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SOME INTEGRAL EQUATIONS WITH KUMMER'S FUNCTIONS IN THE KERNELS

BY
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1. **Introduction.** Since 1963 several authors ([13], [2], [6], [14], [10], [11], [12], [9]) have considered integral equations each one of which is contained as a special case in one of the two equations

$$(1.1)\qquad K^{\beta}(a, b, c)f(x) \equiv \int_x^{\beta} \frac{(t-x)^{b-1}}{\Gamma(b)}\, {}_1F_1(a, b, c(x-t))f(t)\,dt =^{\circ} g(x),$$

$$(1.2)\qquad K_{\alpha}(a, b, c)f(x) \equiv \int_{\alpha}^{x} \frac{(x-t)^{b-1}}{\Gamma(b)}\, {}_1F_1\,(a, b, c(x-t))f(t)\,dt =^{\circ} g(x)$$

for $\operatorname{Re} b>0$ and $x \in [\alpha, \beta]$.

In this paper we make a systematic use of the fractional integration operator J^{μ} (see §2) to prove results on the linear operator $K^{\beta}(a, b, c)$ and to discuss theorems on the solutions of (1.1). The technique used immediately suggests that the corresponding results on $K_{\alpha}(a, b, c)$ and the related equation (1.2) also hold.

The functions f and g belong to the class of measurable (in the Lebesgue sense) complex-valued functions defined for almost all x in $[\alpha, \beta]$. Functions in this class which are equal almost everywhere in $[\alpha, \beta]$ are not regarded as distinct. Indeed, all the integral equations considered are understood to hold for almost all x in $[\alpha, \beta]$.

The restriction $\operatorname{Re} b>0$ is imposed throughout the paper to ensure the convergence of the Lebesgue integrals used but the methods of the paper can be applied to obtain solutions of (1.1) and (1.2) for $\operatorname{Re} b \le 0$ also, provided the integrals are interpreted as convolutions of generalized functions and the discussion is in the domain of generalized functions (see [3] and [5]).

2. **Linear operators J^{μ} and $K^{\beta}(a, b, c)$.** L is the space of (equivalent classes of) complex-valued functions f of a real variable which are L-integrable on a finite interval $[\alpha, \beta]$, $\alpha \ge 0$ and is normed by $\|f\| = \int_{\alpha}^{\beta} |f(t)|\,dt$.

The operator J^{μ}. For complex μ with $\operatorname{Re} \mu>0$ the linear operator J^{μ} on L into itself is defined by

$$(2.1)\qquad (J^{\mu}f)(x) = \int_x^{\beta} \frac{(t-x)^{\mu-1}}{\Gamma(\mu)} f(t)\,dt \quad \text{for almost all } x \in [\alpha, \beta].$$

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It is simple to see that J^{μ} is bounded. If the integral equation

$$J^{\mu}f = g, \quad \operatorname{Re}\mu > 0 \tag{2.2}$$

has a solution $f \in L$, then its uniqueness is ensured by a well-known theorem; also the solution is denoted by $J^{-\mu}g$. Thus when $\operatorname{Re}\mu < 0$, $J^{\mu} = (J^{-\mu})^{-1}$. We set $J^0 f = f$ and denote by L_{μ} the subspace of functions g in L such that $g = J^{\mu}f$ with $f \in L$ and $\operatorname{Re}\mu > 0$. If $0 < \operatorname{Re}\mu < \operatorname{Re}\nu$, then it is easily verified that $L_{\nu} \subset L_{\mu} \subset L$, the inclusion being proper.

It is a standard result: if $\operatorname{Re}\mu > 0$, $\operatorname{Re}\nu > 0$, and $f \in L$, then

$$J^{\mu}J^{\nu}f = J^{\mu+\nu}f. \tag{2.3}$$

Indeed (2.3) holds for all complex μ and ν with $\operatorname{Re}\mu \neq 0$, $\operatorname{Re}\nu \neq 0$, $\operatorname{Re}(\mu+\nu) \neq 0$, provided that f is in a suitable L_{λ} such that $J^{\nu}f$ and $J^{\mu+\nu}f$ both exist in L. If $\operatorname{Re}\mu = 0$ and f is in L_{λ} with $\operatorname{Re}\lambda > 0$, then $J^{\mu}f$ is defined by

$$J^{\mu}f = J^{-1}J^{1+\mu}f \tag{2.4}$$

and is in L.

The operator $K^{\beta}(a, b, c)$. We begin with the following general lemma which will be of frequent use and is a generalization of the standard theorem on the existence and integrability of the fractional integral (2.1).

LEMMA 1. *If* $\operatorname{Re}\mu > 0$ *and* $f \in L$, *then the integral*

$$\int_x^{\beta} \frac{(t-x)^{\mu-1}}{\Gamma(\mu)} \, {}_1F_1(a, b, c(x-t)) f(t)\, dt$$

defines a function in L *where* ${}_1F_1$ *is, in the standard notation, Kummer's confluent hypergeometric function and* a, b, c *are complex parameters,* $b \neq 0, -1, -2, \ldots$

Proof. It suffices to show that

$$\int_{\alpha}^{\beta} dx \int_{\alpha}^{\beta} |(t-x)^{\mu-1} {}_1F_1(a, b, c(x-t)) f(t)| \, dt < \infty. \tag{2.5}$$

Since ${}_1F_1(a, b, z)$ is an entire function of z, it easily follows that the double integral (2.5) does not exceed $M(\operatorname{Re}\mu)^{-1}(\beta^{\operatorname{Re}\mu} - \alpha^{\operatorname{Re}\mu})\|f\|$, M being a bound of $|{}_1F_1(a, b, -cu)|$ for $0 \le u \le \beta - \alpha$.

LEMMA 2. *If* a, b, *and* c *are complex numbers with* $\operatorname{Re} b > 0$, *then for almost all* $x \in [\alpha, \beta]$

$$(K^{\beta}(a, b, c)f)(x) = \int_x^{\beta} \frac{(t-x)^{b-1}}{\Gamma(b)} \, {}_1F_1(a, b, c(x-t)) f(t)\, dt \tag{2.6}$$

defines a linear bounded operator $K^{\beta}(a, b, c)$ *on* L *into* L.

In particular when $c = 0$, *we get that operating on* L

$$K^{\beta}(a, b, 0) = J^{b}. \tag{2.7}$$

Proof. This follows from Lemma 1 with $\mu=b$ and a few simple verifications.

3. **Properties of $K^{\beta}(a, b, c)$.** We require

LEMMA 3. *If* $\operatorname{Re}\mu$ *and* $\operatorname{Re} b$ *are positive, then*

$$\int_x^t \frac{(v-x)^{\mu-1}}{\Gamma(\mu)}\frac{(t-v)^{b-1}}{\Gamma(b)}\, {}_1F_1(a, b, c(v-t))\, dv = \frac{(t-x)^{b+\mu-1}}{\Gamma(b+\mu)}\, {}_1F_1(a, b+\mu, c(x-t)), \tag{3.1}$$

$$\int_x^t \frac{(v-x)^{b-1}}{\Gamma(b)}\frac{(t-v)^{\mu-1}}{\Gamma(\mu)}\, {}_1F_1(a, b, c(x-v))\, dv = \frac{(t-x)^{b+\mu-1}}{\Gamma(b+\mu)}\, {}_1F_1(a, b+\mu, c(x-t)). \tag{3.2}$$

Proof. If $\operatorname{Re} b>0$, $\operatorname{Re}\mu>0$ and z is a complex number, then it is easily verified [9, (3.7)], that

$$\int_0^1 \frac{(1-s)^{\mu-1}}{\Gamma(\mu)}\frac{s^{b-1}}{\Gamma(b)}\, {}_1F_1(a, b, zs)\, ds = \frac{1}{\Gamma(b+\mu)}\, {}_1F_1(a, b+\mu, z). \tag{3.3}$$

Substitutions $s=(t-v)/(t-x)$, $z=c(x-t)$ in (3.3) at once give (3.1) while $s=(v-x)/(t-x)$, $z=c(x-t)$ lead to (3.2).

THEOREM 1. *If* $\operatorname{Re} b>0$ *and* $\operatorname{Re}\mu>-\operatorname{Re} b$, *then operating on* L

$$J^{\mu}K^{\beta}(a, b, c) = K^{\beta}(a, b+\mu, c). \tag{3.4}$$

Proof. (i) Suppose first that $\operatorname{Re}\mu>0$. For $f\in L$, $K^{\beta}(a, b, c)f$ is in L so that $J^{\mu}K^{\beta}(a, b, c)f$ also exists in L and for almost all $x\in[\alpha, \beta]$

$$J^{\mu}K^{\beta}(a, b, c)f(x) = \int_x^{\beta} \frac{(v-x)^{\mu-1}}{\Gamma(\mu)}\, dv \int_v^{\beta} \frac{(t-v)^{b-1}}{\Gamma(b)}\, {}_1F_1(a, b, c(v-t))f(t)\, dt. \tag{3.5}$$

Changing the order of integration in the repeated integral and using (3.1) we immediately get the result in this case; the change is justified since the repeated integral in (3.5) can be easily verified to be absolutely convergent for almost all x in $[\alpha, \beta]$.

(ii) Suppose next that $0>\operatorname{Re}\mu>-\operatorname{Re} b$. Since $\operatorname{Re}(b+\mu)>0$ and $f\in L$, (i) gives

$$J^{-\mu}K^{\beta}(a, b+\mu, c)f = K^{\beta}(a, b, c)f$$

which leads to (3.4) in this case also.

(iii) Finally suppose that $\operatorname{Re}\mu=0$. By case (i)

$$J^{1+\mu}K^{\beta}(a, b, c)f = K^{\beta}(a, b+\mu+1, c)f.$$

Also by case (i), $J^{1}K^{\beta}(a, b+\mu, c)f = K^{\beta}(a, b+\mu+1, c)f.$

Equating the left sides, we find that

$$\begin{aligned} K^{\beta}(a, b+\mu, c)f &= J^{-1}J^{1+\mu}K^{\beta}(a, b, c)f \\ &= J^{\mu}K^{\beta}(a, b, c)f, \end{aligned}$$

using (2.4).

THEOREM 2. *If* Re $b>0$, $f \in L$ *and* μ *is any complex number such that* $J^{\mu}f$ *exists in* L, *then*

$$J^{\mu}K^{\beta}(a, b, c)f = K^{\beta}(a, b, c)J^{\mu}f; \tag{3.6}$$

that is to say, the operator $K^{\beta}(a, b, c)$ *commutes with* J^{μ}.

Proof. (i) Let Re $\mu>0$; then for $f \in L$, $J^{\mu}f$ is in L and for almost all $x \in [\alpha, \beta]$

$$K^{\beta}(a, b, c)J^{\mu}f(x) = \int_x^{\beta} \frac{(u-x)^{b-1}}{\Gamma(b)}\ {}_1F_1(a, b, c(x-u))\, du \int_u^{\beta} \frac{(t-u)^{\mu-1}}{\Gamma(\mu)} f(t)\, dt. \tag{3.7}$$

Provided the order of integration in the repeated integral can be inverted, the right side is, for almost all $x \in [\alpha, \beta]$, equal to

$$\begin{aligned} \int_x^{\beta} f(t)\, dt \int_x^{t} \frac{(u-x)^{b-1}}{\Gamma(b)} \frac{(t-u)^{\mu-1}}{\Gamma(\mu)}\ {}_1F_1(a, b, c(x-u))\, du \\ = \int_x^{\beta} \frac{(t-x)^{b+\mu-1}}{\Gamma(b+\mu)}\ {}_1F_1\,(a, b+\mu, c(x-t))f(t)\, dt, \end{aligned}$$

evaluating the inner integral by (3.2). We thus arrive at

$$K^{\beta}(a, b, c)J^{\mu}f = K^{\beta}(a, b+\mu, c)f$$

which combines with the preceding theorem to give the desired result; as in that theorem the correctness of the inversion in the order of integration can be verified by Fubini's theorem.

(ii) When Re $\mu<0$, set $J^{\mu}f(x)=\phi(x)$ for almost all x in $[\alpha, \beta]$. Since ϕ is in L and $-$Re $\mu>0$, by case (i)

$$J^{-\mu}K^{\beta}(a, b, c)\phi(x) \overset{\circ}{=} K^{\beta}(a, b, c)J^{-\mu}\phi(x);$$

but $K^{\beta}(a, b, c)\phi$ exists in L so that

$$K^{\beta}(a, b, c)\phi = J^{\mu}K^{\beta}(a, b, c)f$$

and the result follows.

(iii) Let Re $\mu=0$. Since $J^{\mu}f$ exists in L, by case (i) and definition of J^{μ},

$$J^{1+\mu}K^{\beta}(a, b, c)f = K^{\beta}(a, b, c)J^{\mu+1}f = K^{\beta}(a, b, c)J^{1}J^{\mu}f.$$

By case (i) again, since $J^{\mu}f \in L$,

$$K^{\beta}(a, b, c)J^{1}(J^{\mu}f) = J^{1}K^{\beta}(a, b, c)(J^{\mu}f).$$

Thus $J^{-1}J^{1+\mu}K^{\beta}(a, b, c)f$ exists and is equal to $K^{\beta}(a, b, c)\ J^{\mu}f$.

THEOREM 3. *If* Re b, Re $b'>0$, *then operating on L*

$$K^{\beta}(a, b, c)K^{\beta}(a', b', c) = K^{\beta}(a+a', b+b', c). \tag{3.8}$$

Proof. For f in L and almost all $x \in [\alpha, \beta]$

$$K^{\beta}(a, b, c)K^{\beta}(a', b', c)f(x)$$

$$= \int_x^{\beta} \frac{(u-x)^{b-1}}{\Gamma(b)}\, {}_1F_1(a, b, c(x-u))\, du \int_u^{\beta} \frac{(t-u)^{b'-1}}{\Gamma(b')} \times {}_1F_1(a', b', c(u-t))f(t)\, dt \tag{3.9}$$

$$= \int_x^{\beta} f(t)\, dt \int_x^{t} \frac{(u-x)^{b-1}}{\Gamma(b)} \frac{(t-u)^{b'-1}}{\Gamma(b')}\, {}_1F_1(a, b, c(x-u)) \times {}_1F_1(a', b', c(u-t))\, du, \tag{3.10}$$

the reversal in the order of integration being valid if the repeated integral in (3.9) is absolutely convergent a.e. in $[\alpha, \beta]$. By Theorem 1

$$\int_u^{\beta} \left| \frac{(t-u)^{b'-1}}{\Gamma(b')}\, {}_1F_1(a', b', c(u-t))f(t) \right| dt$$

exists in L and by another application of the same theorem the absolute convergence, for almost all $x \in [\alpha, \beta]$, of the repeated integral in (3.9) follows.

Putting $v=(u-x)/(t-x)$ in the inner integral in (3.10), we write it as $(t-x)^{b+b'-1}I(x)$ where

$$I(x) = \int_0^1 \frac{v^{b-1}}{\Gamma(b)} \frac{(1-v)^{b'-1}}{\Gamma(b')}\, {}_1F_1(a, b, cv(x-t))\, {}_1F_1(a', b', c(1-v)(x-t))\, dv$$

$$= \int_0^1 \frac{v^{b-1}}{\Gamma(b)} \frac{(1-v)^{b'-1}}{\Gamma(b')} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_m(a')_n\{cv(x-t)\}^m\{c(1-v)(x-t)\}^n}{(b)_m(b')_n m!n!}\, dv \tag{3.11}$$

$$= \sum_{m,n=0}^{\infty} \frac{(a)_m(a')_n c^{m+n}(x-t)^{m+n}}{\Gamma(b+m)\Gamma(b'+n)m!n!} \int_0^1 v^{b+m-1}(1-v)^{b'+n-1}\, dv,$$

writing the series expansions for the Kummer's functions which we recall are entire functions, and inverting the order of integration and double summation. The inversion is correct because the double series in (3.11) converges uniformly with respect to v over the range of integration and for all x and t in $[\alpha, \beta]$.

Evaluating the Eulerian integral in terms of Gamma functions, we deduce that

$$I(x) = \frac{1}{\Gamma(b+b')} \sum_{m,n=0}^{\infty} \frac{(a)_m(a')_n c^{m+n}(x-t)^{m+n}}{(b+b')_{m+n} m!n!}$$

$$= \frac{1}{\Gamma(b+b')} \sum_{m=0}^{\infty} \frac{c^m(x-t)^m}{(b+b')_m} \sum_{n=0}^{m} \frac{(a)_{m-n}(a')_n}{(m-n)!n!} \tag{3.12}$$

$$= \frac{1}{\Gamma(b+b')} \sum_{m=0}^{\infty} \frac{(a+a')_m}{(b+b')_m} \frac{c^m(x-t)^m}{m!},$$

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in which we have summed the double series by diagonals instead of by rows, and used an analogue of Vandermonde's theorem. Thus we have

$$I(x) = \frac{1}{\Gamma(b+b')}\, {}_1F_1(a+a', b+b', c(x-t)) \tag{3.13}$$

from which it follows that the repeated integral in (3.10) is, for almost all $x \in [\alpha, \beta]$, equal to

$$\int_x^\beta \frac{(t-x)^{b+b'-1}}{\Gamma(b+b')}\, {}_1F_1(a+a', b+b', c(x-t))f(t)\, dt$$

and this establishes the theorem.

REMARK. Theorem 2 does not follow as a particular case of Theorem 3 though for $c=0$ both reduce to the standard result (2.3) on the product of fractional integration operators.

THEOREM 4. *If* Re $b>0$ *and* $f \in L$, *then for almost all* $x \in [\alpha, \beta]$

$$J^{-b}K^\beta(a, b, c)f(x) = f(x)-ac\int_x^\beta {}_1F_1(a+1, 2, c(x-t))f(t)\, dt. \tag{3.14}$$

Proof. By Theorem 1, for almost all $x \in [\alpha, \beta]$

$$\begin{aligned} J^{1-b}K^\beta(a, b, c)f(x) &= K^\beta(a, 1, c)f(x) \\ &= \int_x^\beta {}_1F_1(a, 1, c(x-t))f(t)\, dt \end{aligned}$$

which suggests that

$$\begin{aligned} J^{-b}K(a, b, c)f(x) &=^\circ -\frac{d}{dx}\int_x^\beta {}_1F_1(a, 1, c(x-t))f(t)\, dt \\ &= f(x)-ac\int_x^\beta {}_1F_1(a+1, 2, c(x-t))f(t)\, dt. \end{aligned}$$

As a direct verification of this we show that

$$J^b\left\{f(x)-ac\int_x^\beta {}_1F_1(a+1, 2, c(x-t))f(t)\, dt\right\} =^\circ K^\beta(a, b, c)f(x) \quad \text{for } x \in [\alpha, \beta]. \tag{3.15}$$

Since $f \in L$ and Re $b>0$, by suitable applications of Lemma 1 and Lemma 2, it is immediate that the left-hand member in (3.15) exists a.e. in $[\alpha, \beta]$ and can be written as

$$J^bf(x)-ac\int_x^\beta \frac{(u-x)^{b-1}}{\Gamma(b)}\, du \int_u^\beta {}_1F_1(a+1, 2, c(u-t))f(t)\, dt \tag{3.16}$$

$$=^\circ J^bf(x)-ac\int_x^\beta f(t)\, dt\int_x^t \frac{(u-x)^{b-1}}{\Gamma(b)}\, {}_1F_1(a+1, 2, c(u-t))\, du \tag{3.17}$$

by an application of Tonneli's theorem to invert the order of integration.

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It is not difficult to verify that for complex z and $\operatorname{Re} b>0$

$$ {}_1F_1(a,b,z) = 1+az\int_0^1 (1-s)^{b-1}{}_1F_1(a+1,2,zs)\,ds $$

which on putting $s=(t-u)/(t-x)$, $z=c(x-t)$ gives

$$ ac\int_x^t \frac{(u-x)^{b-1}}{\Gamma(b)}\,{}_1F_1(a+1,2,c(u-t))\,du = \frac{(t-x)^{b-1}}{\Gamma(b)}\{1-{}_1F_1(a,b,c(x-t))\}. $$

Since f is in L, for almost all $x\in[\alpha,\beta]$

$$ \begin{aligned} &ac\int_x^\beta f(t)\,dt\int_x^t \frac{(u-x)^{b-1}}{\Gamma(b)}\,{}_1F_1(a+1,2,c(u-t))\,du \\ &\quad= \int_x^\beta \frac{(t-x)^{b-1}}{\Gamma(b)}\{1-{}_1F_1(a,b,c(x-t))\}f(t)\,dt \\ &\quad= \int_x^\beta \frac{(t-x)^{b-1}}{\Gamma(b)}f(t)\,dt-\int_x^\beta \frac{(t-x)^{b-1}}{\Gamma(b)}\,{}_1F_1(a,b,c(x-t))f(t)\,dt \\ &\quad= J^bf(x)-K^\beta(a,b,c)f(x), \end{aligned} \tag{3.18} $$

where the existence of the double integral in (3.18) a.e. in $[\alpha,\beta]$ is ensured by the absolute convergence for almost all x of the repeated integral in (3.16). This also ensures the existence of the integral on the right in (3.18) which is written as the difference of two integrals existing a.e. in $[\alpha,\beta]$. Finally substituting the last expression in (3.17) we get (3.15) as desired.

COROLLARY 4.1. *If* $\operatorname{Re} b>0$ *and* $J^{-b}g$ *exists in* L, *then the integral equation for unknown* f *in* L

$$ \int_x^\beta \frac{(t-x)^{b-1}}{\Gamma(b)}\,{}_1F_1(a,b,c(x-t))f(t)\,dt =^\circ g(x) \quad \text{for } \alpha\le x\le\beta \tag{3.19} $$

is equivalent to the integral equation for unknown f *in* L

$$ f(x)-ac\int_x^\beta {}_1F_1(a+1,2,c(x-t))f(t)\,dt = J^{-b}g(x) \quad \textit{for } \alpha\le x\le\beta. \tag{3.20} $$

LEMMA 4. *If* $\operatorname{Re} b>0$ *and* $f\in L$, *then a.e. in* $[\alpha,\beta]$

$$ K^\beta(a,b,c)f(x)\,e^{cx} = e^{cx}K^\beta(b-a,b,-c)f(x). $$

Proof. It is immediate from Kummer's first formula [4, (6.3(7))].

THEOREM 5. *If* $\operatorname{Re} b>0$ *and* $f\in L$, *then for almost all* $x\in[\alpha,\beta]$

$$ K^\beta(a,b,c)f(x) = J^{b-a}\,e^{cx}\,J^a\,e^{-cx}f(x) \quad \text{for } \operatorname{Re} a>0, \tag{3.21} $$

$$ = e^{cx}\,J^a\,e^{-cx}\,J^{b-a}f(x) \quad \text{for } \operatorname{Re} a<\operatorname{Re} b. \tag{3.22} $$

Proof. (i) When $\operatorname{Re} a>0$, by Theorem 1 applied twice

$$ J^aK^\beta(a,b,c)f = J^bK^\beta(a,a,c)f. \tag{3.23} $$

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Since $f \in L$ and Re b, Re $a > 0$, the functions on both the sides exist in L. We now prove that under this assumption

$$K^{\beta}(a, b, c)f = J^{b-a}K^{\beta}(a, a, c)f. \tag{3.24}$$

If Re $a <$ Re b, (3.23) gives

$$J^{a}K^{\beta}(a, b, c)f = J^{b}K^{\beta}(a, a, c)f = J^{a}J^{b-a}K^{\beta}(a, a, c)f$$

so that

$$K^{\beta}(a, b, c)f = J^{b-a}K^{\beta}(a, a, c)f;$$

the last function exists in L since Re $(b-a) > 0$.

If Re $a >$ Re b, (3.23) leads to

$$J^{b}K^{\beta}(a, a, c)f = J^{a}K^{\beta}(a, b, c)f = J^{b}J^{a-b}K^{\beta}(a, b, c)f,$$

so that

$$K^{\beta}(a, a, c)f = J^{a-b}K^{\beta}(a, b, c)f$$

which gives

$$K^{\beta}(a, b, c)f = J^{-(a-b)}K^{\beta}(a, a, c)f,$$

the existence of the functions involved having already been noted.

If Re $a =$ Re b, instead of (3.23) we use

$$J^{1}K^{\beta}(a, b, c)f = J^{(b-a)+1}K^{\beta}(a, a, c)f$$

also obtained by two applications of Theorem 1. Since $K^{\beta}(a, b, c)f$ exists in L, this gives

$$\begin{aligned} K^{\beta}(a, b, c)f &= J^{-1}J^{(b-a)+1}K^{\beta}(a, a, c)f \\ &= J^{b-a}K^{\beta}(a, a, c)f \end{aligned}$$

using the definition (2.4).

Thus (3.24) is established for any complex a, with Re $a > 0$. Noticing that

$$K^{\beta}(a, a, c)f(x) = \int_{x}^{\beta} \frac{(t-x)^{a-1}}{\Gamma(a)} e^{c(x-t)} f(t)\, dt, \tag{3.25}$$

(3.21) follows immediately.

(ii) When Re $a <$ Re b, by Lemma 4

$$K^{\beta}(a, b, c)f(x) =^{\circ} e^{cx} K^{\beta}(b-a, b, -c)f'(x) \quad \text{for } x \in [\alpha, \beta]$$

where $f'(x) = e^{-cx}f(x)$. But by the above case

$$K^{\beta}(b-a, b, -c)f'(x) = J^{a}\, e^{-cx}\, J^{b-a}\, e^{cx} f'(x)$$

and (3.22) follows.

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4. **Existence and uniqueness of solutions of (1.1).** It is proved that the existence of $J^{-b}g$ in L is a necessary and sufficient condition for (1.1) to possess a solution $f \in L$. Uniqueness of solutions is ensured by

THEOREM 6. *If* $\operatorname{Re} b>0$, *then the integral equation*

$$\int_x^\beta \frac{(t-x)^{b-1}}{\Gamma(b)}\, {}_1F_1(a, b, c(x-t))f(t)\, dt =^\circ g(x) \quad \text{for } \alpha \le x \le \beta \tag{4.1}$$

cannot have more than one solution f in L.

Proof. Since $K^\beta(a, b, c)$ is a linear operator on L, it is enough to prove that $K^\beta(a, b, c)f=0$ implies $f=0$. This, however, follows directly from Theorem 5 and the uniqueness theorem for fractional integrals.

THEOREM 7. *If* $\operatorname{Re} b>0$, *then a necessary and sufficient condition for the integral equation*

$$\int_x^\beta \frac{(t-x)^{b-1}}{\Gamma(b)}\, {}_1F_1(a, b, c(x-t))f(t)\, dt =^\circ g(x) \quad \text{for } x \in [\alpha, \beta] \tag{4.2}$$

to have a solution f in L is the existence of $J^{-b}g$ in L.

Proof. *Necessity.* Suppose (4.2) has a solution f in L. Using (3.15) the equation (4.2) can be written as

$$J^b\left\{f(x)-ac\int_x^\beta {}_1F_1(a+1, 2, c(x-t))f(t)\, dt\right\} =^\circ g(x). \tag{4.3}$$

Since this equation has a solution f in L and the expression in braces exists in L, it readily follows that $J^{-b}g$ exists and is in L.

Sufficiency. Suppose that $J^{-b}g$ exists in L. By Corollary 4.1, the integral equation (4.2) is equivalent to

$$f(x)-ac\int_x^\beta {}_1F_1(a+1, 2, c(x-t))f(t)\, dt =^\circ G(x), \quad x \in [\alpha, \beta] \tag{4.4}$$

with $G=J^{-b}g$. By the transformation $x=\alpha+\beta-x'$, $t=\alpha+\beta-t'$, (4.4) is converted into a Volterra equation of the second kind:

$$f'(x')-ac\int_\alpha^{x'} {}_1F_1(a+1, 2, c(t'-x'))f'(t')\, dt' = G'(x') \quad \text{for } x' \in [\alpha, \beta] \tag{4.5}$$

where $f'(x')=f(x)$ and $G'(x')=G(x)$.

Since $G \in L$, it easily follows that $G' \in L$. Also it is clear from the analytic character of ${}_1F_1$ that

$$|{}_1F_1(a+1, 2, c(t'-x'))| < M \quad \text{for all } x',\quad t' \in [\alpha, \beta],$$

M a constant. Hence by a well-known theorem [8], the integral equation (4.5) has a unique solution f in L and the sufficiency part follows.

5. **Explicit solutions of (1.1).** In Theorem 10 we obtain an explicit solution of (1.1) under the assumptions of Theorem 7. A transform pair comprising (1.1) and an inversion integral is obtained in Theorem 9 under an assumption which has to be slightly more restrictive. We begin with the following theorem which apart from being of use in Theorem 10, is of some independent interest also.

THEOREM 8. *If* Re $\mu>0$, *then the integral equation*

$$e^{-cx} J^{\mu}\phi(x) =^{\circ} J^{\mu} e^{-cx} f(x), \quad x \in [\alpha, \beta] \tag{5.1}$$

has for each $f \in L$ *a solution* ϕ *in* L *expressible by*

$$\phi(x) = f(x) - \mu c \int_x^\beta {}_1F_1(1+\mu, 2, c(x-t)) f(t)\, dt \tag{5.2}$$

and for each ϕ *in* L *a solution* $f \in L$ *given by*

$$f(x) = \phi(x) + \mu c \int_x^\beta {}_1F_1(1-\mu, 2, c(x-t)) \phi(t)\, dt. \tag{5.3}$$

Proof. (i) Suppose that $f \in L$. From (3.15) with both a and b replaced by μ

$$J^{\mu}\left\{f(x) - \mu c \int_x^\beta {}_1F_1(\mu+1, 2, c(x-t)) f(t)\, dt\right\}$$
$$=^{\circ} K^{\beta}(\mu, \mu, c) f(x) \quad \text{for } x \in [\alpha, \beta].$$

Using (3.25) for the right-hand member, it immediately follows that

$$J^{\mu}\phi(x) =^{\circ} e^{cx} J^{\mu} e^{-cx} f(x), \quad x \in [\alpha, \beta]$$

where

$$\phi(x) = f(x) - \mu c \int_x^\beta {}_1F_1(\mu+1, 2, c(x-t)) f(t)\, dt.$$

Also since $f \in L$ is given, it is clear by Lemma 1 that ϕ is expressed as a difference of two functions in L and is indeed itself in L.

(ii) Suppose that $\phi \in L$ is given. If we replace $f(x)$ by $e^{cx}\psi(x)$ and $\phi(x)$ by $e^{cx} g(x)$, then (5.1) becomes

$$J^{\mu} e^{cx} g(x) =^{\circ} e^{cx} J^{\mu} \psi(x).$$

Clearly for $\phi \in L$, g is in L, so by first part with c replaced by $-c$, this equation has a solution $\psi \in L$ given by

$$\psi(x) = g(x) + \mu c \int_x^\beta {}_1F_1(1+\mu, 2, -c(x-t)) g(t)\, dt$$

which becomes (5.3) when put back in terms of f and ϕ and Kummer's first theorem is used.

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THEOREM 9. *If* Re $l>$ Re $b>0$ *and* $J^{-l}g$, $J^{-l}f$ *exist in L, then for* $x \in [\alpha, \beta]$

$$\int_x^\beta \frac{(t-x)^{b-1}}{\Gamma(b)}\, {}_1F_1(a, b, c(x-t))f(t)\, dt \overset{\circ}{=} g(x), \tag{5.5}$$

$$\int_x^\beta \frac{(t-x)^{l-b-1}}{\Gamma(l-b)}\, {}_1F_1(-a, l-b, c(x-t))J^{-l}g(t)\, dt \overset{\circ}{=} f(x) \tag{5.6}$$

imply each other.

Proof. In our notation, we are to show that

$$K^\beta(a, b, c)f = g \tag{5.7}$$

$$K^\beta(-a, l-b, c)J^{-l}g = f \tag{5.8}$$

are equivalent. It is clear that under the assumptions of the theorem, f and g both exist in L.

(i) First suppose that (5.7) holds. By direct substitution the left-hand member of (5.7), which exists in L, equals

$$\begin{aligned} K^\beta(a, b, c)K^\beta(-a, l-b, c)J^{-l}g &= K^\beta(0, l, c)J^{-l}g, \quad \text{by Theorem 3} \\ &= J^l(J^{-l}g) = g, \end{aligned}$$

noting that for $\phi \in L$,

$$K^\beta(0, l, c)\phi(x) = \int_x^\beta \frac{(t-x)^{l-1}}{\Gamma(l)}\phi(t)\, dt.$$

(ii) Suppose now that (5.8) holds. The left side can, again on a direct substitution, be written as

$$\begin{aligned} K^\beta(-a, l-b, c)J^{-l}[K^\beta(a, b, c)f] &= K^\beta(-a, l-b, c)K^\beta(a, b, c)J^{-l}f, \quad \text{by Theorem 2} \\ &= K^\beta(0, l, c)J^{-l}f, \quad \text{by Theorem 1} \\ &= J^l J^{-l} f = f. \end{aligned}$$

THEOREM 10. *If* Re $b>0$ *and* $J^{-b}g$ *exists in L, then the integral equation*

$$\int_x^\beta \frac{(t-x)^{b-1}}{\Gamma(b)}\, {}_1F_1(a, b, c(x-t))f(t)\, dt \overset{\circ}{=} g(x) \quad \text{for } x \in [\alpha, \beta] \tag{5.9}$$

has the solution f in L given by

$$f(x) = e^{cx}J^{-a}e^{-cx}J^{a-b}g(x) \quad \text{for Re } a > 0, \tag{5.10}$$

$$= J^{a-b}e^{cx}J^{-a}e^{-cx}g(x) \quad \text{for Re } a < \text{Re } b. \tag{5.11}$$

Proof. (i) Suppose Re $a>0$. By Theorem 5, equation (5.9) can be written, for $f \in L$, as

$$J^{b-a}e^{cx}J^a e^{-cx}f(x) \overset{\circ}{=} g(x).$$

This requires that

$$J^a e^{-cx} f(x) =^\circ e^{-cx} J^{a-b} g(x),$$

that is,

$$J^a e^{-cx} f(x) =^\circ e^{-cx} (J^a(J^{-b} g(x))).$$

By Theorem 8, this has a solution f in L. By the definition of J^{-a} such a solution is given by

$$e^{-cx} f(x) =^\circ J^{-a} e^{-cx} J^a(J^{-b} g)(x),$$

and this gives (5.10).

(ii) Suppose $\operatorname{Re} a < \operatorname{Re} b$. By Lemma 4, equation (5.9) is equivalent to

$$K^\beta(b-a, b, -c)\{e^{-cx} f(x)\} =^\circ e^{-cx} g(x).$$

Since $\operatorname{Re}(b-a) > 0$, by case (i), this equation has the solution given by

$$e^{-cx} f(x) =^\circ e^{-cx} J^{a-b} e^{cx} J^{-a} e^{-cx} g(x)$$

which gives (5.11).

REMARK 1. By specializing the parameters of ${}_1F_1$, a device employed in [9, §7], it is simple to obtain solutions of integral equations similar to (1.1) with the kernels involving some classical polynomials or special functions. For example, our solution of the equation

$$\int_\sigma^1 (u-\sigma)^\alpha L_n^\alpha(c(u-\sigma)) f(u)\, du = g(\sigma), \quad \operatorname{Re} \alpha > -1$$

solved by Srivastava [11] for $c=1$, is easily seen to be

$$f(x) = \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} J^{-n-\alpha-1} e^{cx} J^n e^{-cx} g(x).$$

REMARK 2. The results involving $K^\beta(a, b, c)$ can be verified to hold analogously for a more general class of linear operators ${}_pK_q^\beta(a_i; b_j, b; c)$ on L defined by

$$\begin{aligned} {}_pK_q^\beta(a_i; b_j, b; c) f(x) = \int_x^\beta & \frac{(t-x)^{b-1}}{\Gamma(b)} \\ & \times {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_{q-1}, b; c(x-t)) f(t)\, dt \end{aligned}$$

where ${}_pF_q$ is a generalized hypergeometric function with suitable p and q such that the series representing ${}_pF_q$ either converges or terminates. Even the conditions for the existence of solutions of

$$(5.12) \qquad {}_pK_q^\beta(a_i; b_j, b; c) f = g$$

are the same as for (1.1); but from a consideration of Cauchy's functional equation [1]

$$(5.13) \qquad f(x+y) = f(x) f(y)$$

and the relation

$$J^{a_1}\,{}_pK_q^\beta(a_i;\,b_j,\,b;\,c)f = J^b\,{}_pK_q^\beta(a_i;\,b_j,\,a_1;\,c)f \tag{5.14}$$

analogous to (3.23) it follows that the solutions of (5.12) cannot be expressed in an explicit form analogous to (5.10) in any case other than that discussed in the paper, viz. $p=q=1$ and indeed the trivial one, $p=q=0$.

REMARK 3. The results can be extended to the case, $b=\infty$ provided for $\operatorname{Re} a>0$, $\operatorname{Re} c>0$ (equivalently for $\operatorname{Re} a<\operatorname{Re} b$ and $\operatorname{Re} c<0$) the discussion is restricted to a subclass R_q[7] of functions f such that $x^q f(x)$ is in L for suitable q; and for $\operatorname{Re} a<\operatorname{Re} b$, $\operatorname{Re} c>0$ (as also for $\operatorname{Re} a>0$, $\operatorname{Re} c<0$) it is restricted to a class of functions of exponential type.

6. **Integral equation (1.2).** Define linear operators I^μ and $K_\alpha(a, b, c)$ on L by

$$I^\mu(f(x)) = \int_\alpha^x \frac{(x-t)^{\mu-1}}{\Gamma(\mu)} f(t)\,dt \quad \text{for } \operatorname{Re}\mu > 0, \tag{6.1}$$

$$K_\alpha(a, b, c)f(x) = \int_\alpha^x \frac{(x-t)^{b-1}}{\Gamma(b)}\,{}_1F_1(a, b, c(x-t))f(t)\,dt \quad \text{for } \operatorname{Re} b > 0. \tag{6.2}$$

It is easy to discuss $K_\alpha(a, b, c)$ using I^μ on the same lines as $K^\beta(a, b, c)$ has been discussed using J^μ; of course

$$K_\alpha(a, b, 0) = I^b, \quad \text{operating on } L.$$

The analogues of all the results proved on the operator $K^\beta(a, b, c)$ and the equation (1.1) are valid for the operator $K_\alpha(a, b, c)$ and the integral equation (1.2). The results of [9] are special cases of only some of these analogues when $\alpha=0$ and $c=\pm 1$.

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