Newton polygon.



$\mathrm{supp}(f)$



$\big(\mathrm{N}(f), \mathrm{supp}(f)\big)$

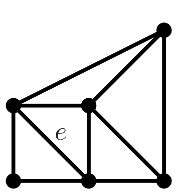
The tropicalization of f is

$$\mathrm{trop}(f) : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto \min\{x + y, 1 + y, 1 + x, 1 + 2x, 1 + 2x + 2y, 3\}.$$

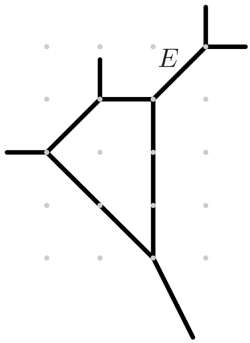
The support and Newton polygon of f respectively of $\mathrm{trop}(f)$ coincide. In order to compute the marked subdivision of the Newton polygon note that the points

$$(0, 1, 1), (1, 0, 1), (2, 0, 1), (2, 2, 1)$$

lie in a plane while watching from below the point $(1, 1, 0)$ sticks out from this plane and the point $(0, 0, 3)$ lies way above it. We therefore get the following subdivision of the Newton polygon:



The polygon spanned by $(0,0)$, $(1,0)$ and $(0,1)$ is dual to the vertex of the tropical curve where the terms 3, $1+x$ and $1+y$ take their common minimum, which is at the point $(x,y)=(2,2)$. Similarly the polygon spanned by $(1,0)$, $(1,1)$ and $(0,1)$ corresponds to the point $(x,y)=(1,1)$, and the common face e of the two polygons then is dual to the edge connecting these two points. Note that the direction vector of this edge E is $v(E)=(1,1)$ is orthogonal to the face e connecting the points $(1,0)$ and $(0,1)$ and points from the starting point $(1,1)$ of E to its end point $(2,2)$. Computing the remaining vertices and edges of $\text{Trop}(C_f)$ we get the following graph.



4. *The tropical j-invariant of an elliptic plane tropical curve*

A toric surface defined by a convex lattice polygon with precisely one interior lattice point is embedded into projective space in such a way that the smooth hyperplane sections are the elliptic curves on the surface. We consider this the *standard* way to obtain an elliptic curve on a toric surface, which inspires the following notation for plane tropical curves where the Newton polygon of the defining tropical Laurent polynomial has precisely one interior point.

Definition 2. A tropical plane curve $\mathcal{C}_F$ defined by a plane tropical Laurent polynomial F whose Newton polygon has precisely one interior lattice point is called *standard elliptic.*

The plane tropical curve $\mathcal{C}_F$ in Example 1 is standard elliptic. Moreover, the graph $\mathcal{C}_F$ has *genus* one, where the genus of a graph is the number of independent cycles of the graph. Obviously a cycle in the graph corresponds to an interior lattice point of the subdivision being a vertex of at least three polygons in the subdivision of the Newton polygon. We want to make this more precise in the following definition.

Definition 3. Let $\mathcal{C}$ be a plane tropical curve with marked Newton polygon $(Q,\mathcal{A})$ and with dual marked subdivision $\{(Q_i,\mathcal{A}_i)\mid i=1,\dots,l\}$. Suppose that $\tilde{\omega}\in \text{Int}(Q)\cap\mathbb{Z}^2$ and that the $(Q_i,\mathcal{A}_i)$ are ordered such that $\tilde{\omega}$ is a vertex of Q_i for $i=1,\dots,k$ $(k\geqslant 1)$ and it is not contained in Q_i for $i=k+1,\dots,l$ (see Figure 2). We then say that $\tilde{\omega}$ *determines a cycle* of $\mathcal{C}$, namely the union of the edges of $\mathcal{C}$ dual to the facets emanating from $\tilde{\omega}$, and we say that these edges *form the cycle* determined by $\tilde{\omega}$. We define the *lattice length of the cycle* to be the sum of the lattice lengths of the edges which form the cycle, where for an edge E with

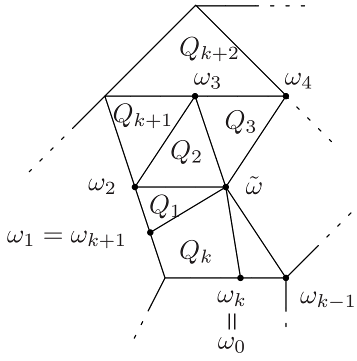


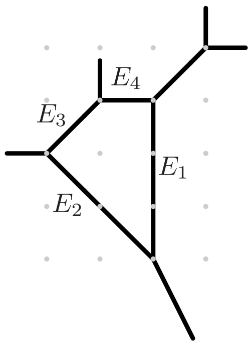
Figure 2: Marked subdivision determining a cycle

direction vector $v(E)$ (see p. 279) the *lattice length of* E is

$$l(E) = \frac{||E||}{||v(E)||}$$

the Euclidean length of E divided by that of $v(E)$.

Example 4. Coming back to our Example 1 the curve has one cycle dual to the interior lattice point $(1, 1)$ and it consists of four edges $E_1, \ldots, E_4$.



The edge E_1 is dual to the edge e_1 from $(0, 1)$ to $(1, 1)$ in the Newton subdivision in Example 1, so that its direction vector is $v(E_1) = (0, -1)$ of Euclidean length 1 and that the lattice length of E_1 is $l(E_1) = ||E_1|| = 3$. Doing similar computations for the other edges the cycle length is

$$l(E_1) + l(E_2) + l(E_3) + l(E_4) = 3 + 2 + 1 + 1 = 7.$$

Definition 5. If $\mathcal{C}$ is a standard elliptic plane tropical curve then $\mathcal{C}$ has at most one cycle, and we define its *tropical* j*-invariant* $j_{\text{trop}}(\mathcal{C})$ to be the lattice length of this cycle if it has one. If $\mathcal{C}$ has no cycle we define its tropical j-invariant to be zero.

In Example 4 the standard elliptic plane tropical curve has tropical j-invariant 7.

If a we fix the part of a Newton subdivision which determines the cycle then there is a nice formula to compute the cycle length, and thus the tropical j-invariant. For the proof we refer to [**9**, Lemma 9]. For the formulation of the statement we use the notation in Figure 2. For this note that the ω_i there are the endpoints of the edges in the Newton subdivision emanating from the vertex $\tilde{\omega}$, and that they are ordered clockwise.

LEMMA 6. *Let $(Q,\mathcal{A})$ be a marked polygon in $\mathbb{R}^2$ with marked subdivision $\mathcal{T} = \{(Q_i,\mathcal{A}_i) \mid i = 1,\dots,l\}$ and suppose that $\tilde{\omega} \in \operatorname{Int}(Q) \cap \mathbb{Z}^2$ is a vertex of Q_i for $i = 1,\dots,k$ and it is not contained in Q_i for $i = k+1,\dots,l$.*

If $u \in \mathbb{R}^{\mathcal{A}}$ is such that the plane tropical curve

$$F = \min\{u_{ij} + i \cdot x + j \cdot y \mid (i,j) \in \mathcal{A}\}$$

induces this subdivision (as described in Section 3), then $\tilde{\omega}$ determines a cycle in the plane tropical curve $\mathcal{C}_F$ and, using the notation in Figure 2, its length is

$$\sum_{j=1}^{k} (u_{\tilde{\omega}} - u_{\omega_j}) \cdot \frac{D_{j-1,j} + D_{j,j+1} + D_{j+1,j-1}}{D_{j-1,j} \cdot D_{j,j+1}}$$

where $D_{i,j} = \det(v_i, v_j)$ with $v_i = \omega_i - \tilde{\omega}$ and $v_j = \omega_j - \tilde{\omega}$.

This formula implies in particular the following corollary, since for a fixed subdivision of $(Q,\mathcal{A})$ it gives the formula for the cycle length j_{trop}.

COROLLARY 7. *If $(Q,\mathcal{A})$ is a marked lattice polygon in $\mathbb{R}^2$ with precisely one interior lattice point, then*

$$j_{\text{trop}} : \mathbb{R}^{\mathcal{A}} \longrightarrow \mathbb{R} : u \mapsto j_{\text{trop}}(u) := j_{\text{trop}}(\mathcal{C}_{F_u})$$

with

$$F_u = \min\{u_{ij} + i \cdot x + j \cdot y \mid (i,j) \in \mathcal{A}\}$$

*is a piecewise linear function which is linear on cones of the secondary fan (cf. [**5**, Chapter 7]) of $\mathcal{A}$.*

5. *Unimodular transformations preserve lattice length*

We want to relate the classical j-invariant to the tropical j-invariant, and we would again like to reduce the consideration of all possible Newton polygons with one interior point to the 16 polygons in Figure 1, or even better, to the three basic ones in the first group there. For that we have to understand the impact of an integral unimodular affine transformation on a plane tropical Laurent polynomial respectively the induced plane tropical curve.

Given a linear form $l = u + i \cdot x + j \cdot y = u + (i,j) \cdot (x,y)^t$ with $i, j \in \mathbb{Z}$ and $u \in \mathbb{R}$ and given an integral unimodular affine transformation

$$\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 : \alpha \mapsto A \cdot \alpha + \tau$$

with $A \in \mathrm{Gl}_2(\mathbb{Z})$ and $\tau \in \mathbb{Z}^2$, we let ϕ act on l via

$$l^{\phi} = u + (x,y) \cdot \phi\big((i,j)^t\big)$$

and we let ϕ act on a plane tropical Laurent polynomial $F = \min\{u_{ij} + i \cdot x + j \cdot y \mid (i,j) \in \mathcal{A}'\}$ via the linear forms, i.e.

$$F^{\phi} = \min\{u_{ij} + (x,y) \cdot \phi\big((i,j)^t\big) \mid (i,j) \in \mathcal{A}'\}.$$

Note that the translation by τ does not change the locus of non-differentiability of the piecewise linear function defined by F and the Newton polygon of F is just translated by τ. So τ has neither any impact on the Newton subdivision of F nor on the tropical curve defined by F. Moreover, it is obvious that if $\{(Q_i, \mathcal{A}_i) \mid i = 1, \ldots, k\}$ is the marked subdivision of $\big(\mathrm{N}(F), \mathrm{supp}(F)\big)$ induced by F, then $\{\phi(Q_i), \phi(\mathcal{A}_i) \mid i = 1, \ldots, k\}$ is the marked subdivision of $\big(\mathrm{N}(F^\phi), \mathrm{supp}(F^\phi)\big)$ induced by F^ϕ.

Note that one can generalize the lattice length of an edge with rational slope in $\mathbb{Z}^2$ in a straight forward way to an edge with rational slope in any given lattice Λ in $\mathbb{R}^2$. Moreover, an affine transformation $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 : x \mapsto A \cdot x + \tau$ with $A \in \mathrm{Gl}_2(\mathbb{Z})$ carries the given lattice $\mathbb{Z}^2$ into some other lattice $\Phi(\mathbb{Z}^2)$, and by definition the lattice length of the image of an edge under Φ with respect to the new lattice will coincide with the lattice length of the edge with respect to the old lattice. If Φ is unimodular, then the new lattice $\Phi(\mathbb{Z}^2)$ is just again the lattice $\mathbb{Z}^2$ and thus lattice length is preserved. This implies the following corollary.

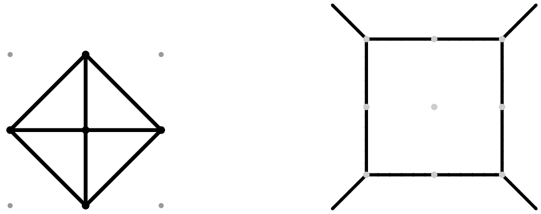
Corollary 8. *Let F be a plane tropical Laurent polynomial such that $\mathcal{C}_F$ is standard elliptic with positive tropical j-invariant and let ϕ be an integral unimodular affine transformation of $\mathbb{R}^2$, then $\mathcal{C}_{F^\phi}$ is standard elliptic with the same tropical j-invariant*

$$j_{\mathrm{trop}}(\mathcal{C}_F) = j_{\mathrm{trop}}(\mathcal{C}_{F^\phi}).$$

Example 9. Consider the polynomial

$$f = x^2y + xy^2 + \frac{1}{t} \cdot xy + x + y$$

inducing the following subdivision of its Newton polygon and the corresponding tropical curve:



$\mathrm{N}(f)$ subdivided $\qquad$ $\mathrm{Trop}(C_f)$

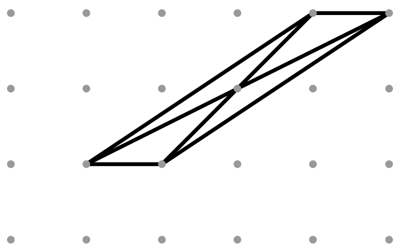
The plane tropical curve $\mathrm{Trop}(C_f)$ is standard elliptic with tropical j-invariant 8. If we now apply the integral unimodular affine transformation

$$\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 : \alpha \mapsto \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \cdot \alpha$$

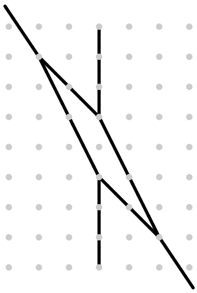
we get

$$f^\phi = x^5y^3 + x^4y^3 + \frac{1}{t} \cdot x^3y^2 + x^2y + xy$$

with the following subdivision of its Newton polygon and the corresponding standard elliptic plane tropical curve having again tropical j-invariant 8.



$\mathrm{N}(f^{\phi})$ subdivided



$\mathrm{Trop}(C_{f^{\phi}})$

The considerations in Section 2 together with this corollary will allow us to reduce the study of the tropicalization of an elliptic curve in a toric surface with an arbitrary Newton polygon with one interior point to the study of those whose Newton polygons are among the 16 polygons in Figure 1.

6. *The main result*

Let us suppose now that $(Q, \mathcal{A})$ is a lattice polygon with only one interior point.

Remark 10. In Section 2 we have seen that the j-invariant of a curve C_f with $\mathrm{supp}(f) \subseteq \mathcal{A}$ can be computed by plugging the coefficients a_{ij} of f into a suitable quotient $j = \frac{A_{\mathcal{A}}}{B_{\mathcal{A}}}$ of homogeneous polynomials $A_{\mathcal{A}}, B_{\mathcal{A}} \in \mathbb{Q}[\underline{a}]$. This means in particular, that the valuation of the j-invariant can be read off $A_{\mathcal{A}}$ and $B_{\mathcal{A}}$ directly, unless some unlucky cancellation of leading terms occurs.

This leads to the following definition.

Definition 11. The *generic valuation* of a polynomial $0 \neq H = \sum_{\omega} H_{\omega} \cdot \underline{a}^{\omega} \in \mathbb{Q}[\underline{a}]$ with $\underline{a} = (a_{ij} \mid (i,j) \in \mathcal{A})$ is

$$\mathrm{val}_H : \mathbb{R}^{\mathcal{A}} \longrightarrow \mathbb{R} : u \mapsto \mathrm{val}_H(u) = \min\{u \cdot \omega \mid H_{\omega} \neq 0\},$$

where

$$u \cdot \omega = \sum_{(i,j) \in \mathcal{A}_c} u_{ij} \cdot \omega_{ij}.$$

The *generic valuation of the j-invariant* is the function

$$\mathrm{val}_j : \mathbb{R}^{\mathcal{A}} \longrightarrow \mathbb{R} : u \mapsto \mathrm{val}_j(u) = \mathrm{val}_{A_{\mathcal{A}}}(u) - \mathrm{val}_{B_{\mathcal{A}}}(u).$$

Note that the tropical j-invariant is a *tropical rational function* in the sense of [**13**, Sec. 2.2] and [**1**, Def. 3.1].

Remark 12. As mentioned above, unless some unlucky cancellation of the leading terms occurs for any $f = \sum_{(i,j) \in \mathcal{A}} a_{ij} \cdot x^i y^j \in \mathbb{K}[x,y]$ with $u_{ij} = \mathrm{val}(a_{ij})$ for all $(i,j) \in \mathcal{A}$ we have

$$\mathrm{val}_j(u) = \mathrm{val}\big(j(f)\big).$$

Note also, that if D is a cone of the Gröbner fan of $A_{\mathcal{A}}$ and D' is a cone of the Gröbner fan of $B_{\mathcal{A}}$ then the restriction

$$\mathrm{val}_{j_{|}} : D \cap D' \longrightarrow \mathbb{R}$$

of val_j to $D \cap D'$ is *linear* by definition, and if both are top-dimensional, then no unlucky cancellation of leading terms can occur. In particular, the *generic valuation of the j-invariant* val_j is a piece linear function.

We can now state the main result of our paper whose proof is discussed in the subsequent sections.

THEOREM 13. *Let $(Q, \mathcal{A})$ be a lattice polygon with only one interior lattice point. If $u \in \mathbb{R}^{\mathcal{A}}$ is such that $\mathcal{C}_F$ with*

$$F : \mathbb{R}^2 \longrightarrow \mathbb{R} : (x, y) \mapsto \min\{u_{ij} + i \cdot x + j \cdot y \mid (i, j) \in \mathcal{A}\}$$

has a cycle, then

$$\mathrm{val}_j(u) = -j_{\mathrm{trop}}(u).$$

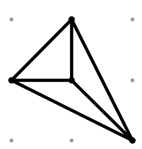
Moreover, if u is in a top-dimensional cone of the secondary fan of $\mathcal{A}$ and $f = \sum_{(i,j)\in\mathcal{A}} a_{ij} \cdot x^i y^j$ with $\mathrm{val}(a_{ij}) = u_{ij}$, then

$$\mathrm{val}\big(j(f)\big) = -j_{\mathrm{trop}}(\mathcal{C}_F) = -j_{\mathrm{trop}}\big(\mathrm{Trop}(C_f)\big).$$

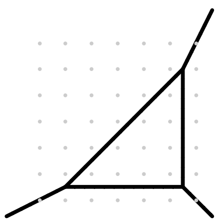
Example 14. Consider the curve C_f defined by

$$f = t^{\frac{3}{2}} \cdot (y + x^2 + xy^2) + xy$$

with the following subdivision of the Newton polygon and the corresponding standard elliptic plane tropical curve $\mathrm{Trop}(C_f)$:



$\mathrm{N}(f) = Q_{cm}$



$\mathrm{Trop}(C_f)$

The vertices of $\mathrm{Trop}(C_f)$ are

$$\left(\frac{3}{2}, 3\right), \quad \left(\frac{3}{2}, -\frac{3}{2}\right) \quad \text{and} \quad \left(-3, -\frac{3}{2}\right),$$

so that its tropical j-invariant is $j_{\mathrm{trop}}\big(\mathrm{Trop}(f)\big) = \frac{27}{2}$, while its j-invariant

$$j(f) = -\frac{1 + 72 \cdot t^{\frac{9}{2}} + 1728 \cdot t^9 + 13824 \cdot t^{\frac{27}{2}}}{t^{\frac{27}{2}} + 27 \cdot t^{18}}$$

has valuation $-\frac{27}{2}$.

An immediate consequence of the above theorem is the following corollary.

COROLLARY 15. *If $f = \sum_{(i,j)\in\mathcal{A}} a_{ij} \cdot x^i y^j \in \mathbb{K}[x, y]$ defines a smooth elliptic curve in $X_{\mathcal{A}}$ whose j-invariant has non-negative valuation, then $\mathrm{Trop}(C_f)$ has no cycle.*

7. *Reduction to $\mathcal{A} \in \{\mathcal{A}_a, \mathcal{A}_b, \mathcal{A}_c\}$*

Using an integral unimodular transformation we may assume that Q is one of the 16 polygons in Figure 1, since the application of such a transformation does not effect the statement of Theorem 13 due to Corollary 8 and Section 2.

Next we want to reduce to the cases $\mathcal{A} \in \{\mathcal{A}_a, \mathcal{A}_b, \mathcal{A}_c\}$.

If $f = \sum_{(i,j)\in\mathcal{A}} a_{ij} \cdot x^i y^j \in \mathbb{K}[x, y]$ with $\mathrm{supp}(f) \subseteq \mathcal{A}$ and we replace f by

$$f' = f + t^\alpha \cdot \sum_{(i,j)\in\mathcal{A}\setminus\mathrm{supp}(f)} x^i y^j$$

where α is much larger than $\max\{\mathrm{val}(a_{ij}) \mid (i, j) \in \mathrm{supp}(f)\}$, then obviously

$$\mathrm{val}\big(j(f)\big) = \mathrm{val}\big(j(f')\big)$$

if f defines a smooth elliptic curve with $j(f) \neq 0$.

Moreover, if we allow to plug in into val_j points u where some of the u_{ij} are ∞ (as long as the result still is a well defined real number), then we can evaluate val_j at u with $u_{ij} = \mathrm{val}(a_{ij}) \in \mathbb{R} \cup \{\infty\}$, f defines a smooth elliptic curve and we get obviously

$$\mathrm{val}_j(u) = \mathrm{val}_j(u')$$

where $u'_{ij} = u_{ij}$ for $(i, j) \in \mathrm{supp}(f)$ and else $u'_{ij} = \alpha$ with α sufficiently large.

Finally, if in the definition of F_u we allow some u_{ij} to be ∞ then with the above notation the cycle of $\mathcal{C}_{F_u}$ and $\mathcal{C}_{F_{u'}}$ will not change, so that

$$j_{\mathrm{trop}}(u) = j_{\mathrm{trop}}(\mathcal{C}_{F_u}) = j_{\mathrm{trop}}(\mathcal{C}_{F_{u'}}) = j_{\mathrm{trop}}(u').$$

This shows that whenever we may as well assume that $\mathcal{A} \in \{\mathcal{A}_a, \mathcal{A}_b, \mathcal{A}_c\}$.

8. *The cases $\mathcal{A}_a$, $\mathcal{A}_b$ and $\mathcal{A}_c$*

The case $\mathcal{A}_c$ has been treated in [**9**], and the two other cases work along the same lines. We therefore will be rather short in our presentation. Instead of considering all the cases by hand, as was done in [**9**] we will refer to computations done using the computer algebra systems `polymake` [**4**], `TOPCOM` [**16**] and SINGULAR [**6**]. The code that we used for this is contained in the SINGULAR library `jinvariant.lib` (see [**10**]) and it is available via the URL

http://www.lms.ac.uk/jcm/12/lms2008-012/appendix-a/jinvariant.lib.

A more detailed explanation of the computations will be given in Section 9.

Fix now $(Q, \mathcal{A})$ with $\mathcal{A} \in \{\mathcal{A}_a, \mathcal{A}_b, \mathcal{A}_c\}$.

We first of all observe that by Corollary 7 the tropical j-invariant is linear on the cones of the secondary fan of $\mathcal{A}$ and that by Lemma 6 we can read off the assignment rule on each cone from the Newton subdivision of $(Q, \mathcal{A})$. Moreover, for the statement in Theorem 13 we only have to consider such cones for which the interior lattice point of Q is visible in the subdivision.

If $U_\mathcal{A} \subseteq \mathbb{R}^\mathcal{A}$ is the union of these cones, then it was in each of the cases $\mathcal{A} \in \{\mathcal{A}_a, \mathcal{A}_b, \mathcal{A}_c\}$ computed by the procedure `testInteriorInequalities` in the library jinvariant.lib that $U_\mathcal{A}$ is contained in a single cone of the Gröbner fan of $A_\mathcal{A}$

and for the restriction of $\mathrm{val}_{A_{\mathcal{A}}}$ to $U_{\mathcal{A}}$ we have

$$\mathrm{val}_{A_{\mathcal{A}}|} : U_{\mathcal{A}} \longrightarrow \mathbb{R} : u \mapsto 12 \cdot u_{11}.$$

It suffices therefore to show that $\mathrm{val}_{B_{\mathcal{A}}}$ is linear on the cones of the secondary fan of $\mathcal{A}$ and to compare the assignment rules for val_j and j_{trop} on each of these cones.

The two marked polygons $(Q_b, \mathcal{A}_b)$ and $(Q_c, \mathcal{A}_c)$ define *smooth* toric surfaces and in these cases $B_{\mathcal{A}} = \Delta_{\mathcal{A}}$ is the $\mathcal{A}$-discriminant of $\mathcal{A}$ (cf. [**5**, Chapter 9]). Therefore, by the Prime Factorization Theorem (see [**5**, Theorem 10.1.2]) the secondary fan of $\mathcal{A}_b$ respectively $\mathcal{A}_c$ is a refinement of the Gröbner fan of the $B_{\mathcal{A}_b}$ respectively $B_{\mathcal{A}_c}$. In view of Remark 12 and by the above considerations this shows in particular that val_j is linear on each cone of the secondary fan of $\mathcal{A}$ for $\mathcal{A} \in \{\mathcal{A}_b, \mathcal{A}_c\}$ which is contained in $U_{\mathcal{A}}$. The comparison of the assignment rules of the two linear functions val_j and j_{trop} on each of the cones contained in $U_{\mathcal{A}}$ was done by the procedure `displayFan` from the SINGULAR library jinvariant.lib using `TOPCOM` and `polymake`. It produces two postscript files which show all the different cases together with the assignment rules. The files are available via

www.lms.ac.uk/jcm/12/lms2008-012/appendix-a/secondary_fan_of_2x2.ps

for $\mathcal{A} = \mathcal{A}_b$ respectively via

www.lms.ac.uk/jcm/12/lms2008-012/appendix-a/secondary_fan_of_cubic.ps

for $\mathcal{A} = \mathcal{A}_c$ respectively. 849 cases have to be considered for $\mathcal{A} = \mathcal{A}_c$ and 255 for $\mathcal{A} = \mathcal{A}_b$.

In the case $\mathcal{A} = \mathcal{A}_a$ the toric surface $X_{\mathcal{A}}$ is *not smooth*, but a quadric cone. Moreover, in this case $B_{\mathcal{A}}$ is *not* the $\mathcal{A}$-discriminant $\Delta_{\mathcal{A}}$, but instead

$$B_{\mathcal{A}} = u_{02}^2 \cdot \Delta_{\mathcal{A}}.$$

Thus, the Gröbner fan of $B_{\mathcal{A}}$ coincides with the Gröbner fan of $\Delta_{\mathcal{A}}$ and it is still true by the Prime Factorization Theorem that the secondary fan of $\mathcal{A}$ is a refinement of the Gröbner fan of $B_{\mathcal{A}}$. We can therefore argue as above, and the case distinction (202 cases) can be viewed via

www.lms.ac.uk/jcm/12/lms2008-012/appendix-a/secondary_fan_of_4x2.ps.

This finishes our proof, where for the "moreover" part we take Remark 12 into account.

Remark 16. It follows from the proof that j_{trop} is indeed linear on each cone of the Gröbner fan of $B_{\mathcal{A}}$ in the above cases. This could have been proved directly with the same argument as in [**9**, Lemma 16].

If we denote by $D_{\mathcal{A}}$ the regular $\mathcal{A}$-determinant (cf. [**5**, Section 11.1]), then $D_{\mathcal{A}_b} = \Delta_{\mathcal{A}_b}$ and $D_{\mathcal{A}_c} = \Delta_{\mathcal{A}_c}$ by [**5**, Theorem 11.1.3] since $X_{\mathcal{A}}$ is smooth in these cases. Even though in general the regular $\mathcal{A}$-determinant is not a polynomial, it is so for $\mathcal{A} = \mathcal{A}_a$ by [**5**, Theorem 11.1.6] since $\mathcal{A}_a$ is quasi-smooth by [**5**, Theorem 5.4.12] in the sense of that theorem. More precisely, we have

$$D_{\mathcal{A}_a} = u_{02} \cdot \Delta_{\mathcal{A}_a}$$

and by [**5**, Theorem 11.1.3] it is a divisor of the principal $\mathcal{A}$-determinant $E_{\mathcal{A}}$ (cf. [**5**, Chapter 9]). Therefore, the secondary fan of $\mathcal{A}$ (which is the Gröbner fan of $E_{\mathcal{A}}$) is a refinement of the Gröbner fan of $D_{\mathcal{A}}$ and thus of the Gröbner fan of $B_{\mathcal{A}}$.

One therefore could have used the description of the vertices of the Newton polytope of $D_{\mathcal{A}}$ in [**5**, Theorem 11.3.2] in order to show that on each cone of the Gröbner fan of $B_{\mathcal{A}}$ contained in $U_{\mathcal{A}}$ the two functions val_j and j_{trop} coincide by a direct case study as was done in [**9**, Lemma 19] for the case $\mathcal{A} = \mathcal{A}_c$.

9. *How computer algebra was used to prove Theorem 13.*

In this section we will explain in more detail how methods from computer algebra were used to prove the main result Theorem 13. We explain what objects have to be computed (e.g. secondary fans, Gröbner fans, etc.) and how this can be done. Moreover, we give a short description of the procedures and programs used for the computations. The first subsection is illustrated by a pet example which is simple but shows the features of interest.

9.1. *Computing the secondary fan*

The program `TOPCOM` by Jörg Rambau (see [**16**]) is a command line tool to compute all triangulations of a given configuration of lattice points. In the following we will restrict to configurations in the plane. The procedure `points2triangs` of this package takes a lattice point configuration, say $\mathcal{A}$, as input and computes the regular triangulations of the point configuration. E.g. if the configuration consists of the points $\{(0,0),(1,0),(2,0),(1,1)\}$ (see Figure 3) the input will be given as a matrix of homogeneous coordinates

```
[[0,0,1],[1,0,1],[2,0,1],[1,1,1]]
```

and the output

```
T[1]:={{0,1,3},{1,2,3}};
T[2]:={{0,2,3}};
```

shows for each triangulation the points which form the triangles, where 0 corresponds to the first point, i.e. $(0,0)$, 1 to the second point, i.e. $(1,0)$, and so on.

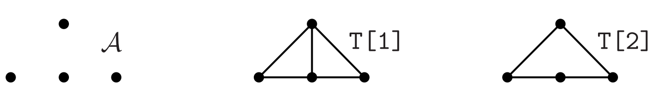


Figure 3: A point configuration and its triangulations

Once we have the regular triangulations we can compute the vertices of the secondary polytope $\Sigma_{\mathcal{A}}$ of $\mathcal{A}$. It is a lattice polytope in $\mathbb{R}^n$ where n is the number of lattice points in the given configuration $\mathcal{A}$. For this we have to fix an ordering of the lattice points in the configuration, which amounts to fixing coordinates on $\mathbb{R}^n$. Given a triangulation $T = \{T_1, \ldots, T_k\}$ then the vertex $v_T = (x_1, \ldots, x_n)$ of $\Sigma_{\mathcal{A}}$ corresponding to T is computed as follows: the i-th coordinate x_i is twice the sum of the areas of all triangles in T for which the i-th point in the configuration $\mathcal{A}$ is a vertex (see [**5**, Def. 7.1.6]). In our pet example above the secondary polytope consists of the line segment joining the points $(1,2,1,2)$ and $(2,0,2,2)$ in $\mathbb{R}^4$. It is a theorem that the regular triangulations correspond to the vertices of the secondary polytope (see [**5**, Thm. 7.1.7]).

The computation of the vertices of the secondary polytope for the lattice point configurations $\mathcal{A}_a$, $\mathcal{A}_b$ and $\mathcal{A}_c$ that we are interested in is done via the Singular procedure `secondaryPolytope` in the library `polymake.lib`. It calls the external program `points2triangs` in order to compute the regular triangulations.

The secondary fan $\mathrm{SF}(\mathcal{A})$ of $\mathcal{A}$ is then defined as the normal fan of the secondary polytope $\Sigma(\mathcal{A})$. In particular, the vertices of $\Sigma(\mathcal{A})$ are in one-to-one correspondence with the cones of $\mathrm{SF}(\mathcal{A})$. We can describe such a cone by linear inequalities. If we know for a vertex v of $\Sigma(\mathcal{A})$ to which other vertices, say $v_1, \ldots, v_k$, it is connected along an edge of $\Sigma(\mathcal{A})$, then the vectors $v - v_1, \ldots, v - v_k$ interpreted as inequalities define the cone of $\mathrm{SF}(\mathcal{A})$ dual to the vertex v of $\Sigma(\mathcal{A})$: for this we interpret a vector $v - v_i = (a_1, \ldots, a_n)$ as the inequality

$$a_1 \cdot x_1 + \ldots + a_n \cdot x_n \geqslant 0.$$

The above mentioned procedure `secondaryPolytope` computed the vertices of the secondary polytopes in the three cases we were interested in. We then used the program `polymake` by Ewgenij Gawrilow and Michael Joswig (see [4]) to compute the so called vertex-edge-graph of the secondary polytope. It contains the information which vertices are connected to each other. For this one stores the vertices of the secondary polytope in a file `aaa.pm` and calls `polymake` with this file. In our pet example the file would contain the following data:

```
_application polytope
_version 2.3
_type RationalPolytope

POINTS
1 1 2 1 2
1 2 0 2 2
```

Again the points are handed over in homogeneous coordinates, but unlike with `TOPCOM` here the first coordinate is the homogenizing one. The command

```
polymake aaa.pm VERTICES AFFINE_HULL GRAPH
```

then computes the equations of the affine hull, i.e. the smallest affine space containing the polytope, and the vertex-edge-graph of the polytope. For our pet example the output would be:

```
VERTICES
1 1 2 1 2
1 2 0 2 2

GRAPH
{1}
{0}

AFFINE_HULL
-4 2 1 0 0
0 -1 0 1 0
-2 0 0 0 1
```

Each row below AFFINE_HULL defines one of the equations of the affine hull, and the first row should be interpreted as:

$$0 = -4 + 2 \cdot x_1 + x_2.$$

Each row below VERTICES gives one vertex of the polytope, which are of course just the points we handed over. GRAPH describes the vertex-edge-graph, where the ith row below graph lists the vertices to which the vertex of the ith row below VERTICES is connected. Note that vertices are labelled starting from 0. In our pet example the first vertex is connected to the vertex with label 1, i.e. the second vertex, while the second vertex is connected to the one with label 0, i.e. the first one. Technically the computations are realized by calling the SINGULAR procedure polymakePolytope from the library polymake.lib, which in turn invokes polymake. Computing the inequalities of the cones of the secondary fan is then an easy task.

If the secondary polytope is contained in an affine space of codimension d, its affine hull, then there is a linear space of dimension d which is contained in each cone of the secondary fan. This space is called the linearity space of the secondary fan and it is spanned by the homogeneous part of the equations of the affine hull of the secondary polytope. In our pet example the linearity space is spanned by the vectors

$$w_1 = (2, 1, 0, 0)^t, w_2 = (-1, 0, 1, 0)^t, w_3 = (0, 0, 0, 1)^t.$$

We consider the secondary fan always modulo this linearity space, so that a cone in the secondary fan of dimension $d + 1$ will be called a ray.

From the inequalities of a cone one can then compute the extremal rays of the cone, i.e. the $d + 1$-dimensional faces of the cone. The computations can be done with polymake, handing over the inequalities and the generators of the linearity space and using the option VERTICES. In our pet example the input for the second cone would be:

```
_application polytope
_version 2.3
_type RationalPolytope

INEQUALITIES
0 1 -2 1 0

EQUATIONS
0 2 1 0 0
0 -1 0 1 0
0 0 0 0 1
```

This results in the output:

```
VERTICES
0 1 -2 1 0
1 0 0 0 0
```

A "vertex" with first coordinate zero corresponds here to an extremal ray. The cone in the example thus consists only of one ray, which is given as

$$\{\lambda \cdot (1, -2, 1, 0)^t + \kappa_1 \cdot w_1 + \kappa_2 \cdot w_2 + \kappa_3 \cdot w_3 \mid \lambda \in \mathbb{R}_{\geqslant 0}, \kappa_1, \kappa_2, \kappa_3 \in \mathbb{R}\},$$

and is thus actually a half space of dimension four.

This finishes the computation of the cones of the secondary fan, and for each such cone we know the corresponding triangulation of the convex hull of $\mathcal{A}$.

9.2. *Dealing with the Gröbner fan of A*

In Section 8 we claim that the union of all cones of the secondary fan corresponding to Newton subdivisions that are triangulations where the interior point is a vertex of some triangle is in a single cone of the Gröbner fan of the numerator A of the j-invariant.

In the secondary polytope of a lattice point configuration $\mathcal{A} \in \{\mathcal{A}_a, \mathcal{A}_b, \mathcal{A}_c\}$ we can easily determine the vertices which correspond to regular triangulations where the interior lattice point is a vertex of some triangle, since then the corresponding coordinate is non-zero. Considering the convex hull, say U, of the union of all the extremal rays in these cones we can be sure that U contains each of these cones, i.e. $U_{\mathcal{A}} \subseteq U$ in the notation of Section 8. This can again be done by `polymake` in a similar way as above, where rays are considered as points with homogenizing component 0.

We then have to compute the Gröbner fan of A. For this we compute first its Newton polytope by handing the exponent vectors of A as points to `polymake` with the option `VERTICES`. It turns out that the exponent vector of a_{11}^{12} is a vertex of the Newton polytope in each of the three cases $\mathcal{A} \in \{\mathcal{A}_a, \mathcal{A}_b, \mathcal{A}_c\}$. The Gröbner fan of $A_{\mathcal{A}}$ is the normal fan of its Newton polytope, and we are only interested in the cone dual to the vertex given by the exponent vector of a_{11}^{12}. Computing first the vertex-edge-graph of the Newton polytope we can then compute inequalities describing this cone with the aid of `polymake` in the way explained above.

Note that the coordinates for both the secondary fan of $\mathcal{A}$ and the Gröbner fan of $A_{\mathcal{A}}$ correspond to the lattice points in $\mathcal{A}$. In order to check if the cone U is completely contained in the one cone of the Gröbner fan of $A_{\mathcal{A}}$ computed above it suffices to check if the extremal rays of U satisfy the inequalities of the cone in the Gröbner fan. This is an easy task, and it shows that on each cone of interest in the secondary fan the generic valuation of the numerator $A_{\mathcal{A}}$ of the j-invariant will be $12 \cdot u_{11}$.

All computations are carried out by `testInteriorInequalities`, a Singular procedure from the library jinvariant.lib which in turn invokes `polymake` for several computations.

9.3. *Comparing j_{trop} and* val_j

It remains to compare the generic valuation of $B_{\mathcal{A}}$ to the cycle length on each cone of the secondary fan of $\mathcal{A}$ corresponding to a Newton subdivision, which is a triangulation where the interior point is a vertex of some triangle. They should differ by $12 \cdot u_{11}$.

We have already computed the interesting cones of the secondary fan (see Section 9.1). Thus the formula in Lemma 6 allows us to compute the formula of the linear function j_{trop} on each such cone by just computing certain determinants.

In the cases we are interested in to each regular triangulation of the convex hull of $\mathcal{A}$ we can associate a vertex of the Newton polytope of the regular $\mathcal{A}$-determinant $D_{\mathcal{A}}$. This is no one-to-one correspondence, but given the triangulation there is an easy formula for computing the coordinates of the associated vertex in the Newton polytope of $D_{\mathcal{A}}$ (see [5, Theorem 11.3.2]). Moreover, the polynomial $B_{\mathcal{A}}$ is either equal to $D_{\mathcal{A}}$ (if $\mathcal{A} = \mathcal{A}_b$ or $\mathcal{A} = \mathcal{A}_c$) or it differs from this one by a factor u_{02} (if $\mathcal{A} = \mathcal{A}_a$) (see Remark 16). In any case the generic valuation of $B_{\mathcal{A}}$ on a full-dimensional cone C of the secondary fan can (up to a summand u_{02} in the case $\mathcal{A} = \mathcal{A}_a$) be deduced from the coordinates of the vertex of the Newton polytope of $D_{\mathcal{A}}$ which corresponds to the triangulation corresponding to C. For this we only have to interpret the vector as a linear function by matrix multiplication.

The computation of the linear functions j_{trop} and val_j on each cone of interest of the secondary fan are done by the SINGULAR procedure `displayFan` in the library jinvariant.lib. The procedure tests that the linear functions coincide, and moreover, it produces for each of the cases $\mathcal{A}_a$, $\mathcal{A}_b$ and $\mathcal{A}_c$ a latex file which shows all possible triangulations together with the formulae of the linear functions j_{trop} and val_j on the corresponding cone of the secondary fan and some additional information.

Appendix A. *jinvariant.lib*

The file is a SINGULAR library containing most of the SINGULAR procedures described above. Note that the libraries `tropical.lib` and `polymake.lib` are an integral part of each SINGULAR distribution from Version 3.1 on.

Appendix B. *secondary_fan_of_4x2.pdf*

This file contains the output of the SINGULAR procedure `displayFan` showing the 296 different regular triangulations of the secondary fan of $\mathcal{A}_a$ together with the formulae for val_j and for j_{trop} on each cone of the secondary fan where the interior lattice point is a vertex of some triangle. These are the first 202 cones discussed in the file.

Appendix C. *secondary_fan_of_2x2.pdf*

This file contains the output of the SINGULAR procedure `displayFan` showing the 387 different regular triangulations of the secondary fan of $\mathcal{A}_c$ together with the formulae for val_j and for j_{trop} on each cone of the secondary fan where the interior lattice point is a vertex of some triangle. These are the first 255 cones discussed in the file.

Appendix D. *secondary_fan_of_cubic.pdf*

This file contains the output of the SINGULAR procedure `displayFan` showing the 1166 different regular triangulations of the secondary fan of $\mathcal{A}_c$ together with the formulae for val_j and for j_{trop} on each cone of the secondary fan where the interior lattice point is a vertex of some triangle. These are the first 849 cones discussed in the file.

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