Can. J. Math., Vol. XXII, No. 5, 1970, pp. 939–952

INCIDENCE SYSTEMS ASSOCIATED WITH NON-PLANAR NEARFIELDS

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1. Introduction. In [**2**] Sandler studied generalizations of projective planes called pseudo planes. These structures gave rise to ternary operations for which addition and multiplication are loop operations. Our aim in this paper is to investigate the pseudo planes for which the operations of addition and multiplication give rise to nearfields. The pseudo planes for which this is true will be called π-systems in this paper, and a list of postulates for such systems is given in § 2. In that section some results on collineations of π-systems are given which are stronger than analogous results for general pseudo planes.

The key result is contained in Theorem 3.2, which gives a certain uniformity to lines of non-zero slope. This leads to the result that any collineation of a π-system coordinatized by a non-planar nearfield either interchanges X and Y or fixes them. This is a very desirable result in the study of collineations of projective planes, and it is helpful in this case also.

Section 4 contains a formulation of the collineations of a π-system coordinatized by a non-planar nearfield. These are identical with the collineations of projective planes coordinatized by planar nearfields given by André in [**1**].

In [**3**], Zemmer gave a construction of a class of non-planar nearfields. Two classes of non-planar nearfields are given in § 4, one of which properly contains the nearfields of Zemmer. It is hoped that these nearfields will be helpful in the study of sharply doubly and sharply triply transitive groups.

2. π-systems.

Definition 2.1. A *π-system* consists of a set, π, whose elements are called points, and certain subsets of π, called lines, subject to the following:

(i) If P, $Q \in \pi$, $P \neq Q$, there is a unique line containing both P and Q, denoted PQ;

(ii) If m and n are distinct lines, then there is at most one point in $m \cap n$;

(iii) There are four points, no three of which are in the same line;

(iv) Lines form four disjoint non-empty classes;

(a) 1-lines all intersect in a common point, Y, they are intersected by all lines, and all but one line on Y is a 1-line.

Received September 22, 1969. These results are part of the author's Ph.D. dissertation written at the University of Missouri, Columbia, Missouri, under the direction of Professor J. L. Zemmer.

(b) 2-lines all intersect in a common point, $X \neq Y$, they are intersected by all lines, and all but one line on X is a 2-line;
(c) l_∞ is the line containing X and Y, and it is intersected by all lines,
(d) 3-lines.

Definition 2.2 (Sandler). A set, π, of points and lines, together with an incidence relation, I, will be called a *pseudo plane* if there are two distinct points $P_1, P_2 \in \pi$, and two distinct lines $L_1, L_2 \in \pi$ such that $P_1 \mathrm{I} L_2$, $P_1 \mathrm{I} L_1$, $P_2 \mathrm{I} L_1$, and such that

(1) For any point P such that $P \mathrm{I} L_1$ or $P \mathrm{I} L_2$ and any point $Q \in \pi$, there is a unique line $L \in \pi$ with $P \mathrm{I} L$ and $Q \mathrm{I} L$.
(2) For any line L such that $P_1 \mathrm{I} L$ or $P_2 \mathrm{I} L$ and any line $M \in \pi$, there is a unique point $P \in \pi$ with $P \mathrm{I} L$ and $P \mathrm{I} M$.
(3) There are four points in π, no three of which are incident with the same line.

Let π be a π-system. For P a point and L a line, define $P \mathrm{I} L$ if and only if $P \in L$. Let $P_1 = Y$, $P_2 = X$, $L_1 = l_\infty$, and L_2 be any 1-line. With this identification, we can see that every π-system is a pseudo plane. Postulate (1) for pseudo planes is satisfied by virtue of postulate (i) for π-systems. Postulate (3) for pseudo planes is identical with postulate (iii) for π-systems. If L is a line such that $P_1 \mathrm{I} L$ or $P_2 \mathrm{I} L$, then L is either a 1-line, a 2-line, or l_∞. In any case, L intersects all lines, and so postulate (2) for pseudo planes is satisfied.

Before examining π-systems in the light of nearfields, we will give some properties of general π-systems.

Definition 2.3. A *collineation* of a π-system is a one-to-one mapping, say α, of points onto points, and lines onto lines such that $P \in m$ if and only if $P\alpha \in m\alpha$. α is called *central* with centre Q and axis n if α fixes every line on Q and every point on n. Such a collineation is also called a perspectivity.

LEMMA 2.1. *A collineation which fixes every point on each of two distinct lines is the identity.*

Proof. Let α be a collineation which fixes every point on the two distinct lines m and n. Let P be a point not on m or n.

Case 1. m is not a 3-line. Let U and V be distinct points of n, which are also distinct from $m \cap n$. Let UP and VP intersect m in R and S, respectively. U, V, R, and S are fixed by α, and so UR and VS are fixed lines. Thus P is fixed by α, as $P = (UR) \cap (VS)$. Hence α is the identity.

Case 2. n is not a 3-line. Interchange the roles of m and n in Case 1.

Case 3. m and n are 3-lines. First, we will show that X and Y are fixed by α. Let P be either X or Y, and let R and S be points on n, $R \neq P \neq S$. Let $U = m \cap (PR)$ and $V = m \cap (PS)$. The points R, S, U, and V are fixed by α, and so $P = (UR) \cap (VS)$ is fixed.

https://doi.org/10.4153/CJM-1970-108-1 Published online by Cambridge University Press

Now let P be a point not on m, n, or l_∞. Then $R = m \cap (XP)$ and $S = m \cap (YP)$ are fixed by α, and so $P = (RX) \cap (SY)$ is fixed by α. Since $l_\infty = XY$, l_∞ is fixed by α. Let $P \in l_\infty$, $X \neq P \neq Y$. Let k be any line on P, $k \neq l_\infty$. There are points R and S on k, $R, S \notin l_\infty$. Hence $k = RS$ is fixed. Thus $P = l_\infty \cap k$ is fixed by α. Therefore α is the identity.

The following two results are stronger than similar ones for general pseudo planes which are found in [**2**].

LEMMA 2.2. *Let α be a collineation of the π-system, π, which fixes every point on the line m, and the two points P, $Q \notin m$.*

(i) *If m is not a 3-line, then α is the identity.*

(ii) *If P, $Q \notin l_\infty$, then α is the identity.*

(iii) *If at least one of P or Q is X or Y, then α is the identity.*

Proof. Case 1. m is not a 3-line. Let $R \notin m$, and $R \notin PQ$. Then RP and RQ intersect m. Let U and V be the intersection points, respectively. Then PU and VQ are fixed lines, and so $R = (PU) \cap (VQ)$ is fixed by α. Thus α fixes every point not on PQ. Let n be any line on P, $n \neq PQ$. Then α fixes all points on n, and so by Lemma 2.1, α is the identity.

Case 2. P, $Q \notin l_\infty$. If m is not a 3-line, Case 1 suffices. Let m be a 3-line, and $p = PY$ and $q = QY$. Then $m \cap p$ and $m \cap q$ are fixed. Thus, PY and QY are fixed lines. Hence Y is fixed by α. Let T be a point on PQ, $P \neq T \neq Q$, $T \neq Y$. Let $T' = m \cap (TY)$. Then T' is fixed by α, and so $T = (PQ) \cap (T'Y)$ is fixed. Thus α is the identity, by Lemma 2.1, since points on m and PQ are fixed.

Case 3. At least one of P or Q is X or Y, and call it P. If m is not a 3-line, Case 1 suffices. Let m be a 3-line. Let n be any line on Q which intersects m. Then $n\alpha = n$. Let $R \in n$, $R \neq Q$. Then PR intersects m at a point which is fixed. Hence PR is fixed. Thus, $R = n \cap (PR)$ is fixed. Therefore again, by Lemma 2.1, α is the identity.

LEMMA 2.3. *A central collineation, α, is completely determined by its centre, C, axis, m, and the mapping, $P \to P\alpha$, of any point P not on m different from C such that $P\alpha \in CP$.*

Proof. Let α_1 and α_2 be collineations with centres, C, axes, m, and $P\alpha_1 = P\alpha_2$ for $P \neq C$, $P \notin m$, and $P\alpha_1 \in PC$. Then $\alpha_1\alpha_2^{-1}$ fixes P, has centre, C, and axis, m.

Let $R \in m$, $R \neq C$, and $R \notin CP$. Let $Q \in RP$ with $P \neq Q \neq R$. $\alpha_1\alpha_2^{-1}$ fixes P and R, and so it fixes RP. $\alpha_1\alpha_2^{-1}$ has centre, C, and so it fixes QC. Thus, $Q = (QC) \cap (PR)$ is fixed by $\alpha_1\alpha_2^{-1}$. Thus all points on m and PR are fixed by $\alpha_1\alpha_2^{-1}$. Thus, by Lemma 2.1, $\alpha_1\alpha_2^{-1}$ is the identity. Hence $\alpha_1 = \alpha_2$.

3. Nearfields and π-systems. Sandler has shown in [**2**] that a pseudo plane may be coordinatized; a ternary, T, defined; and binary operations, $+$ and $\cdot$ defined on the coordinatizing set in a manner analogous to that used with projective planes. If R is the coordinatizing set, then $(R, +)$ is a loop with identity 0. If $R' = R - \{0\}$, then $(R', \cdot)$ is a loop with identity 1. The point Y has coordinate ∞, all other points on l_∞ have coordinates (m), with X having (0). Points not on l_∞ have coordinates (a, b). 1-lines have equations $x = c$, 2-lines have equations $y = c$, 3-lines have equations $y = T(x, m, k)$.

Definition 3.1. A *nearfield* is a triple $(R, +, \cdot)$ for which

(1) $(R, +)$ is an abelian group with identity 0,

(2) $(R', \cdot)$ is a group with identity 1,

(3) $0 \cdot a = a \cdot 0 = 0$ for all $a \in R$, and

(4) $a \cdot (b + c) = a \cdot b + a \cdot c$.

A nearfield is called *planar* if each equation of the form $x = a \cdot x + b$, $a \neq 1$, $b \in R$, has a unique solution. Otherwise it is called *non-planar*. The *kern* of R, $\operatorname{kern}(R) = \{a | (x + y) \cdot a = x \cdot a + y \cdot a, x, y \in R\}$, the *centre* of R, $Z(R) = \{a | x \cdot a = a \cdot x, x \in R\}$.

We will say that a π-system, π, is coordinatized by a nearfield if R is the coordinatizing set; $+$ and $\cdot$ are the binary operations defined by the ternary, T; $T(a, b, c) = ba + c$; and $(R, +, \cdot)$ is a nearfield. The line whose equation is E will be denoted (E). When no confusion is possible, we will use ab for $a \cdot b$.

THEOREM 3.1. *The following are groups of collineations of a π-system coordinatized by a nearfield, $(R, +, \cdot)$.*

(1) $\Upsilon = \{\tau_{a,b}\}$, *the set of translations.*

$$\begin{array}{lll}
\tau_{a,b}\colon & (x, y) \to (x + a, y + b) & (y = mx + k) \to (y = mx - ma + k + b) \\
& (x) \to (x) & (x = k) \to (x = k + a) \\
& \infty \to \infty & l_\infty \to l_\infty.
\end{array}$$

(2) $A = \{\alpha\}$, *the set of automorphisms, where α' is an automorphism of* $(R, +, \cdot)$.

$$\begin{array}{lll}
\alpha\colon & (x, y) \to (x\alpha', y\alpha') & (y = mx + k) \to (y = (m\alpha')x + k\alpha') \\
& (x) \to (x\alpha') & (x = k) \to (x = k\alpha') \\
& \infty \to \infty & l_\infty \to l_\infty.
\end{array}$$

(3) $S = \{\mu_a\}$, *the set of stretchings. For a in the kern of R, $a \neq 0$, define μ_a as follows:*

$$\begin{array}{lll}
\mu_a\colon & (x, y) \to (xa, ya) & (y = mx + k) \to (y = mx + ka) \\
& (x) \to (x) & (x = k) \to (x = ka) \\
& \infty \to \infty & l_\infty \to l_\infty.
\end{array}$$

(4) $M = \{m_{a,b}\}$, *the set of multiplications, for $a \neq 0 \neq b$.*

$$\begin{array}{lll}
m_{a,b}\colon & (x, y) \to (ax, by) & (y = mx + k) \to (y = bma^{-1}x + bk) \\
& (x) \to (bxa^{-1}) & (x = k) \to (x = ak) \\
& \infty \to \infty & l_\infty \to l_\infty.
\end{array}$$

https://doi.org/10.4153/CJM-1970-108-1 Published online by Cambridge University Press

(5) $L = \{m_{a,1}\}$ *and* $N = \{m_{1,a}\}$.

(6) Z, *the set consisting of the identity and the mapping* ρ, *where* ρ *is defined as follows:*

$$\begin{array}{ll} \rho\colon (x, y) \to (y, x) & (y = mx + k) \to (y = m^{-1}x - m^{-1}k) \text{ for } m \neq 0 \\ \quad (m) \to (m^{-1}) \text{ for } m \neq 0 & (y = k) \to (x = k) \\ \quad \infty \to (0) & (x = k) \to (y = k) \\ \quad (0) \to \infty & l_\infty \to l_\infty \end{array}$$

Proof. Since each mapping is one-to-one and onto the points and lines of π, it suffices to show that each set is a group, and each mapping preserves incidence.

(1) The only incidence that needs to be checked is $(u, v) \in (y = mx + k)$.

$(u, v)\tau_{a,b} = (u + a, v + b)$, and

$$(y = mx + k)\tau_{a,b} = (y = mx - ma + k + b).$$

$v = mu + k$, and so

$$\begin{aligned} m(u + a) - ma + k + b &= mu + ma - ma + k + b \\ &= mu + k + b = v + b. \end{aligned}$$

Thus $(u, v)\tau_{a,b} \in (y = mx + k)\tau_{a,b}$.

It is easily seen that the inverse of $\tau_{a,b}$ is $\tau_{-a,-b}$, and so

$$\begin{aligned} (x, y)\tau_{c,d}\tau_{-a,-b} &= (x + c, y + d)\tau_{-a,-b} \\ &= (x + c - a, y + d - b) \\ &= (x, y)\tau_{c-a,d-b}. \end{aligned}$$

Thus Υ is a group.

(2) Again, we need only check $(u, v) \in (y = mx + k)$.

$$(m\alpha')(u\alpha') + k\alpha' = (mu + k)\alpha' = v\alpha'.$$

Thus $(u, v)\alpha$ is on the line $(y = mx + k)\alpha$. The group nature of A is clear.

(3) Checking $(u, v) \in (y = mx + k)$, we have $v = mu + k$ implies that $m(ua) + ka = (mu)a + ka = (mu + k)a = va$ since $a \in \operatorname{kern}(R)$. Thus μa is a collineation.

It is known [**3**] that the kern of a nearfield is a division ring, and so the non-zero elements form a multiplicative group. It is easily seen that the inverse of μ_a is μ_b, where $ab = 1$.

$$(x, y)\mu_c\mu_b = (xc, yc)\mu_b = ((xc)b, (yc)b) = (x(cb), y(cb)) = \mu_{cb}.$$

Thus S is a group.

(4) Checking $(u, v) \in (y = mx + k)$, we have $v = mu + k$ implies that $(bma^{-1})(au) + bk = bmu + bk = b(mu + k) = bv$. Thus $m_{a,b}$ is a collineation.

M is shown to be a group exactly as in (3). In fact, $m_{a,b}m_{c,d} = m_{ac,bd}$.

(5) The fact that L and N are groups is immediate.

(6) Let $(u, v) \in (y = mx + k)$, $m \neq 0$. $(u, v)\rho = (v, u)$, and

$$(y = mx + k)\rho = (y = m^{-1}x - m^{-1}k).$$

Since $v = mu + k$, $m^{-1}v - m^{-1}k = u$. Thus ρ is a collineation. Clearly, $\rho^2 = I$, the identity, and so Z is a group.

It is a routine calculation to see that each group in Theorem 3.1 commutes with the other groups in that theorem, except Z with L and N. $ZL = NZ$.

We will now consider only the π-systems which can be coordinatized by non-planar nearfields. Let $(R, +, \cdot)$ be a non-planar nearfield, and $x = rx + t$, $r \neq 1$, be an equation which fails to have a solution. It follows that $t \neq 0$.

THEOREM 3.2. *If $(a, b) \notin (y = nx + k)$, $n \neq 0$, then there is a line containing (a, b) which does not intersect $(y = nx + k)$.*

Proof. We will first show that if $b \neq 0$, then there is a line on $(0, b)$ which does not intersect $(y = x)$. Suppose the contrary. Then for every $m \neq 1$, the equation $x = mx + b$ must have a solution. Thus, the equation $tb^{-1}x = tb^{-1}(mx + b) = tb^{-1}mx + t$ must have a solution. In particular, let $m = bt^{-1}rtb^{-1}$.

$$\begin{aligned} tb^{-1}x &= tb^{-1}(bt^{-1}rtb^{-1})x + t \\ &= r(tb^{-1}x) + t \end{aligned}$$

has a solution. That is, $z = rz + t$ has a solution, which is a contradiction to the choice of r and t.

Next, let $a \neq b$. We will show that there is a line on (a, b) which does not intersect $(y = x)$. By the above, there is a line, p, on $(0, b - a)$ which does not intersect $(y = x)$. The translation $\tau_{a,a}$ sends $(0, b - a)$ to (a, b), and $(y = x)$ to $(y = x)$. Thus $p\tau_{a,a}$ is the required line.

Now, let $b \neq a + c$. We will show that there is a line on (a, b) which does not intersect $(y = x + c)$. Since $b \neq a + c$, by the preceding paragraph, there is a line, q, on $(a, b - c)$ which does not intersect $(y = x)$. The translation $\tau_{0,c}$ sends $(a, b - c)$ to (a, b) and $(y = x)$ to $(y = x + c)$. Thus the line $q\tau_{0,c}$ is the required line.

Finally, let $(a, b) \notin (y = nx + k)$, $n \neq 0$. Then $b \neq na + k$. Thus by the preceding paragraph, there is a line, w, on (na, b) not intersecting the line $(y = x + k)$. Let $d = n^{-1}$. The collineation, $m_{d,1}$ sends (na, b) to (a, b), and $(y = x + k)$ to $(y = nx + k)$. Thus $wm_{d,1}$ is the required line on (a, b) which fails to intersect $(y = nx + k)$.

THEOREM 3.3. *Every collineation of a π-system coordinatized by a non-planar nearfield fixes l_∞.*

Proof. Let π be coordinatized by a non-planar nearfield, and let α be a collineation of π.

First, we will show that the set

$$S = \{1\text{-lines}\} \cup \{2\text{-lines}\} \cup \{l_\infty\}$$

is fixed by α. Let $m \in S$, and suppose that $m\alpha = n$, a 3-line. Let $(a, b) \notin n$. By Theorem 3.2, there is a line, p, on (a, b) such that $p \cap n = \emptyset$. Thus, $p\alpha^{-1} \cap n\alpha^{-1} = p\alpha^{-1} \cap m = \emptyset$. This is contrary to the fact that 1-lines, 2-lines, and l_∞ intersect every line. Thus $m\alpha = n$ is not a 3-line, and $S\alpha = S$.

Consequently, α maps 3-lines to 3-lines.

By the above, if $l_\infty\alpha = m \neq l_\infty$, then m is either a 1-line or a 2-line, and $(y = x)\alpha$ is a 3-line. Thus $m \cap (y = x)\alpha = (1)\alpha = (a, b)$ for some a and b. Now $X\alpha \neq (a, b)$, and $Y\alpha \neq (a, b)$. At least one of X and Y is mapped onto a point, (c, d), by α since $l_\infty\alpha \neq l_\infty$. Now, $(c, d) \neq (a, b)$, and $(c, d) \in m$. Thus $(c, d) \notin (y = x)\alpha$. Every line on X or Y intersects $(y = x)$, and so every line on (c, d) intersects $(y = x)\alpha$. This is contrary to Theorem 3.2. Thus $l_\infty\alpha = l_\infty$.

COROLLARY 3.1. *If α is a collineation of a π-system coordinatized by a non-planar nearfield, then α either fixes X and Y or interchanges X and Y.*

4. Determination of the collineation group. In this section we will determine all collineations of a π-system coordinatized by a non-planar nearfield. In fact, as is the case with projective planes coordinatized by planar nearfields [**1**], it will be shown that any collineation can be written as a product of those collineations determined in Theorem 3.1. The following notation will be used: C is the set of all collineations; P is the set of all perspectivities; Q is the set of all projections, that is, products of elements of P; D is the set of all collineations fixing $(0, 0)$, (0), $(1, 1)$, and ∞.

Let α be a non-identity collineation which fixes X. Then by Corollary 3.1, α fixes Y. Let m be a 3-line. Then by Lemma 2.2, m is not fixed pointwise by α.

We will first determine P. Let $\alpha \in P$, $\alpha \neq I$. α either fixes X and Y or interchanges them.

Case 1. α fixes X and Y. By the above remark, the axis of α must be l_∞, a 1-line, or a 2-line.

(a) The axis of α is l_∞. The centre may or may not be on l_∞. If the centre of α is on l_∞, then $(0, 0)\alpha = (a, b) \neq (0, 0)$. Such a perspectivity will be shown to be a translation in Lemma 4.1.

If the centre of α is not on l_∞, let it be (a, b). $\beta = \tau_{a,b}\alpha\tau_{-a,-b}$ has axis l_∞ and centre $(0, 0)$. Thus we may assume that $(a, b) = (0, 0)$. Such a perspectivity will be shown to be a stretching in Lemma 4.2.

(b) The axis of α is not l_∞. Then it is either a 1-line or a 2-line. If the centre is not (0) or ∞, it must be on the axis by Lemma 2.2. If the centre is not (0) or ∞, let $(y = x + k)$ be a line on the centre. Then,

$$(y = x + k)\alpha = (y = x + k),$$

and $l_\infty\alpha = l_\infty$, and so $(1)\alpha = (1)$. This means that $\alpha = I$, by Lemma 2.2. Thus the centre of α is (0) or ∞.

Furthermore, if the axis is not on $(0, 0)$, let (a, b) be a point on the axis. Then, $\beta = \tau_{a,b}\alpha\tau_{-a,-b}$ has axis on $(0, 0)$ and centre (0) or ∞. Such perspectivities will be determined in Lemma 4.3.

Case 2. α interchanges (0) and ∞. Then the axis must be a 3-line. If the centre of α is a point, (a, b), then $(x = a)$ and $(y = b)$ are fixed lines. Since $l_\infty\alpha = l_\infty$, then (0) and ∞ are fixed. This is a contradiction. Thus the centre of α is on l_∞. As in the preceding cases, we may assume that the axis is on $(0, 0)$. Such perspectivities will be determined in Lemma 4.4.

LEMMA 4.1. *Let α have axis l_∞ and centre on l_∞, then α is given by*

$$(x, y) \to (x + a, y + b).$$

Proof. Let $(0, 0)\alpha = (a, b)$, and α have centre (m). Then

$$(y = mx)\alpha = (y = mx),$$

and so $(a, b) \in (y = mx)$. Thus, $b = ma$. $\tau_{a,b}$ has axis l_∞ and sends $(0, 0)$ to (a, b). Also,

$$(y = mx + k)\tau_{a,b} = (y = mx - ma + k + b) = (y = mx + k).$$

Therefore (m) is the centre of $\tau_{a,b}$. Thus, by Lemma 2.3, $\alpha = \tau_{a,b}$.

A similar argument holds if the centre of α is ∞.

LEMMA 4.2. *Let α have axis l_∞ and centre $(0, 0)$. Then $\alpha = \mu_s$, for some $s \in \operatorname{kern}(R) - \{0\}$.*

Proof. Since the centre of α is $(0, 0)$, for every $m \neq 0$,

$$(y = mx)\alpha = (y = mx).$$

Thus, $(1, 1)\alpha = (s, s)$ for some $s \neq 0$. Also, $(a, a)\alpha = (a', a')$, and so α induces a mapping, $a \to a'$, on R. We will show that $a' = as$, and $s \in \operatorname{kern}(R) - \{0\}$.

Let $b, c \in R$, $(b, b)\alpha = (b', b')$, and the line $(y = bx)$ is fixed by α. $(x = 1)\alpha = (x = s)$, and so

$$\begin{aligned}(1, b)\alpha = [(x = 1) \cap (y = bx)]\alpha &= (x = 1)\alpha \cap (y = bx)\alpha\\ &= (x = s) \cap (y = bx)\\ &= (s, bs).\end{aligned}$$

Thus $b' = bs$.

Hence $(b + c, b + c)\alpha = ((b + c)s, (b + c)s)$, but we also have

$$(y = x + c)\alpha = (y = x + cs),$$

and so $(b + c)' = bs + cs$. Thus, $(b + c)s = bs + cs$.

LEMMA 4.3. *Let α have its axis on $(0, 0)$ and fix (0); then α is given by $(x, y) \to (ax, y)$ or $(x, y) \to (x, ay)$ for some $a \in R'$.*

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Proof. By the discussion preceding Lemma 4.1, α has centre (0) or ∞.

Case 1. α has centre (0). By Lemma 2.3, the axis of α must be $(x = 0)$. We have $(y = 1)\alpha = (y = 1)$, and so $(1, 1)\alpha = (a, 1)$ for some $a \neq 0$. The mapping $m_{a,1}$ has centre (0), axis $(x = 0)$, and sends $(1, 1)$ to $(a, 1)$. Thus $\alpha = m_{a,1}$ by Lemma 2.3.

Case 2. α has centre ∞. By an argument similar to that in Case 1, $\alpha = m_{1,a}$.

LEMMA 4.4. *Let α interchange (0) and ∞, and have its axis on $(0, 0)$, then for some $a \in R'$, α is given by $(x, y) \to (a^{-1}y, ax)$.*

Proof. By the discussion preceding Lemma 4.1, α has axis $(y = ax)$ for some $a \neq 0$. $(1, a) \in (y = ax)$, and $(1, a)\alpha = (1, a) = (a', 1')$ for some permutation sending $x \to x'$ as (0) and ∞ are interchanged.

Let $b = a^{-1}$. The mapping $m_{a,b}\rho$ sends (x, y) to $(a^{-1}y, ax)$, and interchanges (0) and ∞. If $(u, v) \in (y = ax)$, then $m_{a,b}\rho$: $(u, v) = (u, au) \to (u, au)$. Thus the axis of $m_{a,b}\rho$ is $(y = ax)$.

Now consider $\alpha^{-1}m_{a,b}\rho$. This product fixes every point on $(y = ax)$, and (0) and ∞. Hence by Lemma 2.2, $\alpha^{-1}m_{a,b}\rho = I$. Thus, $(x, y) \to (a^{-1}y, ax)$.

We now have the following result on P.

THEOREM 4.1. *If $\alpha \in P$, $\alpha = \tau_{c,d}\alpha'\tau_{-c,-d}$, where α' is given by one of the following:*

(1) $(x, y) \to (x + a, y + b)$;
(2) $(x, y) \to (xs, ys)$, $s \in \operatorname{kern}(R) - \{0\}$;
(3) $(x, y) \to (ax, y)$, $a \neq 0$;
(4) $(x, y) \to (x, ay)$, $a \neq 0$;
(5) $(x, y) \to (a^{-1}y, ax)$, $a \neq 0$;

and $\tau_{c,d}$ is some translation, possibly the identity.

We now turn to a determination of C.

LEMMA 4.5. *Let $\alpha \in D$; then α induces an automorphism, α', on $(R, +, \cdot)$ such that $(x, y)\alpha = (x\alpha', y\alpha')$.*

Proof. Since $(0, 0)\alpha = (0, 0)$, $(1, 1)\alpha = (1, 1)$, and $l_\infty\alpha = l_\infty$, we have $(1)\alpha = (1)$. Since $(0)\alpha = (0)$ and $\infty\alpha = \infty$, we have $(x, y)\alpha = (x\alpha', y\alpha'')$ for some permutations α' and α'' of R.

$(y = x)\alpha = (y = x)$, and so $(x, x)\alpha = (x\alpha', x\alpha'') \in (y = x)$. Thus $x\alpha' = x\alpha''$ for all x. Hence $\alpha' = \alpha''$. Thus, $(x, y)\alpha = (x\alpha', y\alpha')$.

First we will show that $(ab)\alpha' = (a\alpha')(b\alpha')$.

$$\begin{aligned}(b, ab) &= (x = b) \cap (y = ax)\\ &= (x = b) \cap [(0, 0)(1, a)]\\ &= (x = b) \cap [(0, 0)((x = 1) \cap (y = a))].\end{aligned}$$

Applying α to this we obtain

$$\begin{aligned}(b\alpha', (ab)\alpha') &= (x = b\alpha') \cap [(0, 0)((x = 1) \cap (y = a\alpha'))]\\ &= (b\alpha', (a\alpha')(b\alpha')).\end{aligned}$$

Thus $(ab)\alpha' = (a\alpha')(b\alpha')$.

Now we will show that $(a + b)\alpha' = a\alpha' + b\alpha'$.

$$\begin{aligned}(a, a+b) &= (x = a) \cap (y = x + b)\\ &= (x = a) \cap [(0, b)(1)].\end{aligned}$$

Applying α to this, we obtain,

$$\begin{aligned}(a\alpha', (a+b)\alpha') &= (x = a\alpha') \cap [(0, b\alpha')(1)]\\ &= (a\alpha', a\alpha' + b\alpha').\end{aligned}$$

Therefore $(a + b)\alpha' = a\alpha' + b\alpha'$, and thus, α' is an automorphism of $(R, +, \cdot)$.

Notice that by Lemma 4.5, $A = D$. We will use A instead of D in the determination of C.

THEOREM 4.2. *If $\alpha \in C$, then $\alpha = \gamma\beta\tau_{a,b}$ or $\alpha = \gamma\beta\tau_{a,b}\rho$, where $\beta \in M$ and $\gamma \in A$.*

Proof. Case 1. Let α: $(0, 0) \to (a, b)$, $(1, 1) \to (c, d)$, $(0) \to (0)$, and $\infty \to \infty$. Then $\alpha\tau_{-a,-b}$: $(0, 0) \to (a, b) \to (0, 0)$ and

$$(1, 1) \to (c, d) \to (c - a, d - b).$$

Now, $(y = x)\alpha$ is a 3-line, and so $c - a \neq 0 \neq d - b$. Thus let $\beta = m_{c-a,d-b}$. Then, $\alpha\tau_{-a,-b}\beta^{-1}$: $(0, 0) \to (0, 0) \to (0, 0)$, and

$$(1, 1) \to (c - a, d - b) \to (1, 1).$$

Therefore, if we let $\gamma = \alpha\tau_{-a,-b}\beta^{-1}$, then $\gamma \in A$, and $\alpha = \gamma\beta\tau_{a,b}$.

Case 2. Let α: $(0, 0) \to (b, a)$, $(1, 1) \to (d, c)$, $(0) \to \infty$, and $\infty \to (0)$. Then $\alpha\rho = \gamma\beta\tau_{a,b}$ as in Case 1. Thus $\alpha = \gamma\beta\tau_{a,b}\rho$ since $\rho^2 = I$.

Therefore, by Theorem 4.2 and the commutativity of the groups in Theorem 3.1, we have the following result.

THEOREM 4.3. *If $\alpha \in C$, then α is given by one of the following:*

(1) $(x, y) \to (a(x\alpha') + b, c(y\alpha') + d)$, $a \neq 0 \neq c$;

(2) $(x, y) \to (a(y\alpha') + b, c(x\alpha') + d)$, $a \neq 0 \neq c$;

and α' is an automorphism of $(R, +, \cdot)$.

5. Classes of non-planar nearfields. In this section we will give the constructions of two classes of non-planar nearfields.

Construction 5.1. Let F be a field of characteristic 0; I an index set; λ_i, $i \in I$, indeterminates over F; $F[\lambda_i]$ the ring of polynomials in the λ_i over F; and $F(\lambda_i)$ the field of quotients of $F[\lambda_i]$. Let T_j, $j \in I$, be mappings from $F(\lambda_i)$ to itself such that $\alpha(\lambda_i)T_j = \alpha(\lambda_i, \lambda_j + 1)$, the rational function obtained by replacing λ_j with $\lambda_j + 1$. Let δ_j, $j \in I$, be mappings from $F(\lambda_i)$ to Z, the ring of integers, such that if $p(\lambda_i)$, $q(\lambda_i) \in F[\lambda_i]$ with

$$\alpha(\lambda_i) = p(\lambda_i)/q(\lambda_i),$$

then $(\alpha(\lambda_i))\delta_j$ = degree of $p(\lambda_i)$ in λ_j − degree of $q(\lambda_i)$ in λ_j. We will say $j \in \alpha(\lambda_i)$ if λ_j appears in $\alpha(\lambda_i)$ with a non-zero coefficient. Define $0 \circ \beta = 0$ for all β, and for $\alpha \neq 0$,

$$\alpha \circ \beta = \alpha[\beta \prod T_i^{(\alpha)\delta_i}] \qquad (i \in \alpha(\lambda_i)).$$

Let $(N, +, \circ) = (F(\lambda_i), +, \circ)$.

The following facts about the δ_i and T_i are easily verified and used without mention in the following theorem.

(1) $(\alpha\beta)\delta_i = (\alpha)\delta_i + (\beta)\delta_i$;
(2) $(\alpha)\delta_i = (\alpha T_i)\delta_i$;
(3) $(\alpha)\delta_j = (\alpha T_i)\delta_j$ for $i \neq j$;
(4) $T_j T_i = T_i T_j$ for all i and j;
(5) Each T_i is an automorphism of $(N, +, \circ)$.

THEOREM 5.1. $(N, +, \circ)$ *is a non-planar nearfield.*

Proof. First, we will show that $(N, +, \circ)$ is a nearfield. Let $\alpha \neq 0$; then

$$\alpha \circ (\alpha^{-1} \prod T_i^{-(\alpha)\delta_i}) = \alpha\alpha^{-1} = 1.$$

Thus non-zero elements of N have inverses with respect to $\circ$.

$$\begin{aligned}
(\alpha \circ \beta) \circ \gamma &= (\alpha(\beta \prod T_i^{(\alpha)\delta_i})) \circ \gamma \\
&= \alpha(\beta \prod T_i^{(\alpha)\delta_i})\gamma \prod T_j^{(\alpha(\beta \prod T_i^{(\alpha)\delta_i}))\delta_j} \\
&= \alpha(\beta \prod T_i^{(\alpha)\delta_i})\gamma \prod T_j^{(\alpha)\delta_j + (\beta)\delta_j} \\
&= \alpha(\beta\gamma \prod T_j^{(\beta)\delta_j}) \prod T_i^{(\alpha)\delta_i} \\
&= \alpha \circ (\beta \circ \gamma), \\
\alpha \circ (\beta + \gamma) &= \alpha(\beta + \gamma) \prod T_i^{(\alpha)\delta_i} \\
&= \alpha(\beta \prod T_i^{(\alpha)\delta_i} + \gamma \prod T_i^{(\alpha)\delta_i}) \\
&= \alpha(\beta \prod T_i^{(\alpha)\delta_i}) + \alpha(\gamma \prod T_i^{(\alpha)\delta_i}) \\
&= \alpha \circ \beta + \alpha \circ \gamma.
\end{aligned}$$

Thus $(N, +, \circ)$ is a nearfield.

Next we will show that each of the equations $x = \lambda_i \circ x + 1$ has no solution. Suppose that there is a solution $p(\lambda_i)/q(\lambda_i)$, with $(p(\lambda_i), q(\lambda_i)) = 1$ in $F[\lambda_i]$, for the equation $x = \lambda_j \circ x + 1$. Then

$$p(\lambda_i)/q(\lambda_i) = \lambda_j(p(\lambda_i, \lambda_j + 1)/q(\lambda_i, \lambda_j + 1)) + 1.$$

Thus, we have

$$p(\lambda_i)q(\lambda_i, \lambda_j + 1) = \lambda_j p(\lambda_i, \lambda_j + 1)q(\lambda_i) + q(\lambda_i)q(\lambda_i, \lambda_j + 1).$$

Therefore, $q(\lambda_i)$ divides $q(\lambda_i, \lambda_j + 1)$, which implies that

$$q(\lambda_i) = q(\lambda_i, \lambda_j + 1).$$

Upon applying T_j to this equation, we obtain $q(\lambda_i, \lambda_j + 1) = q(\lambda_i, \lambda_j + 2)$. Thus $q(\lambda_i) = q(\lambda_i, \lambda_j + 2)$. Continuing this process, for each n, we have

$q(\lambda_i) = q(\lambda_i, \lambda_j + n)$. Thus the equation, $q(\lambda_i) = q(\lambda_i, 0 + x)$, obtained by setting $\lambda_j = 0$ has infinitely many solutions. So $j \notin q(\lambda_i)$. Thus, $p(\lambda_i) = \lambda_j p(\lambda_i, \lambda_j + 1) + q(\lambda_i)$ must hold. This is impossible since

$$(\lambda_j p(\lambda_i, \lambda_j + 1) + q(\lambda_i))\delta_j = (p(\lambda_i))\delta_j + 1 \neq (p(\lambda_i))\delta_j.$$

Hence we have a contradiction, and thus $x = \lambda_j \circ x + 1$ has no solution.

The non-planar nearfields constructed by Zemmer [**3**] are the nearfields of Construction 5.1 with the cardinality of I, $\text{card}(I) = 1$. Now we will show that the class of nearfields of Construction 5.1 contains at least one nearfield which is not isomorphic to any of those in [**3**].

THEOREM 5.2. *Let $(N, +, \circ)$ be a nearfield constructed as in Construction 5.1. The centre $Z(N)$ of N is F.*

Proof. Certainly if $a \in F$, then $a \in Z(N)$.

Suppose that there is an $\alpha(\lambda_i) \in Z(N) - F$. Then there is a λ_t appearing in $\alpha(\lambda_i)$ whose coefficients are not all zero. First, we will suppose that $(\alpha(\lambda_i))\delta_t \neq 0$.

$$(\dagger) \qquad \alpha(\lambda_i) \circ \lambda_t = \lambda_t \circ \alpha(\lambda_i),$$

$$(*) \qquad \alpha(\lambda_i) \circ (\lambda_t - (\alpha)\delta_t) = (\lambda_t - (\alpha)\delta_t) \circ \alpha(\lambda_i).$$

From these equations, we obtain

$$(\dagger\dagger) \qquad \lambda_t\alpha(\lambda_i) + (\alpha)\delta_t\alpha(\lambda_i) = \lambda_t\alpha(\lambda_i, \lambda_t + 1),$$

$$(**) \qquad \alpha(\lambda_i)\lambda_t = \lambda_t\alpha(\lambda_i, \lambda_t + 1) - (\alpha)\delta_t\alpha(\lambda_i, \lambda_t + 1).$$

Combining $(\dagger\dagger)$ and $(**)$, we obtain

$$\lambda_t\alpha(\lambda_i, \lambda_t + 1) - (\alpha)\delta_t\alpha(\lambda_i, \lambda_t + 1) + (\alpha)\delta_t\alpha(\lambda_i) = \lambda_t\alpha(\lambda_i, \lambda_t + 1).$$

Thus, $(\alpha)\delta_t\alpha(\lambda_i, \lambda_t + 1) = (\alpha)\delta_t\alpha(\lambda_i)$, and so we have $\alpha(\lambda_i, \lambda_t + 1) = \alpha(\lambda_i)$.

Let $\alpha(\lambda_i) = p(\lambda_i)/q(\lambda_i)$ with $(p(\lambda_i), q(\lambda_i)) = 1$. Then $q(\lambda_i)$ divides $q(\lambda_i, \lambda_t + 1)$ and $p(\lambda_i)$ divides $p(\lambda_i, \lambda_t + 1)$ since

$$p(\lambda_i, \lambda_t + 1)q(\lambda_i) = p(\lambda_i)q(\lambda_i, \lambda_t + 1).$$

Thus, as in the proof of Theorem 5.1, $p(\lambda_i), q(\lambda_i) \in F(\lambda_i)$, where $i \in I - \{t\}$. This is contrary to the choice of λ_t. Hence $\alpha(\lambda_i) \in F$.

Now we will suppose that $(\alpha(\lambda_i))\delta_t = 0$. Considering $(\dagger)$, we have $\alpha(\lambda_i)\lambda_t = \lambda_t\alpha(\lambda_i, \lambda_t + 1)$. Thus, in this case also, $\alpha(\lambda_i) = \alpha(\lambda_i, \lambda_t + 1)$, which as above implies that $\alpha(\lambda_i) \in F$.

Isomorphic nearfields must have isomorphic centres. Thus let $(F(\lambda), +, \circ)$ be a nearfield as constructed in [**3**], and let Q be the set of rational numbers. Let $\text{card}(I) = c$, the cardinality of the real numbers. Suppose that there is a field, F, such that $(F(\lambda), +, \circ)$ is isomorphic to $(Q(\lambda_i), +, \circ)$, $i \in I$. Then $Z(F(\lambda)) = F$ and $Z(Q(\lambda_i)) = Q$ by Theorem 5.2. Thus $F = Q$. But, card$(Q(\lambda))$ is less than card$(Q(\lambda_i)) = c$. Hence no isomorphism is possible. Thus, no nearfield as constructed by Zemmer is isomorphic to $(Q(\lambda_i), +, \circ)$, $i \in I$.

Construction 5.2. Let F be a field of characteristic 0; λ an indeterminate over F;

$$M = \left\{ \sum_{i=k}^{\infty} \phi_i(\lambda)t^i \mid \phi_i(\lambda) \in F(\lambda) \right\};$$

$$\sum \phi_i(\lambda)t^i + \sum \psi_i(\lambda)t^i = \sum (\phi_i(\lambda) + \psi_i(\lambda))t^i;$$

and

$$\left(\sum_{i=k}^{\infty} \phi_i(\lambda)t^i \right) \cdot \left(\sum_{i=n}^{\infty} \psi_i(\lambda)t^i \right) = \sum_{i=n}^{\infty} \left(\sum_{j=k}^{i} \phi_j(\lambda)\psi_{i-j}(\lambda + j) \right) t^i.$$

Let $(\sum \phi_i(\lambda)t^i)\delta =$ degree of numerator of $\phi_k(\lambda)$ − degree of denominator of $\phi_k(\lambda)$, where $\phi_k(\lambda)$ is the first non-zero $\phi_i(\lambda)$. Let

$$(\sum \phi_i(\lambda)t^i)T = \sum \phi_i(\lambda + 1)t^i.$$

For $\alpha, \beta \in M$ define $\alpha \circ \beta = \alpha(\beta T^{(\alpha)\delta})$ if $\alpha \neq 0$. For $\alpha = 0$, $\alpha \circ \beta = 0$.

THEOREM 5.3. *$(M, +, \circ)$ is a non-planar nearfield.*

Proof. It is routine to see that $(M, +, \cdot)$ is a nearfield. We will give only the proof that multiplicative inverses of non-zero elements exist.

Let $\alpha = \sum_{i=k}^{\infty} \phi_i(\lambda)t^i \neq 0$, with $\phi_k(\lambda) \neq 0$. We will use induction on i to find $\psi_{-k+i}(\lambda)$ such that $(\sum \phi_i(\lambda)t^i)(\sum \psi_{-k+i}(\lambda)t^{-k+i}) = 1$.

Let $\psi_{-k}(\lambda) = \phi_k^{-1}(\lambda - k)$. Then

$$\begin{aligned} \phi_k(\lambda)t^k\psi_{-k}(\lambda)t^{-k} &= \phi_k(\lambda)t^k\phi_k^{-1}(\lambda - k)t^{-k} \\ &= \phi_k(\lambda)\phi_k^{-1}(\lambda) \\ &= 1. \end{aligned}$$

Let $\psi_{-k+1}(\lambda) = -\phi_k^{-1}(\lambda - k)\phi_{k+1}(\lambda - k)\psi_{-k}(\lambda + 1)$. Then

$$\phi_k(\lambda)t^k\psi_{-k+1}(\lambda)t^{-k+1} + \phi_{k+1}(\lambda)t^{k+1}\psi_{-k}(\lambda)t^{-k} = 0.$$

Suppose that $\psi_{-k+i}(\lambda)$ have been determined for $0 \leqq i < n$ such that

$$\phi_k(\lambda)t^k\psi_{-k+i}(\lambda)t^{-k+i} + \ldots + \phi_{k+i}(\lambda)t^{k+i}\psi_{-k}(\lambda)t^{-k} = 0.$$

Let

$$\begin{aligned} \psi_{-k+n}(\lambda) = -\phi_k^{-1}(\lambda - k)[\phi_{k+1}(\lambda - k)\psi_{-k+n-1}(\lambda + 1) + \ldots \\ + \phi_{k+n}(\lambda - k)\psi_{-k}(\lambda + n)]. \end{aligned}$$

It is routine to see that $\sum \psi_{-k+i}(\lambda)t^i$ suffices.

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Now we will turn to the $\circ$ operation. Let $\alpha, \beta, \gamma \in M - \{0\}$.

$$\begin{aligned}(\alpha \circ \beta) \circ \gamma &= \alpha(\beta T^{(\alpha)\delta}) \circ \gamma \\ &= [\alpha(\beta T^{(\alpha)\delta})][\gamma T^{(\alpha(\beta T^{(\alpha)\delta}))\delta}] \\ &= [\alpha(\beta T^{(\alpha)\delta})][\gamma T^{(\alpha)\delta+(\beta)\delta}] \\ &= \alpha(\beta(\gamma T^{(\beta)\delta}))T^{(\alpha)\delta} \\ &= \alpha(\beta \circ \gamma)T^{(\alpha)\delta} \\ &= \alpha \circ (\beta \circ \gamma).\end{aligned}$$

Thus $\circ$ is an associative operation.

Let $\alpha \neq 0$, $\alpha \circ (\alpha^{-1})T^{-(\alpha)\delta} = \alpha(\alpha^{-1})T^{-(\alpha)\delta}T^{(\alpha)\delta} = 1$. Thus $(M', \circ)$ is a group. $\alpha \circ (\beta + \gamma) = \alpha \circ \beta + \alpha \circ \gamma$ is clear since T is an automorphism of $(M, +, \cdot)$. Thus $(M, +, \circ)$ is a nearfield.

The equation $x = \lambda \circ x + \lambda$ has no solution. Suppose it does and let it be $\sum \phi_i(\lambda)t^i$. Then

$$\begin{aligned}\sum \phi_i(\lambda)t^i &= \lambda \circ \sum \phi_i(\lambda)t^i + \lambda \\ &= \sum \lambda\phi_i(\lambda + 1)t^i + \lambda.\end{aligned}$$

Thus $\phi_0(\lambda) = \phi_0(\lambda + 1) + \lambda$. There is no such $\phi_0(\lambda)$, [**3**]. Thus $(M, +, \circ)$ is non-planar.

References

1. J. André, *Projective Ebenen über Fastkorpern*, Math. Z. *62* (1955), 137–160.
2. R. Sandler, *Pseudo planes and pseudo ternaries*, J. Algebra *4* (1966), 300–316.
3. J. Zemmer, *Nearfields, planar, and non-planar*, Math. Student *31* (1964), 145–150.

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https://doi.org/10.4153/CJM-1970-108-1 Published online by Cambridge University Press