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Integrating Lie algebroids via stacks

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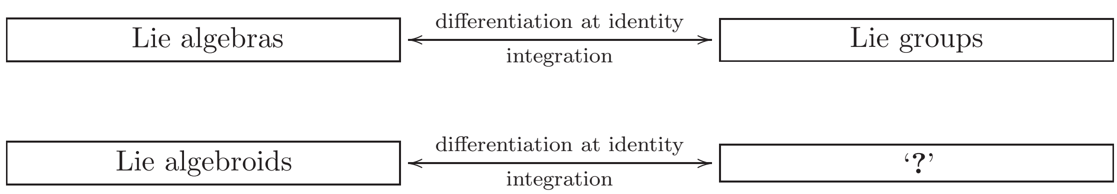
Abstract

Lie algebroids cannot always be integrated into Lie groupoids. We introduce a new structure, 'Weinstein groupoid', which may be viewed as stacky groupoids. We use this structure to present a solution to the integration problem of Lie algebroids. It turns out that every Weinstein groupoid has a Lie algebroid and every Lie algebroid can be integrated into such a groupoid.

1. Introduction

In this paper, we present a new viewpoint to integrate (finite-dimensional) Lie algebroids: unlike (finite-dimensional) Lie algebras which always have their associated Lie groups, Lie algebroids do not always have their associated Lie groupoids [AM84, AM85]. So the Lie algebroid version of Lie's third theorem poses the question indicated by the following chart.



Pradines posed the above question in [Pra68] and constructed local Lie groupoids (formulated in [CDW87, Kar86, van84]) as the integration object '**?**'. However, a global object for '**?**' is still required: not only would it give a conceptually better answer to the diagram above (Lie groups are global objects), but it also has profound applications in Poisson geometry, such as Weinstein's symplectic groupoids [Wei87], Xu's Morita equivalence of Poisson manifolds [Xu91b, Xu91a], symplectic realizations [Wei83], Picard groups [BW04] and the linearization problem of Poisson manifolds [CF04].

After Pradines' local groupoids, progress towards special cases of the above integration problem was made by [Daz90, Deb00, Mac87, Nis00, Wei89], among others. An important approach to finding a global object is the use of path spaces. This idea is not new, see [Wei03] for a nice discussion. We pay particular attention to the recent work of Crainic and Fernandes [CF03] and of Cattaneo and Felder [CF01]. For a Lie algebroid A, they study the space of A-paths. They are able to give a negative answer to the integrability problem: not every Lie algebroid can be integrated into a Lie groupoid. From the space of A-paths they construct a *topological* groupoid and determine equivalent conditions for this groupoid to be a Lie groupoid that integrates the given Lie algebroid A. So their work shows that every Lie algebroid can be integrated into a topological groupoid, but in general this topological groupoid does not have enough information to recover the Lie algebroid we start with. As conjectured by Weinstein, one hopes that there are additional structures on this topological groupoid, which allow us to recover the Lie algebroid. As the topological groupoid is a space of leaves, one natural approach is via étale groupoids [Hae84]. It turns out that differentiable stacks discussed

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in recent papers [BX, Met03, Pro96] also provide a suitable structure to the above conjecture posed by Weinstein.

We introduce the notion of *Weinstein groupoid*, which formalizes the additional structures to put on this topological groupoid. By allowing Weinstein groupoids, we answer the integrability problem positively – every Lie algebroid can be integrated into a Weinstein groupoid.

Definition 1.1 (Weinstein groupoid). A Weinstein groupoid over a manifold M consists of the following data:

(i) an étale differentiable stack $\mathcal{G}$ (see § 3 for the definition);

(ii) (source and target) maps $\bar{\mathbf{s}}$, $\bar{\mathbf{t}}$: $\mathcal{G} \to M$, which are surjective submersions between differentiable stacks;

(iii) (multiplication) a map $\bar{m}$: $\mathcal{G} \times_{\bar{\mathbf{s}},\bar{\mathbf{t}}} \mathcal{G} \to \mathcal{G}$, satisfying the following properties:

- $\bar{\mathbf{t}} \circ \bar{m} = \bar{\mathbf{t}} \circ pr_1$, $\bar{\mathbf{s}} \circ \bar{m} = \bar{\mathbf{s}} \circ pr_2$, where $pr_i : \mathcal{G} \times_{\bar{\mathbf{s}},\bar{\mathbf{t}}} \mathcal{G} \to \mathcal{G}$ is the ith projection $\mathcal{G} \times_{\bar{\mathbf{s}},\bar{\mathbf{t}}} \mathcal{G} \to \mathcal{G}$;
- associativity up to a 2-morphism, i.e. there is a unique 2-morphism α between maps $\bar{m} \circ (\bar{m} \times id)$ and $\bar{m} \circ (id \times \bar{m})$;

(iv) (identity section) an injective immersion $\bar{e}$: $M \to \mathcal{G}$ such that, up to 2-morphisms, the following identities

$$\bar{m} \circ ((\bar{e} \circ \bar{\mathbf{t}}) \times id) = id, \bar{m} \circ (id \times (\bar{e} \circ \bar{\mathbf{s}})) = id,$$

hold (in particular, by combining with the surjectivity of $\bar{\mathbf{s}}$ and $\bar{\mathbf{t}}$, one has $\bar{\mathbf{s}} \circ \bar{e} = id$, $\bar{\mathbf{t}} \circ \bar{e} = id$ on M);

(v) (inverse) an isomorphism of differentiable stacks $\bar{i}$: $\mathcal{G} \to \mathcal{G}$ such that, up to 2-morphisms, the following identities

$$\bar{m} \circ (\bar{i} \times id \circ \Delta) = \bar{e} \circ \bar{\mathbf{s}}, \bar{m} \circ (id \times \bar{i} \circ \Delta) = \bar{e} \circ \bar{\mathbf{t}},$$

hold, where Δ is the diagonal map: $\mathcal{G} \to \mathcal{G} \times \mathcal{G}$.

Moreover, restricting to the identity section, the above 2-morphisms between maps are the id 2-morphisms. Namely, for example, the 2-morphism α induces the id 2-morphism between the following two maps:

$$\bar{m} \circ ((\bar{m} \circ (\bar{e} \times \bar{e} \circ \delta)) \times \bar{e} \circ \delta) = \bar{m} \circ (\bar{e} \times (\bar{m} \circ (\bar{e} \times \bar{e} \circ \delta)) \circ \delta),$$

where δ is the diagonal map: $M \to M \times M$.

The terminology involving stacks in the above definition will be explained in detail in § 3. For now, to get a general idea of these statements, one can take stacks simply to be manifolds.

Our main result is the following theorem.

Theorem 1.2 (Lie's third theorem). *With each Weinstein groupoid one can associate a Lie algebroid. For every Lie algebroid A, there are naturally two Weinstein groupoids $\mathcal{G}(A)$ and $\mathcal{H}(A)$ with Lie algebroid A.*

We can apply our result to the classical integrability problem, which studies exactly when a Lie algebroid can be integrated into a Lie groupoid.

Theorem 1.3. *A Lie algebroid A is integrable in the classical sense if and only if $\mathcal{H}(A)$ is representable, i.e. it is an honest (smooth) manifold. In this case $\mathcal{H}(A)$ is the source-simply connected Lie groupoid of A (it is also called the Weinstein groupoid of A in [CF03]).*

We can also relate our work to the previous work on the integration of Lie algebroids via the following two theorems.

Theorem 1.4. *Given a Weinstein groupoid $\mathcal{G}$, there is an associated local Lie groupoid G_{loc} (canonical up to isomorphisms near the identity section) that has the same Lie algebroid as $\mathcal{G}$.*

Theorem 1.5. *The orbit spaces of $\mathcal{H}(A)$ and $\mathcal{G}(A)$ as topological spaces are both isomorphic to the topological groupoid of A constructed in [CF03].*

2. Path spaces

We define the A_0-path space, which is very similar to the A-paths defined in [CF03] (it is, in fact, a submanifold of the A-path space).

Definition 2.1. Given a Lie algebroid $A \xrightarrow{\pi} M$ (A is assumed to be a Hausdorff manifold) with anchor $\rho : A \to TM$, a C^1 map a: $I = [0, 1] \to A$ is an A_0*-path* if it satisfies the equation

$$\rho(a(t)) = \frac{d}{dt}(\pi \circ a(t)),$$

with boundary conditions $a(0) = 0$, $a(1) = 0$, $\dot{a}(0) = 0$, $\dot{a}(1) = 0$. We often denote the base path $\pi \circ a(t)$ in M by $\gamma(t)$. We denote by P_0A the set of all A_0-paths of A. It is a topological space with topology given by uniform convergence of maps. Omitting the boundary condition above, we get the definition of A-*paths*, and we denote the space of A-paths by P_aA.

We can equip P_0A with the structure of a smooth (Banach) manifold modeled by $P\mathbb{R}^n = C^1(I, \mathbb{R}^n)$ with norm $\|f\|^2 = \sup\{|f|^2 + |f'|^2\}$. P_0A is defined by equations on PA, so it inherits the structure of a Banach manifold from PA. See [CF03] and [Zhu04] for details.

Definition 2.2. Let $a(\epsilon, t)$ be a family of A_0-paths of class C^2 in ϵ and assume that their base paths $\gamma(\epsilon, t)$ have fixed end points. Let ∇ be a connection on A with torsion T_∇ defined as

$$T_\nabla(\alpha, \beta) = \nabla_{\rho(\beta)}\alpha - \nabla_{\rho(\alpha)}\beta + [\alpha, \beta].$$

Then the solution $b = b(\epsilon, t)$ of the differential equation

$$\partial_t b - \partial_\epsilon a = T_\nabla(a, b), \quad b(\epsilon, 0) = 0 \tag{1}$$

does not depend on the choice of connection ∇. Furthermore, $b(\cdot, t)$ is an A-path for every fixed t, i.e. $\rho(b(\epsilon, t)) = (d/d\epsilon)\gamma(\epsilon, t)$. If the solution b satisfies $b(\epsilon, 1) = 0$, for all ϵ, then a_0 and a_1 are said to be *equivalent* and we write $a_0 \sim a_1$.

Remark 1.

(i) Here, $T_\nabla(a, b)$ is not quite well defined. We need to extend a and b by sections of A, α and β, such that $a(t) = \alpha(\gamma(t), t)$ and the same for b. Then $T_\nabla(a, b)|_{\gamma(t)} := T_\nabla(\alpha, \beta)|_{\gamma(t)}$ at every point on the base path. However, the choice of extending sections does not affect the result.

(ii) A homotopy of A-paths [CF03] is defined by replacing A_0 by A in the definition above. A similar result as above holds for A-paths [CF03]. So the above statement holds viewing A_0-paths as A-paths.

This flow of A_0-paths $a(\epsilon, t)$ generates a foliation $\mathcal{F}$. The A_0-path space is a Banach submanifold of the A-path space and $\mathcal{F}$ is the restricted foliation of the foliation defined in [CF03, § 4]. For any foliation, there is an associated *monodromy groupoid* [MM03] (or *fundamental groupoid* as in [CW99]): the arrows are paths within a leaf up to homotopies with fixed end points inside the leaf. For any regular foliation on a smooth manifold its monodromy groupoid is a Lie groupoid in the sense of [CF03]. In our case, it is an infinite-dimensional groupoid equipped with a Banach manifold structure. Here, we slightly generalized the definition of Lie groupoids to the category of Banach manifolds by requiring the same conditions, but in the sense of Banach manifolds. Denote the

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monodromy groupoid of $\mathcal{F}$ by $\mathrm{Mon}(P_0A) \overset{\mathbf{s}_M}{\underset{\mathbf{t}_M}{\rightrightarrows}} P_0A$. In a very similar way [MM03], one can also define the *holonomy groupoid* $\mathrm{Hol}(\mathcal{F})$ of $\mathcal{F}$: the arrows are equivalence classes of paths with the same holonomy.

To obtain a finite-dimensional description we take an open cover $\{U_i\}_{i\in I}$ of P_0A such that in each chart U_i one can choose a transversal P_i of the foliation $\mathcal{F}$. We can assume I is countable as P_0A is second countable. By [CF03, Proposition 4.8], each P_i is a smooth manifold of dimension equal to that of A. Let $P = \coprod P_i$, which is a smooth immersed submanifold of P_0A. We can choose $\{U_i\}$ and transversal $\{P_i\}$ to satisfy the following conditions.

(i) If U_i contains the constant path 0_x for some $x \in M$, then U_i has the transversal P_i containing all constant paths 0_y in U_i for $y \in M$.

(ii) If $a(t) \in P_i$ for some i, then $a(1-t) \in P_j$ for some j.

It is possible to meet the above two conditions: for (i) we refer readers to [CF03, Proposition 4.8]. There the result is for P_aA. For P_0A, one has to use a smooth reparametrization τ with the properties:

(i) $\tau(t) = 1$ for all $t \geqslant 1$ and $\tau(t) = 0$ for all $t \leqslant 0$;

(ii) $\tau'(t) > 0$ for all $t \in (0,1)$.

Then $a^\tau(t) := \tau(t)' a(\tau(t))$ is in P_0A for all $a \in P_aA$. $\phi_\tau : a \mapsto a^\tau$ defines an injective bounded linear map from $P_aA \to P_0A$. Therefore, we can adapt the construction for P_aA to our case by using ϕ_τ. For (ii), we define a map $\mathrm{inv} : P_0A \to P_0A$ by $\mathrm{inv}(a(t)) = a(1-t)$. Obviously inv is an isomorphism. In particular, it is open. So we can add $\mathrm{inv}(U_i)$ and $\mathrm{inv}(P_i)$ to the collection of open sets and transversals. The new collection will have the desired property.

The restriction $\mathrm{Mon}(P_0A)|_P$ of $\mathrm{Mon}(P_0A)$ to P is a finite-dimensional étale Lie groupoid (i.e. the source (hence, the target) map is a local diffeomorphism) [MRW87], which we denote by $\Gamma \rightrightarrows P$. For a different transversal P' the restriction of $\mathrm{Mon}(P_0A)$ to P' is another finite-dimensional étale Lie groupoid. All of these groupoids are related by 'Morita equivalence'. One can do the same to $\mathrm{Hol}(P_0A)$ and obtain a finite-dimensional étale Lie groupoid, which we denote by $\Gamma^h \overset{\mathbf{s}_1}{\underset{\mathbf{t}_1}{\rightrightarrows}} P$. Although these groupoids are Morita equivalent to each other, they are in general not Morita equivalent to the groupoids induced from $\mathrm{Mon}(P_0A)$.

We will build a Weinstein groupoid of A based on this path space P_0A. One can interpret the 'identity section' as an embedding obtained from taking constant paths 0_x, for all $x \in M$; the 'inverse' of a path $a(t)$ as $a(1-t)$; the 'source and target maps' $\mathbf{s}$ and $\mathbf{t}$ as taking the end points of the base path $\gamma(t)$. According to the two conditions above, these maps are also well defined on the finite-dimensional space P. As reparametrizations and projections are bounded linear operators in Banach space $C^\infty(I, \mathbb{R}^n)$, the maps defined above are smooth maps in P_0A, hence in P.

To define the multiplication, notice that for two A-paths a_1, a_0 in P_0A such that the base paths satisfy $\gamma_0(1) = \gamma_1(0)$, one can define a 'concatenation' [CF03]:

$$a_1 \odot a_0 = \begin{cases} 2a_0(2t), & 0 \leqslant t \leqslant \frac{1}{2} \\ 2a_1(2t-1), & \frac{1}{2} < t \leqslant 1. \end{cases}$$

Concatenation is a bounded linear operator in the local charts, hence is a smooth map. However, it is not associative. Moreover, it is not well defined on P. If we quotient out by the equivalence relation induced by $\mathcal{F}$, concatenation is associative and well defined. However, after quotienting out by the equivalence, we may no longer end up with a smooth manifold. To overcome the difficulty, our solution is to pass to the world of differentiable stacks.

3. Differentiable stacks and Lie groupoids

The notion of stacks has been extensively studied in algebraic geometry for the past few decades (see, for example, [DM69, Vis89, LM00, BEFFGK]). Stacks can also be defined over other categories, such as the category of topological spaces and the category of smooth manifolds (see, for example, [AGV72, Pro96, Vis02, BX, Met03]). In this section we collect certain facts about stacks in the differentiable category that will be used in next sections. Detailed treatments of these can be found in the literature (see, for example, [Pro96, BX, Met03]). Some recent application of differentiable stacks can be found in [BX03].

3.1 Definitions

Let $\mathcal{C}$ be the category of differentiable manifolds (second countable but not necessarily Hausdorff). A stack over $\mathcal{C}$ is a category fibred in groupoids satisfying two conditions: 'isomorphism is a sheaf', and 'descent datum is effective', see [BX] and [Met03] for the complete definition.

A manifold is a stack over $\mathcal{C}$. We call such stacks *representable*. Morphisms between stacks are just functors between fibred categories. A morphism $f : \mathcal{X} \to \mathcal{Y}$ is a *representable submersion* if for any morphism $M \to \mathcal{Y}$ from a manifold M, the fiber product $\mathcal{X} \times_{\mathcal{Y}} M$ is representable and the induced morphism $\mathcal{X} \times_{\mathcal{Y}} M \to M$ is a submersion (between manifolds). If, in addition, $\mathcal{X} \times_{\mathcal{Y}} M \to M$ is surjective, then f is called a *representable surjective submersion* [BX].

A *differentiable stack* is a stack $\mathcal{X}$ over $\mathcal{C}$ together with a representable surjective submersion $\pi : X \to \mathcal{X}$ from a Hausdorff smooth manifold X (called an *atlas*). A manifold is a differentiable stack by identifying it with its functor of points. For a Lie group G, the category of principal G-bundles is an example of a differentiable stack.

Properties of morphisms between differentiable stacks can be defined by considering pullbacks to atlases. In this way one can define what it means for a morphism to be *smooth*, *étale*, *an immersion*, *a closed immersion* and *an injective immersion*; see, for instance, [BX] and [Met03]. To check whether a morphism has these properties it suffices to check on a particular atlas. Compositions of representable (surjective) submersions are still representable (surjective) submersion. Representable (surjective) submersions are stable under base-change.

A differentiable stack $\mathcal{X}$ is called *étale* if it has an atlas $\pi : X \to \mathcal{X}$ such that π is étale.

3.2 Stacks and groupoids

One can go between differentiable stacks and Lie groupoids. For a differentiable stack $\mathcal{X}$ with an atlas $X_0 \to \mathcal{X}$, there is a Lie groupoid $X_1 := X_0 \times_{\mathcal{X}} X_0 \rightrightarrows X_0$ with the two maps being projections. This groupoid is called a groupoid presentation of $\mathcal{X}$. An étale differentiable stack can be presented by an étale groupoid. Given a Lie groupoid $G = (G_1 \rightrightarrows G_0)$, the category of principal G bundles BG is a differentiable stack with an atlas $G_0 \to BG$ such that $G_1 = G_0 \times_{BG} G_0$. See [Vis89, Pro96, BX, Met03] for details.

There are some ambiguities: Different atlases give different groupoids, and two Lie groupoids may represent the same stack. The following result clarifies this.

Proposition 3.1 (see [Pro96, BX, Met03]). *Two Lie groupoids present isomorphic differential stacks if and only if they are Morita equivalent.*

In other words, differentiable stacks correspond to Morita equivalence classes of Lie groupoids. Also, smooth 1-morphisms between differentiable stacks correspond to *Hilsum–Skandalis* (HS) morphisms given by one-side principal bibundles of the groupoids, see [Pro96] for details. 2-morphisms of differentiable stacks correspond to 2-morphisms of Lie groupoids (see [Pro96] and [Met03] for a definition).

3.3 Fibre products and submersions

It is convenient to use invariant maps to produce maps between stacks. In this section we state some results concerning invariant maps and fiber products which will be used later in the construction of the Weinstein groupoids. The proofs are standard and can be found in [Zhu04].

LEMMA 3.2. *Given a Lie groupoid $G := (G_1 \rightrightarrows G_0)$ and a manifold M, any G-invariant map $f : G_0 \to M$ induces a morphism between differentiable stacks $\bar{f} : BG \to M$ such that $f = \bar{f} \circ \phi$, where $\phi : G_0 \to BG$ is the covering map of atlases.*

DEFINITION 3.3 ((Surjective) submersions). A morphism $f : \mathcal{X} \to \mathcal{Y}$ of differentiable stacks is called a submersion if for any atlas $M \to \mathcal{X}$, the composition $M \to \mathcal{X} \to \mathcal{Y}$ satisfies the following: for any atlas $N \to \mathcal{Y}$ the induced morphism $M \times_{\mathcal{Y}} N \to N$ is a submersion. A surjective submersion is a submersion that is also an epimorphism.

Note that this definition is different from that in [Met03]. A representable submersion is a submersion, but the converse is not true.

PROPOSITION 3.4 (Fibred products). *Let Z be a manifold and $f : \mathcal{X} \to Z$ and $g : \mathcal{Y} \to Z$ be morphisms of differentiable stacks. If either f or g is a submersion, then $\mathcal{X} \times_Z \mathcal{Y}$ is a differentiable stack. Moreover, let $X \to \mathcal{X}, Y \to \mathcal{Y}$ be atlases for $\mathcal{X}$ and $\mathcal{Y}$, respectively. Then $X \times_Z Y \to \mathcal{X} \times_Z \mathcal{Y}$ is an atlas for $\mathcal{X} \times_Z \mathcal{Y}$. Moreover, put $X_1 = X \times_{\mathcal{X}} X$ and $Y_1 = Y \times_{\mathcal{Y}} Y$, then $\mathcal{X} \times_Z \mathcal{Y}$ is presented by the groupoid $(X_1 \times_Z Y_1 \rightrightarrows X \times_Z Y)$.*

LEMMA 3.5. *If a G-invariant map $f : G_0 \to M$ is a submersion, then the induced map $\bar{f} : BG \to M$ is a submersion of differentiable stacks.*

It is not hard to see that the construction of stacks in the category of smooth manifolds can be extended to the category of Banach manifolds, yielding the notion of Banach stacks. Many properties of differentiable stacks, including those discussed here, are also shared by Banach stacks. Also, the 2-category of differentiable stacks can be obtained from the 2-category of Banach stacks by restricting the base category.

4. The Weinstein groupoids of Lie algebroids

4.1 The construction

Recall that in § 2.1, given a Lie algebroid A, we constructed an étale groupoid $\Gamma \rightrightarrows P$. We obtain an étale differential stack $\mathcal{G}(A)$ presented by $\Gamma \rightrightarrows P$. For a different transversal P', the restriction $\Gamma' = \mathrm{Mon}(P_0 A)|_{P'}$ is Morita equivalent to Γ through the finite-dimensional bibundle $\mathbf{s}_M^{-1}(P) \cap \mathbf{t}_M^{-1}(P')$. So they represent isomorphic differential stacks. Therefore, we might base our discussion on $\Gamma \rightrightarrows P$. As $\mathrm{Mon}(P_0 A) \rightrightarrows P_0 A$ is Morita equivalent to $\Gamma \rightrightarrows P$ through the Banach bibundle $\mathbf{s}_M^{-1}(P)$, $\mathcal{G}(A)$ can also be presented by $\mathrm{Mon}(P_0 A)$ as a Banach stack.

In this section, we construct two Weinstein groupoids $\mathcal{G}(A)$ and $\mathcal{H}(A)$ for every Lie algebroid A and prove Theorem 1.3.

We begin with $\mathcal{G}(A)$. We first define the inverse, identity section, source and target maps on the level of groupoids.

DEFINITION 4.1. Define the following.

- $i : (\Gamma \rightrightarrows P) \to (\Gamma \rightrightarrows P)$ by $g = [a(\epsilon, t)] \mapsto [a(\epsilon, 1 - t)]$, where $[\cdot]$ denotes the homotopy class in $\mathrm{Mon}(P_0 A)$.
- $e : M \to (\Gamma \rightrightarrows P)$ by $x \mapsto 1_{0_x}$, where 1_{0_x} denotes the identity homotopy of the constant path 0_x.

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- $\mathbf{s} : (\Gamma \rightrightarrows P) \to M$ by $g = [a(\epsilon, t)] \mapsto \gamma(0,0)(= \gamma(\epsilon, 0), \forall \epsilon)$, where γ is the base path of a.
- $\mathbf{t} : (\Gamma \rightrightarrows P) \to M$ by $g = [a(\epsilon, t)] \mapsto \gamma(0,1)(= \gamma(\epsilon, 1), \forall \epsilon)$.

These maps can be defined similarly on $\mathrm{Mon}(P_0A) \rightrightarrows P_0A$. These maps are all bounded linear maps in the local charts of $\mathrm{Mon}(P_0A)$. Therefore, they are smooth homomorphisms between Lie groupoids. Hence, they defined smooth morphisms between differentiable stacks. We denote the maps corresponding to i, e, $\mathbf{s}$ and $\mathbf{t}$ on the stack level by $\bar{i}$, $\bar{e}$, $\bar{\mathbf{s}}$ and $\bar{\mathbf{t}}$, respectively.

LEMMA 4.2. *The maps $\bar{\mathbf{s}}$ and $\bar{\mathbf{t}}$ are surjective submersions. The map $\bar{e} : M \to \mathcal{G}(A)$ is an injective immersion. The map $\bar{i}$ is an isomorphism.*

Proof. As any path through x in M can be lifted to a path in P passing through any given preimage of x, $\mathbf{s}$ and $\mathbf{t}$ restricted to P are Γ-invariant and submersions. According to Lemmas 3.2 and 3.5, the induced maps $\bar{\mathbf{s}}$ and $\bar{\mathbf{t}}$ are submersions.

Denote by e_0 the restricted map of e on the level of objects: $e_0 : M \to P$. Note that e_0 fits into the following diagram (which is not commutative).

$$\begin{array}{ccc} M \times_{\mathcal{G}(A)} P & \xrightarrow{pr_2} & P \\ {\scriptstyle pr_1}\downarrow & \overset{e_0}{\nearrow} & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{\bar{e}} & \mathcal{G}(A) \end{array} \qquad (2)$$

Consider $x = (f : U \to M) \in M$, $\bar{e}(x) = U \times_{e_0 \circ f, G_0} G_1$ as a G-torsor, and $e_0(x) = (e_0 \circ f : U \to G_0) \in G_0$. Consider also $y = (g : U \to G_0) \in G_0$, $\pi(y) = U \times_{g, G_0} G_1$. A typical object of $M_i \times_{\mathcal{G}} G_0$ is (x, η, y) where η is a morphism of G-torsors from $\bar{e}(x)$ to $\pi(y)$ over id_U of U. Then by the equivariancy of η, we have a map ϕ: $U \to G_1$, such that $e_0 \circ f = g \cdot \phi$. Therefore, we have a map $\alpha : M \times_{\mathcal{G}(A)} G_0 \to G_1$ given by $\alpha(x, \eta, y) = \phi$, such that

$$e_0 \circ pr_1 = pr_2 \cdot \alpha.$$

As π is étale, so is pr_1. Moreover, as e_0 is an embedding, pr_2 must be an immersion. This shows that $\bar{e}$ is an immersion for one atlas, hence $\bar{e}$ is an immersion.

As $\mathbf{s} \circ e = \mathbf{t} \circ e = id$ on the level of groupoids, the same identity passes to identity on the level of differentiable stacks. As $\bar{\mathbf{s}} \circ \bar{e} = \bar{\mathbf{t}} \circ \bar{e} = id$, it is easy to see that $\bar{e}$ must be monomorphic and $\bar{\mathbf{s}}$ (and $\bar{\mathbf{t}}$) must be epimorphic.

The map i is an isomorphism of groupoids, hence it induces an isomorphism at the level of stacks. □

Now we define the multiplication in the infinite-dimensional presentation. First we extend 'concatenation' to $\mathrm{Mon}(P_0A)$. Consider two elements $g_1, g_0 \in \mathrm{Mon}(P_0A)$ whose base paths on M are connected at the end points. Suppose g_i is represented by $a_i(\epsilon, t)$. Define

$$g_1 \odot g_0 = [a_1(\epsilon, t) \odot_t a_0(\epsilon, t)],$$

where $\odot_t$ means concatenation with respect to the parameter t and the $[\cdot]$ denotes the equivalence class of homotopies.

Note that $\mathbf{s} \circ \mathbf{s}_M = \mathbf{s} \circ \mathbf{t}_M$ and $\mathbf{t} \circ \mathbf{s}_M = \mathbf{t} \circ \mathbf{t}_M$ are surjective submersions by reasoning similar to that in the above, where $\mathbf{t}_M$ and $\mathbf{s}_M$ are source and target maps of $\mathrm{Mon}(P_0A) \rightrightarrows P_0A$. Hence, by Proposition 3.4,

$$\mathrm{Mon}(P_0A) \times_{\mathbf{s} \circ \mathbf{s}_M, M, \mathbf{t} \circ \mathbf{t}_M} \mathrm{Mon}(P_0A) \rightrightarrows P_0A$$

with source and target maps $\mathbf{s}_M \times \mathbf{s}_M$ and $\mathbf{t}_M \times \mathbf{t}_M$ is a Lie groupoid and it presents the stack $\mathcal{G} \times_{\bar{\mathbf{s}},M,\bar{\mathbf{t}}} \mathcal{G}$.

Finally, let m be the following smooth homomorphism between Lie groupoids.

$$\begin{array}{ccc} \mathrm{Mon}(P_0A) \underset{\mathbf{s}\circ\mathbf{s}_M, M, \mathbf{t}\circ\mathbf{t}_M}{\times} \mathrm{Mon}(P_0A) & \xrightarrow{\odot} & \mathrm{Mon}(P_0A) \\ {\scriptstyle \mathbf{t}_M\times\mathbf{t}_M} \Big\Downarrow {\scriptstyle \mathbf{s}_M\times\mathbf{s}_M} & & {\scriptstyle \mathbf{t}_M} \Big\Downarrow {\scriptstyle \mathbf{s}_M} \\ P_0A \times P_0A & \xrightarrow{\odot} & P_0A \end{array}$$

Multiplication is less obvious for the étale presentation $\Gamma \rightrightarrows P$. We will have to define the multiplication through a HS morphism.

Viewing P as a submanifold of P_0A, let $E = \mathbf{s}_M^{-1}(P) \cap \mathbf{t}_M^{-1}(m(P \times_M P)) \subset \mathrm{Mon}(P_0A)$. As $\mathbf{s}_M$ and $\mathbf{t}_M$ are surjective submersions and $m(P \times_M P) \cong P \times_M P$ is a submanifold of P_0A, E is a smooth manifold. As P is a transversal, $\mathbf{t}_M : E \to m(P \times_M P)$ is étale. Moreover, $\dim m(P \times_M P) = 2 \dim P - \dim M$. So E is finite dimensional. Further, note that $m : P_0A \times P_0A \to P_0A$ is injective and its 'inverse' m^{-1} defined on the image of m is given by

$$m^{-1} : b(t) \mapsto (b(2t_1), b(1 - 2t_2)), \quad t_1 \in [0, \tfrac{1}{2}], \quad t_2 \in [\tfrac{1}{2}, 1],$$

which is bounded linear in a local chart. Let $\pi_1 = m^{-1} \circ \mathbf{t}_M : E \to P \times_M P$ and $\pi_2 = \mathbf{s}_M : E \to P$. Then it is routine to check that the bibundle (E, π_1, π_2) gives a HS morphism from $\Gamma \times_M \Gamma \rightrightarrows P \times_M P$ to $\Gamma \rightrightarrows P$. It is not hard to verify that on the level of stacks (E, π_1, π_2) and m give two 1-morphisms differed by a 2-morphism. Thus, after modifying E by this 2-morphism, we get another HS morphism (E_m, π_1', π_2'), which presents the same map as m. Moreover, $E_m \cong E$ as bibundles.

Therefore, we have the following definition.

Definition 4.3. Define $\bar{m} : \mathcal{G}(A) \times_{\bar{\mathbf{s}},\bar{\mathbf{t}}} \mathcal{G}(A) \to \mathcal{G}(A)$ to be the smooth morphism between étale stacks presented by (E_m, π_1', π_2').

Remark 2. If we use $\mathrm{Mon}(P_0A)$ as the presentation, $\bar{m}$ is also presented by m.

Lemma 4.4. *The multiplication $\bar{m} : \mathcal{G}(A) \times \mathcal{G}(A) \to \mathcal{G}(A)$ is a smooth morphism between étale stacks and is associative up to a 2-morphism. That is, the diagram*

$$\begin{array}{ccc} \mathcal{G}(A) \underset{\mathbf{s},\mathbf{t}}{\times} \mathcal{G}(A) \underset{\mathbf{s},\mathbf{t}}{\times} \mathcal{G}(A) & \xrightarrow{id\times\bar{m}} & \mathcal{G}(A) \underset{\mathbf{s},\mathbf{t}}{\times} \mathcal{G}(A) \\ \Big\downarrow {\scriptstyle \bar{m}\times id} & & \Big\downarrow {\scriptstyle \bar{m}} \\ \mathcal{G}(A) \underset{\mathbf{s},\mathbf{t}}{\times} \mathcal{G}(A) & \xrightarrow{\bar{m}} & \mathcal{G}(A) \end{array}$$

is 2-commutative, i.e. there exists a 2-morphism $\alpha : \bar{m} \circ (\bar{m} \times id) \to \bar{m} \circ (id \times \bar{m})$.

Before the proof, we give a general remark about 2-morphisms.

Remark 3. For two groupoid homomorphisms f and g between G and H, a 2-morphism between the induced maps on the level of stacks represented by f to g is just a smooth map $\alpha : G_0 \to H_1$ such that $\alpha(\gamma x) = g(\gamma)\alpha(x)f(\gamma)^{-1}$, where $x \in G_0$ and $\gamma \in G_1$. This in particular gives us $f(x) = g(x) \cdot \alpha(x)$. So it is easy to see that not every two morphisms can be connected by a 2-morphism and when they do, the 2-morphism may not be unique (for example, this happens when the isotropy group is non-trivial and abelian).

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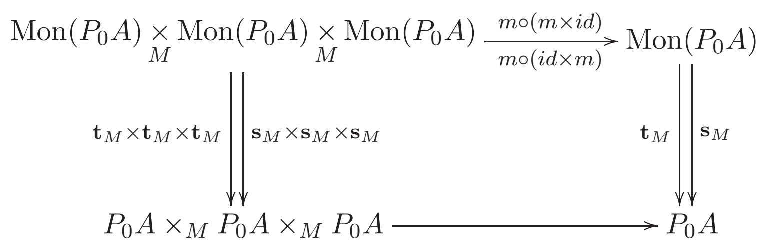
Proof. We will establish the 2-morphism on the level of Banach stacks. Note that a smooth morphism in the category of Banach manifolds between finite-dimensional manifolds is a smooth morphism in the category of finite-dimensional smooth manifolds. Therefore, the 2-morphism we will establish gives a 2-morphism for the étale stacks.

Take the Banach presentation $\mathrm{Mon}(P_0A)$, then $\bar{m}$ can simply be presented as a homomorphism between groupoids as in (4.1). According to Remark 3, we now construct a 2-morphism $\alpha : P_0A \times_M P_0A \times_M P_0A \to \mathrm{Mon}(P_0A)$ in the following diagram.



Let $\alpha(a_1, a_2, a_3)$ be the natural rescaling between $a_1 \odot (a_2 \odot a_3)$ and $(a_1 \odot a_2) \odot a_3$. Namely, $\alpha(a_1, a_2, a_3)$ is the homotopy class represented by

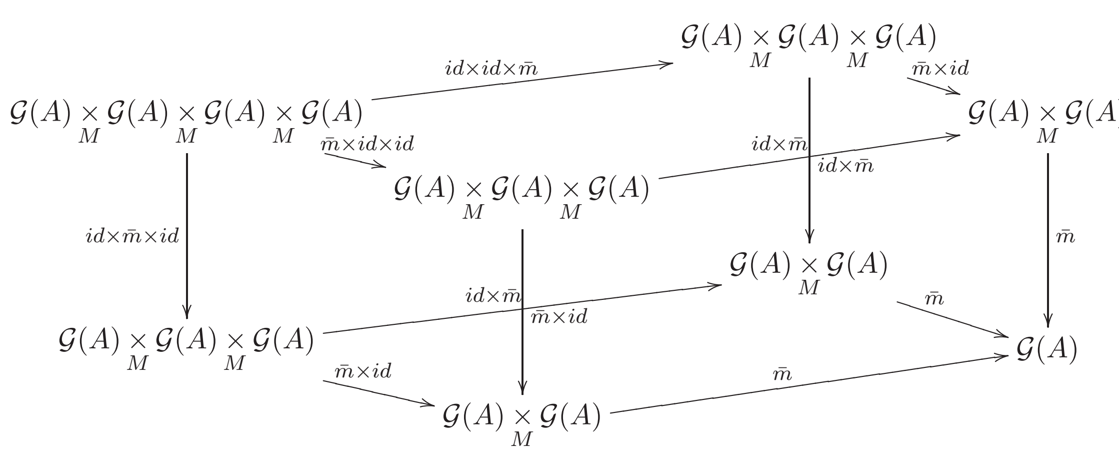
$$a(\epsilon, t) = ((1-\epsilon) + \epsilon\sigma'(t))a((1-\epsilon)t + \epsilon\sigma(t)), \tag{3}$$

where $\sigma(t)$ is a smooth reparametrization such that $\sigma(1/4) = 1/2$, $\sigma(1/2) = 3/4$. In local charts, α is a bounded linear operator. Therefore, it is a smooth morphism between Banach spaces. Moreover, for $x \in P_0A \times_M P_0A \times_M P_0A$ and $g \in \mathrm{Mon}(P_0A) \times_M \mathrm{Mon}(P_0A) \times_M \mathrm{Mon}(P_0A)$, $\alpha(g \cdot x) = m \circ (id \times m)(g) \cdot \alpha(x) \cdot (m \circ (m \times id))^{-1}(g)$. In fact, it is not hard to see $m \circ (m \times id)(x) = m \circ (id \times m)(x) \cdot \alpha(x)$. Counting homotopy inside and noting that we quotient out the homotopies of homotopies, the former equation is also true. Therefore, α serves as the desired 2-morphism. □

One might be curious about whether there are further obstructions to associativity. There are six ways to multiply four elements in $\mathcal{G}(A)$. Put

$$\begin{aligned}
F_1 &= \bar{m} \circ \bar{m} \times id \circ \bar{m} \times id \times id,\\
F_2 &= \bar{m} \circ id \times \bar{m} \circ \bar{m} \times id \times id,\\
F_3 &= \bar{m} \circ \bar{m} \times id \circ id \times id \times \bar{m},\\
F_4 &= \bar{m} \circ id \times \bar{m} \circ id \times id \times \bar{m},\\
F_5 &= \bar{m} \circ id \times \bar{m} \circ id \times \bar{m} \times id,\\
F_6 &= \bar{m} \circ \bar{m} \times id \circ id \times \bar{m} \times id.
\end{aligned}$$

These morphisms fit into the following commutative cube.



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There is a 2-morphism on each face of the cube to connect F_i and F_{i+1} ($F_7 = F_1$), constructed as in the last lemma. Let $\alpha_i : F_i \to F_{i+1}$. Will the composition $\alpha_6 \circ \alpha_6 \circ \cdots \circ \alpha_1$ be the identity 2-morphism? If so, given any two different ways of multiplying four (hence any number of) elements, different methods to obtain 2-morphisms between them will give rise to the same 2-morphism. As 2-morphisms between two 1-morphisms are not unique if our differential stacks are not honest manifolds, it is necessary to study the existence of further obstructions.

Proposition 4.5. *There is no further obstruction to associativity of $\bar{m}$ in $\mathcal{G}(A)$.*

Proof. In the presentation $\mathrm{Mon}(P_0A)$ of $\mathcal{G}(A)$, the α_i constructed above can be explicitly expressed as a smooth morphism: $P_0A \times_M P_0A \times_M P_0A \times_M P_0A \to \mathrm{Mon}(P_0A)$. More precisely, according to the lemma above, $\alpha_i(a_1, a_2, a_3, a_4)$ is the natural rescaling between $F_i(a_1, a_2, a_3, a_4)$ and $F_{i+1}(a_1, a_2, a_3, a_4)$. Here, by abuse of notation, we also denote the homomorphism on the groupoid level by F_i. It is not hard to see that $\alpha_6 \circ \alpha_5 \circ \cdots \circ \alpha_1$ is represented by a rescaling that is homotopic to the identity homotopy between A_0-paths.

Therefore, the composed 2-morphism is actually identity as $\mathrm{Mon}(P_0A)$ is made up by the homotopy of homotopy of A_0-paths. We also note that identity morphism in the category of Banach manifolds between two finite-dimensional manifolds is identity morphism in the category of finite-dimensional smooth manifolds. Therefore, there is no further obstructions even for 2-morphisms of étale stacks. ☐

Now to show $\mathcal{G}(A)$ is a Weinstein groupoid, it remains to show that the identities in items (4) and (5) in Definition 1.1 hold and the 2-morphisms in these identities are identity 2-morphisms when restricted to M. Note that for any A_0-path $a(t)$, we have

$$a(t) \odot_t 1_{\gamma(0)} \sim a(t), \quad a(1-t) \odot_t a(t) \sim \gamma(0),$$

where γ is the base path of $a(t)$. Using Remark 3(i), we can see that on the groupoid level $m \circ ((e \circ \mathbf{t}) \times id)$ and id only differ by a 2-morphism, and the same for the pairs $m \circ (i \times id)$ and $e \circ \mathbf{s}$, $\mathbf{s} \circ m$ and $\mathbf{s} \circ pr_1$. Therefore the corresponding identities hold on the level of differentiable stacks. Transform them to stacks and the rest of the identities also follow. Moreover, the 2-morphisms (in all presentations of $\mathcal{G}(A)$ we have described above) are formed by rescalings. When they restrict to constant paths in M, they are just id.

Summing up what we have discussed above, $\mathcal{G}(A)$ with all the structures we have given is a Weinstein groupoid over M.

We further comment that one can construct another natural Weinstein groupoid $\mathcal{H}(A)$ associated with A exactly in the same way as $\mathcal{G}(A)$ by the Lie groupoid $\mathrm{Hol}(P_0A)$ or $\Gamma^h \underset{\mathbf{t}_1}{\overset{\mathbf{s}_1}{\rightrightarrows}} P$ as they are Morita equivalent by a similar reason as their monodromy counterparts. One can establish the identity section, the inverse, etc., even the multiplication in exactly the same way. One only has to note that in the construction of the multiplication, the 2-morphism in the associativity diagram is the holonomy class (instead of homotopy class) of the reparametrization (3). One can do so because homotopic paths have the same holonomy. Moreover, by the same reason, there is no further obstructions to the multiplication on $\mathcal{H}(A)$.

Finally, we want to comment about the Hausdorffness of the source fibres (hence, the target fibres by the inverse) of $\mathcal{G}(A)$ and $\mathcal{H}(A)$.

Definition 4.6. An étale differentiable stack $\mathcal{X}$ is Hausdorff if and only if the diagonal map

$$\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X},$$

is a closed immersion.

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Remark 4. In the case when $\mathcal{X}$ is a manifold, the diagonal map being a closed immersion is equivalent to its image being closed. Hence, this notion coincides with the usual Hausdorffness for manifolds.

Unlike the case of Lie groupoids, the source fibre of $\mathcal{G}(A)$ or $\mathcal{H}(A)$ is not, in general, Hausdorff. (see Example 1). The obstruction lies inside the foliation $\mathcal{F}$ defined in § 2.1.

Proposition 4.7. *The source fibre of $\mathcal{G}(A)$ is Hausdorff if and only if the leaves of the foliation $\mathcal{F}$ are closed. The same is true for $\mathcal{H}(A)$.*

Proof. We prove this for $\mathcal{G}(A)$. The proof for $\mathcal{H}(A)$ is similar. Let P be the étale atlas we have chosen. Then the source fibre $\bar{\mathbf{s}}^{-1}(x) = x \times_{M,\bar{\mathbf{s}}} \mathcal{G}(A)$ is a differentiable stack presented by $\mathbf{s}^{-1}(x)$ by Proposition 3.4. Consider the following diagram.

$$\begin{array}{ccc}
\bar{\mathbf{s}}^{-1}(x) \times_{\bar{\mathbf{s}}^{-1}(x)\times\bar{\mathbf{s}}^{-1}(x)} \mathbf{s}^{-1}(x)\times\mathbf{s}^{-1}(x) & \xrightarrow{\delta} & \mathbf{s}^{-1}(x)\times\mathbf{s}^{-1}(x) \\
\downarrow & & \downarrow \\
\bar{\mathbf{s}}^{-1}(x) & \xrightarrow{\Delta} & \bar{\mathbf{s}}^{-1}(x)\times\bar{\mathbf{s}}^{-1}(x)
\end{array}$$

As $\Gamma = P \times_{\mathcal{G}(A)} P$, it is not hard to check that $\bar{\mathbf{s}}^{-1}(x) \times_{\bar{\mathbf{s}}^{-1}(x)\times\bar{\mathbf{s}}^{-1}(x)} \mathbf{s}^{-1}(x)\times\mathbf{s}^{-1}(x)$ is isomorphic to $\Gamma|_{\mathbf{s}^{-1}(x)}$ and δ is just $\mathbf{s}_1 \times \mathbf{t}_1$, where $\mathbf{s}_1$ and $\mathbf{t}_1$ are the source and target maps of Γ. Obviously $\mathbf{s}_1 \times \mathbf{t}_1$ is an immersion as Γ is an étale groupoid. Moreover, the image of δ is closed by the following argument: take a convergent sequence $(a_0^i(t), a_1^i(t))$ of A_0-path with the limit $(a_0(t), a_1(t))$. Suppose that $(a_0^i(t), a_1^i(t))$ is inside the image of δ, i.e. $a_0^i(t) \sim a_1^i(t)$. Let $\bar{a}$ denote the inverse path of a, we have $\bar{a}_0^i(t) \odot a_1^i(t) \sim 1_x$, i.e. they stay in the same leaf of the foliation $\mathcal{F}$. Hence, the limit path $\bar{a}_0(t) \odot a_1(t) \sim 1_x$ (i.e. $(a_0(t), a_1(t))$ is also inside the image of δ) if and only if the leaves of $\mathcal{F}$ are closed. □

Example 1 (Non-Hausdorff source fibres). Let M be $S^2 \times S^2$ with 2-form $\Omega = (\omega, \mu\omega)$. Let the Lie algebroid A over M be $TM \times \mathbb{R}$ with Lie bracket

$$[(V, f), (W, g)] = ([V, W], L_V(g) - L_W(h) + \Omega(V, W)),$$

and anchor the projection onto TM (see [CW99, ch. 16] or [AM84]). Let $(a(\epsilon, t), u(\epsilon, t))$ be an A_0-homotopy, where the first component is in TM and the second component is in the trivial bundle $\mathbb{R}$. The condition of being an A_0-path here is equivalent to $a = (d/dt)\gamma$ and boundary conditions, where γ is the base path. Moreover, the first component of (1) is the usual A_0-homotopy equation for TM, which simply induces the homotopy of the base paths. The second component of (1) is

$$\partial_t v - \partial_\epsilon u = \Omega(a, b),$$

where b in (1) is (b, v) above. Hence, $b = (d/d\epsilon)\gamma$. Integrating the above equation and using the boundary condition of v, we have

$$\int_0^1 u(0, t)\, dt - \int_0^1 u(1, t)\, dt = \int_\gamma \Omega.$$

Let the period group Λ of Ω at a point $x \in M$ be

$$\Lambda_x = \int_\gamma \Omega, \quad [\gamma] \in \pi_2(M, x).$$

As M is simply connected, one can actually show that $(\gamma(0, t), u(0, t)) \sim (\gamma(1, t), u(1, t))$ if and only if $\gamma(0, t)$ and $\gamma(1, t)$ have the same end points and $\int_0^1 (u_0 - u_1)\, dt \in \Lambda$. Then, in the case when μ is irrational, Λ_x is dense in $\mathbb{R}$ for all x. Hence, there exist sequences $u_0^i \to u_0$ and $u_1^i \to u_1$ such that $\int_0^1 (u_0^i - u_1^i) \in \Lambda$ but the limit $\int_0^1 (u_0 - u_1) \notin \Lambda$. Hence the leaves of the foliation $\mathcal{F}$ are not closed.

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In particular the $\mathcal{G}(A)$ and $\mathcal{H}(A)$ are not Lie groupoids. In the case when μ is rational, Λ is discrete. In fact, the Lie algebroid $TM \times \mathbb{R}$ is isomorphic to the Lie algebroid associated to M viewed as a Jacobi manifold. As in [CZ04], $\mathcal{G}(A)$ and $\mathcal{H}(A)$ are both the Lie groupoid $(S^3 \otimes S^3) \otimes (S^3 \otimes S^3)$, where $\otimes$ denotes the tensor product of S^1 bundles. Therefore, by varying $\mu \in \mathbb{R}$ we obtain a series of Weinstein groupoids which are only Lie groupoids for a measure 0 set in $\mathbb{R}$.

4.2 The Integrability of Lie algebroids

The integrability of A and the representability of $\mathcal{G}(A)$ are not exactly the same, due to the presence of isotropy groups. However, as holonomy groupoids are always effective [MM03], we will show that the integrability of A is equivalent to the representability of $\mathcal{H}(A)$.

DEFINITION 4.8 (Orbit spaces). Let $\mathcal{X}$ be a differentiable stack presented by a Lie groupoid $X = (X_1 \rightrightarrows X_0)$. The orbit space of $\mathcal{X}$ is defined as the topological quotient X_0/X_1. Throughout the paper, when we mention the orbit space is a smooth manifold, we mean it has the natural smooth manifold structure induced from X_0 (i.e. the projection $X_0 \to X_0/X_1$ is smooth).

Proof. We have to show the topological quotient is independent of choice of presentations. Suppose that there is another presentation Y that is Morita equivalent to X through (E, J_X, J_Y). Let O_x be the orbit of X_1 in X_0 through point x. By the fact that both groupoid actions are free and transitive fiber-wise, $J_Y \circ J_X^{-1}(O_x)$ is another orbit O_y of Y. In this way, there is a one-to-one correspondence between orbits of X and Y. Hence, Y_0/Y_1 understood as the space of orbits is the same as X_0/X_1. □

THEOREM 4.9. *A Lie algebroid A is integrable in the classical sense, i.e. there is a Lie groupoid whose Lie algebroid is A, if and only if the orbit space of $\mathcal{G}(A)$ is a smooth manifold. Moreover, in this case the orbit space of $\mathcal{G}(A)$ is the unique source-simply connected Lie groupoid integrating A.*

Proof. First, let $\mathrm{Mon}(P_aA)$ be the monodromy groupoid of the foliation introduced by homotopy of A-paths in § 2.1. We will show that $\mathrm{Mon}(P_aA)$ is Morita equivalent to $\mathrm{Mon}(P_0A)$. Note that P_0A is a submanifold of P_aA, so there is another groupoid $\mathrm{Mon}(P_aA)|_{P_0A}$ over P_0A. We claim it is the same as $\mathrm{Mon}(P_0A)$. Namely, an A-homotopy $a(\epsilon, t)$ between two A_0 paths a_0 and a_1 can be homotopic to an A_0-homotopy $\tilde{a}(\epsilon, t)$ between a_0 and a_1. The idea is to divide $\tilde{a}$ into three parts.

(i) First deform a_0 to a_0^τ through $a_0(\epsilon, t)$ which is defined as $(1 - \epsilon + \epsilon\tau'(t))a_0((1-\epsilon)t + \epsilon\tau(t))$, where τ is the reparametrization induced in § 2.1.

(ii) Then deform a_0^τ to a_1^τ through $a(\epsilon, t)^\tau$.

(iii) Lastly, connect a_1^τ to a_1 through $a_1(\epsilon, t)$, which is defined as $a_1((1-\epsilon)\tau'(t)+\epsilon)a_1(\epsilon t+(1-\epsilon)\tau(t))$. Then connect those three pieces by a similar method in the construction of concatenation (although it might be only piecewise smooth at the joints). Obviously, $\tilde{a}$ is a homotopy through A_0-paths and it is homotopic to a rescaling (over ϵ) of $a(\epsilon, t)$ through the concatenation of $a_0((1-\lambda)\epsilon, t)$, $(\lambda + (1-\lambda)\tau'(t))a(\epsilon, \lambda + (1-\lambda)\tau'(t))$ and $a_1((1-\lambda)\epsilon + \lambda, t)$. Eventually, we can smooth out everything to make the homotopy and the homotopy of homotopy both smooth so that they are as desired.

Then, it is routine to check that $\mathrm{Mon}(P_aA)|_{P_0A}$ is Morita equivalent to $\mathrm{Mon}(P_aA)$ through $\mathbf{t}_a^{-1}(P_0A)$, where $\mathbf{t}_a$ is the target $\mathrm{Mon}(P_aA)$.

So the orbit space of $\mathcal{G}(A)$ can be realized as $P_aA/\mathrm{Mon}(P_aA)$. According to the main result in [CF03], $P_aA/\mathrm{Mon}(P_aA)$ is a smooth manifold if and only if A is integrable and, if so, $P_aA/\mathrm{Mon}(P_aA)$ is the unique source-simply connected Lie groupoid integrating A. □

Proof of Theorem 1.3. First of all, by the same argument given in the proof above, one can see that $\mathrm{Hol}(P_0A) = \mathrm{Hol}(P_aA)|_{P_0A}$. Hence, $\mathrm{Hol}(P_0A)$ is Morita equivalent to $\mathrm{Hol}(P_aA)$.

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Moreover, if the orbit space of a holonomy groupoid is a manifold then it is Morita equivalent to the holonomy groupoid itself (see [MM03]).

Hence, a differentiable stack $\mathcal{X} = BG$ presented by a holonomy groupoid G is representable if and only if the orbit space G_0/G_1 is a smooth manifold. One direction is obvious because $G_0/G_1 \rightrightarrows G_0/G_1$ is Morita equivalent to $G = (G_1 \rightrightarrows G_0)$ if the orbit space is a manifold. The converse direction is not hard to establish by examining the Morita equivalence diagram of G and $\mathcal{X} \rightrightarrows \mathcal{X}$. The Morita bibundle has to be G_0 as $\mathcal{X}$ is a manifold. Therefore, G_0 is a principal G bundle over $\mathcal{X}$. This implies that G_0/G_1 is the manifold $\mathcal{X}$.

Note that, in general, the orbit spaces of monodromy groupoids and holonomy groupoids of a foliation are the same. By Theorem 4.9 and argument above, we conclude that A is integrable if and only if $\mathcal{H}(A)$ is representable and in this case, $\mathcal{H}(A)$ is $P_aA/\operatorname{Hol}(P_aA)$, the unique source-simply connected Lie groupoid integrating A. $\square$

Combining the proofs of Theorems 4.9 and 1.3, Theorem 1.5 follows naturally.

So far we have constructed $\mathcal{G}(A)$ and $\mathcal{H}(A)$ for every Lie algebroid A and verified that they are Weinstein groupoids. Basically, we have done half of Theorem 1.2. For the other half of the proof, we first introduce some properties of Weinstein groupoids. Before doing so, we give an example.

Example 2 ($B\mathbb{Z}_2$). $B\mathbb{Z}_2$ is a Weinstein group (i.e. its base space is a point) integrating the trivial Lie algebra 0. The étale differentiable stack $B\mathbb{Z}_2$ is presented by $\mathbb{Z}_2 \rightrightarrows pt$ (here pt represents a point). We establish all of the structure maps on this presentation.

The source and target maps are just projections from $B\mathbb{Z}_2$ to a point. The multiplication m is defined by

$$m : (\mathbb{Z}_2 \rightrightarrows pt) \times (\mathbb{Z}_2 \rightrightarrows pt) \to (\mathbb{Z}_2 \rightrightarrows pt), \quad \text{by } m(a,b) = a \cdot b,$$

where $a, b \in \mathbb{Z}_2$. As $\mathbb{Z}_2$ is commutative, the multiplication is a groupoid homomorphism (hence gives rise to a stack homomorphism). It is easy to see that $m \circ (m \times id) = m \circ (id \times m)$, i.e. we can choose the 2-morphism α inside the associativity diagram to be id.

The identity section e is defined by

$$e : (pt \rightrightarrows pt) \to (\mathbb{Z}_2 \rightrightarrows pt), \quad \text{by } e(pt) = 1,$$

where 1 is the identity element in the trivial group pt and $\mathbb{Z}_2$. The inverse i is defined by

$$i : (\mathbb{Z}_2 \rightrightarrows pt) \to (\mathbb{Z}_2 \rightrightarrows pt), \quad \text{by } i(a) = a^{-1},$$

where $a \in \mathbb{Z}_2$. It is a groupoid homomorphism because $\mathbb{Z}_2$ is commutative.

It is routine to check whether these maps satisfy the axioms of Weinstein groupoids. The local Lie groupoid associated with $B\mathbb{Z}_2$ is just a point. Therefore, the Lie algebra of $B\mathbb{Z}_2$ is 0. Moreover, note that we have only used the commutativity of $\mathbb{Z}_2$. So for any discrete commutative group G, BG is a Weinstein group with Lie algebra 0.

Example 3 ($\mathbb{Z}_2 * B\mathbb{Z}_2$). This is an example where Proposition 4.5 does not hold. Consider the groupoid $\Gamma = (\mathbb{Z}_2 \times \mathbb{Z}_2 \rightrightarrows \mathbb{Z}_2)$. It is an action groupoid with trivial $\mathbb{Z}_2$-action on $\mathbb{Z}_2$. We claim that the presented étale differential stack $B\Gamma$ is a Weinstein group. We establish all of the structure maps on the presentation Γ.

The source and target maps are projections to a point. The multiplication $m : \Gamma \times \Gamma \to \Gamma$ is defined by

$$m((g_1, a_1), (g_2, a_2)) = (g_1 g_2, a_1 a_2).$$

It is a groupoid morphism because $\mathbb{Z}_2$ (the second copy) is commutative. We have $m \circ (m \times id) = m \circ (id \times m)$. However, we can construct a non-trivial 2-morphism $\alpha : \Gamma_0 (= \mathbb{Z}_2) \times \Gamma_0 \times \Gamma_0 \to \Gamma_1$

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defined by

$$\alpha(g_1, g_2, g_3) = (g_1 \cdot g_2 \cdot g_3, g_1 \cdot g_2 \cdot g_3).$$

As the $\mathbb{Z}_2$ action on $\mathbb{Z}_2$ is trivial, we have $m \circ (m \times id) = m \circ (id \times m) \cdot \alpha$.

The identity section e is defined by

$$e : (pt \rightrightarrows pt) \to \Gamma, \quad \text{by } e(pt) = (1, 1),$$

where 1 is the identity element in $\mathbb{Z}_2$. The inverse i is defined by

$$i : \Gamma \to \Gamma, \quad \text{by } i(g, a) = (g^{-1}, a^{-1}).$$

It is a groupoid morphism because $\mathbb{Z}_2$ (the second copy) is commutative.

It is not hard to check whether $B\Gamma$ with these structures maps is a Weinstein group. However, it does not satisfy the further obstruction of the associativity described in Proposition 4.5, we found failure. Let F_i be the six different ways of composing four elements as defined in Proposition 4.5, then the 2-morphisms α_i (basically coming from α) satisfy

$$F_{i+1} = F_i \cdot \alpha_i, \quad i = 1, \ldots, 6 (F_7 = F_1).$$

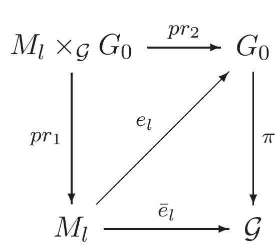
However, $\alpha_i(1, 1, 1, -1) = (-1, -1)$ for all i except that $\alpha_2 = id$. Therefore, $\alpha_6 \circ \alpha_5 \circ \cdots \circ \alpha_1(1, 1, 1, -1) = (-1, -1)$, which is not $id(1, 1, 1, -1) = (-1, 1)$.

5. Weinstein groupoids and local groupoids

In this section, we examine the relation between abstract Weinstein groupoids and local groupoids. Let us first show a useful lemma.

Lemma 5.1. *Given any étale atlas G_0 of $\mathcal{G}$, there exists an open covering $\{M_l\}$ of M such that the immersion $\bar{e} : M \to \mathcal{G}$ can be lifted to embeddings $e_l : M_l \to G_0$. On the overlap $M_l \cap M_j$, there exist an isomorphism φ_{lj}: $e_j(M_j \cap M_l) \to e_l(M_j \cap M_l)$, such that $\varphi_{lj} \circ e_j = e_l$ and the φ_{lj} satisfy cocycle conditions.*

Proof. Let (E_e, J_M, J_G) be the HS bibundle presenting the immersion $\bar{e} : M \to \mathcal{G}$. As a right G-principal bundle over M, E_e is locally trivial, i.e. we can pick an open covering $\{M_l\}$ so that J_M has a section $\tau_l : M_l \to E_e$ when restricted to M_l. As $\bar{e}_l := \bar{e}|_{M_l}$ is an immersion (the composition of immersions $M_l \to M$ and $\bar{e}$ is still an immersion), it is not hard to see that $pr_2 : M_l \times_{\mathcal{G}} G_0 \to G_0$ transformed by base change $G_0 \to \mathcal{G}$ is an immersion. Note that $e_l = J_G \tau_l : M_l \to G_0$ fits inside a similar diagram as (2).



Following a similar argument as in the proof of Lemma 4.2, we can find a map $\alpha : M_l \times_{\mathcal{G}} G \to G_1$ such that

$$e_l \circ pr_1 = pr_2 \cdot \alpha.$$

As π is étale, so is pr_1. Therefore e_l is an immersion.

As an immersion is locally an embedding, we can choose an open covering M_{ik} of $\{M_l\}$ so that $e_l|_{M_{ik}}$ is actually an embedding. To simplify the notation, we can choose a finer covering $\{M_l\}$ at the beginning and make e_l an embedding. Moreover, using the fact that G acts on E_e

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transitively (fiberwise), it is not hard to find a local bisection g_{lj} of $G_1 := G_0 \times_{\mathcal{G}} G_0$, such that $e_l \cdot g_{lj} = e_j$. Then $\varphi_{lj} = \cdot g_{lj}^{-1}$ satisfies that $\varphi_{lj} \circ e_j = e_l$. As e_l are embeddings, ϕ_{lj} naturally satisfy the cocycle condition. $\square$

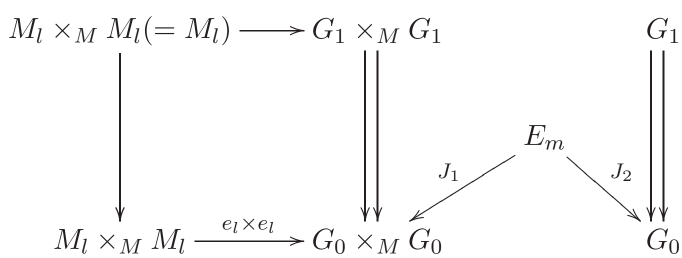
Before the proof of Theorem 1.4, we need a local statement.

Theorem 5.2. *For every Weinstein groupoid $\mathcal{G}$, there exists an open covering $\{M_l\}$ of M such that one can associate a local Lie groupoid U_l over each open set M_l.*

Proof. Let $\mathcal{G}$ be presented by $G = (G_1 \rightrightarrows G_0)$, and $\{M_l\}$ be an open covering as in Lemma 5.1. Let (E_m, J_1, J_2) be the HS bibundle from $G_1 \times_M G_1 \rightrightarrows G_0 \times_M G_0$ to G, which presents the stack morphism $\bar{m} : \mathcal{G} \times_M \mathcal{G} \to \mathcal{G}$. Note that M is the identity section, i.e.

$$\begin{array}{ccc} M_l \times_M M_l (= M_l) & \xrightarrow{\bar{m}=id} & M_l \\ \downarrow & \curvearrowright & \downarrow \\ \mathcal{G} \times_M \mathcal{G} & \xrightarrow{\bar{m}} & \mathcal{G} \end{array}$$

Translate this commutative diagram into groupoids. Then the composition of HS morphisms



(4)

is the same (up to a 2-morphism) as $e_l : M_l \to G_0$. Therefore, composing the HS maps in (4) gives a HS bibundle $J_1^{-1}((e_l \times e_l)(M_l \times_M M_l))$, which is isomorphic (as a HS bibundle) to $M_l \times_{G_0} G_1$, which represents the embedding e_l. Therefore, one can easily find a global section

$$\sigma_l : M_l \to M_l \times_{G_0} G_1 \cong J_1^{-1}((e_l \times e_l)(M_l \times_M M_l)) \subset E_m$$

defined by $x \mapsto (x, 1_{e_l(x)})$. Furthermore, we have $J_2 \circ \sigma_l(M_l) = e_l(M_l)$. As G is an étale groupoid, E_m is an étale principal bundle over $G_0 \times_M G_0$. Hence, J_1 is a local diffeomorphism. Therefore, one can choose two open neighborhoods $V_l \subset U_l$ of M_l in G_0 such that there exists a unique section σ_l' extending σ_l over $(M_l = M_l \times_M M_l \subset) V_l \times_{M_l} V_l$ in E_m and the image of $J_2 \circ \sigma_l'$ is U_l. The restriction of σ_l' on M_l is exactly σ_l. As $U_l \rightrightarrows U_l$ acts freely and transitively fiberwise on $\sigma_l'(V_l \times_{M_l} V_l)$ from the right, $\sigma_l'(V_l \times_{M_l} V_l)$ can serve as a HS bibundle from $V_l \times_{M_l} V_l$ to U_l. (Here, we view manifolds as groupoids.) In fact, it is the same as the morphism

$$m_l := J_2 \circ \sigma_l' : V_l \times_{M_l} V_l \to U_l.$$

By a similar method, we can define the inverse as follows. By (3), (4) and (5) in Definition 1.1, we have $\bar{i} \circ \bar{e}_l = \bar{e}_l$, so the following diagram commutes.

$$\begin{array}{ccc} M_l & \xrightarrow{\bar{m}=id} & M_l \\ \downarrow & \curvearrowright & \downarrow \\ \mathcal{G} & \xrightarrow{\bar{i}} & \mathcal{G} \end{array}$$

Suppose (E_i, J_1, J_2) is the HS bibundle representing $\bar{i}$. Translate the above diagram into groupoids,

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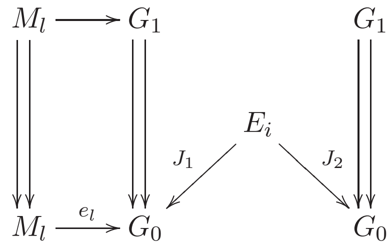
we have the composition of the following HS morphisms:



(5)

is the same (up to a 2-morphism) as $e_l : M_l \to G_0$. Therefore, composing the HS maps in (5) gives a HS bibundle $J_1^{-1}(e_l(M_l))$ that is isomorphic (as a HS bibundle) to $M_l \times_{G_0} G_1$, which represents the embedding e_l. Therefore, one can easily find a global section

$$\tau_l : M_l \to M_l \times_{G_0} G_1 \cong J_1^{-1}(e_l(M_l)) \subset E_i$$

defined by $x \mapsto (x, 1_{e_l(x)})$. Furthermore, we have $J_2 \circ \sigma_l(M_l) = e_l(M_l)$. As G is an étale groupoid, E_i is an étale principal bundle over G_0. Hence, J_1 is a local diffeomorphism. Therefore, one can choose an open neighborhood of M_l in G_0, which we might assume as U_l as well, such that there exists a unique section τ_l' extending τ_l over $(M_l \subset)U_l$ in E_i and the image of $J_2 \circ \tau_l'$ is in U_l. The restriction of τ_l' on M_l is exactly τ_l. So we can define

$$i_l := J_2 \circ \tau_l' : U_l \to U_l.$$

As M is a manifold, examining the groupoid picture of maps $\bar{\mathbf{s}}$ and $\bar{\mathbf{t}}$, one finds that they actually come from two maps $\mathbf{s}$ and $\mathbf{t}$ from G_0 to M. Hence, we define source and target maps of U_l as the restriction of $\mathbf{s}$ and $\mathbf{t}$ on U_l and denote them by $\mathbf{s}_l$ and $\mathbf{t}_l$, respectively.

The 2-associative diagram of $\bar{m}$ tells us that $m_l \circ (m_l \times id)$ and $m_l \circ (id \times m_l)$ differ in the following way: there exists a smooth map from an open subset of $V_l \times_{M_l} V_l \times_{M_l} V_l$, over which both of the above maps are defined, to G_1 such that

$$m_l \circ (m_l \times id) = m_l \circ (id \times m_l) \cdot \alpha.$$

As the 2-morphism in the associative diagram restricting to M is id, we have

$$\alpha(x, x, x) = 1_{e_l(x)}.$$

As G is étale and α is smooth, the image of α is inside the identity section of G_1. Therefore, m_l is associative.

It is not hard to verify other groupoid properties in a similar way by translating corresponding properties on $\mathcal{G}$ to U_l. Therefore, U_l with maps defined above is a local Lie groupoid over M_l. □

To prove the global result, we need the following proposition.

Proposition 5.3. *Given U_l and U_j constructed as above (one can shrink them if necessary), there exists an isomorphism of local Lie groupoids $\tilde{\varphi}_{lj} : U_j \to U_l$ extending the isomorphism φ_{lj} in Lemma 5.1. Moreover, $\tilde{\varphi}_{lj}$ also satisfy cocycle conditions.*

Proof. As we restrict the discussion on $M_l \cap M_j$, we may assume that $M_l = M_j$. According to Lemma 5.1, there is a local bisection g_{lj} of G_1 such that $e_l \cdot g_{lj} = e_j$. Extend the bisection g_{lj} to U_l (we denote the extension still by g_{lj}, and shrink V_k and U_k if necessary for $k = l, j$) so that

$$(V_l \times_{M_l} V_l) \cdot (g_{lj} \times g_{lj}) = V_j \times_{M_j} V_j \quad \text{and} \quad U_l \cdot g_{lj} = U_j.$$

Note that as G_1 is étale, the source map is an local isomorphism. Therefore, by choosing small enough neighborhoods of M_l, the extension of g_{lj} is unique. Let $\tilde{\varphi}_{lj} = \cdot g_{lj}^{-1}$. Then it is naturally an extension of φ_{lj}. Moreover, by uniqueness of the extension, $\tilde{\varphi}_{lj}$ satisfy cocycle conditions as φ_{lj} do.

Now we show that $\tilde{\varphi}_{lj} = \cdot g_{lj}$ is a morphism of local groupoids. It is not hard to see that $\cdot g_{lj}$ preserves source, target and identity embeddings. So we only have to show that

$$i_l \cdot g_{lj} = i_j, \quad m_l \cdot g_{lj} = m_j.$$

For this purpose, we have to recall the construction of these two maps. i_l is defined as $J_2 \circ \tau_l'$. As there is a global section of J_1 over U_l in E_i, we have $J_1^{-1}(U_l) \cong U_l \times_{i_l, G_0} G_1$ as G torsors. Under this isomorphism, we can write τ_l' as

$$\tau_l'(x) = (x, 1_{e_l(x)}).$$

The G action on $U_l \times_{i_l, G_0} G_1$ gives $(x, 1_{e_l(x)}) \cdot g_{lj} = (x, g_{lj})$. Moreover, we have

$$J_2((x, g_{lj})) = J_2(x, 1_{e_j(x)}) = \mathbf{s}_G(g_{lj}),$$

where $\mathbf{s}_G$ is the source map of G. Combining all these, we have shown that $i_l \cdot g_{lj} = i_j$. The other identity for multiplications follows in a similar way. □

Proof of Theorem 1.4. Now it is easy to construct G_{loc} as in the statement of the theorem. Note that the set of $\{U_l\}$ with isomorphisms φ_{lj} which satisfy cocycle conditions serve as a chart system. Therefore, after gluing them together, we arrive at a global object G_{loc}. As φ_{lj} are isomorphisms of local Lie groupoids, the local groupoid structures also glue together. Therefore, G_{loc} is a local Lie groupoid.

If we choose two different open covering $\{M_l\}$ and $\{M_l'\}$ of M for the same étale atlas G_0 of $\mathcal{G}$, we arrive at two systems of local groupoids $\{U_l\}$ and $\{U_l'\}$. As $\{M_l\}$ and $\{M_l'\}$ are compatible chart systems for M, combining them and using Proposition 5.3, $\{U_l\}$ and $\{U_l'\}$ are also compatible chart systems. Therefore, they glue into the same global object up to isomorphisms near the identity section.

If we choose two different étale atlases G_0' and G_0'' of $\mathcal{G}$, we can take their refinement $G_0 = G_0' \times_{\mathcal{G}} G_0''$ and we can take a fine enough open covering $\{M_l\}$ so that it embeds into all three atlases. As $G_0 \to G_0'$ is an étale covering, we can choose U_l in G_0' small enough so that they still embed into G_0. So the groupoid constructed from the presentation G_0 with the covering U_l is the same as the groupoid constructed from the presentation G_0' with the covering U_l. The same is true for G_0'' and G_0. Therefore, our local groupoid G_{loc} is canonical.

We finish the proof of the Lie algebroid part in the next section. □

6. Weinstein groupoids and Lie algebroids

In this section, we define the Lie algebroid of a Weinstein groupoid $\mathcal{G}$. An obvious choice is to define the Lie algebroid of $\mathcal{G}$ as the Lie algebroid of the local Lie groupoid G_{loc}.We give an equivalent definition in a more direct way.

Definition 6.1. Given a Weinstein groupoid $\mathcal{G}$ over M, there is a canonically associated Lie algebroid A over M.

Proof. We just have to examine the second part of proof of Theorem 1.4 more carefully. Choose an étale groupoid presentation G of $\mathcal{G}$ and an open covering M_l as in Lemma 5.1. According to Theorem 1.4, we have a local groupoid U_l and its Lie algebroid A_l over each M_l. Differentiating the $\tilde{\varphi}_{lj}$ in Proposition 5.3, we can achieve algebroid isomorphisms $T\tilde{\varphi}_{lj}$, which also satisfy cocycle conditions. Therefore, using these data, we can glue the A_l into a vector bundle A. Moreover, as the $T\tilde{\varphi}_{lj}$ are Lie algebroid isomorphisms, we can also glue the Lie algebroid structures. Therefore, A is a Lie algebroid.

Following the same arguments as in the proof of Theorem 1.4, we can show uniqueness. If we choose a different presentation G' and a different open covering M_l, we can choose the refinement

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of these two systems and will arrive at a Lie algebroid which is glued from a refinement of both systems. Therefore, it is isomorphic to both Lie algebroids constructed from these two systems. Hence, the construction is canonical. □

Then it is easy to see that the following proposition holds.

Proposition 6.2. *Given a Weinstein groupoid $\mathcal{G}$, it has the same Lie algebroid as its associated local Lie groupoid G_{loc}.*

Together with the Weinstein groupoid $\mathcal{G}(A)$ that we have constructed in § 4, we are now ready to complete the proof of Theorem 1.2.

Proof of the second half of Theorem 1.2. We take the étale presentation P of $\mathcal{G}(A)$ and $\mathcal{H}(A)$ as we constructed in § 2.1. Let us recall how we construct local groupoids from $\mathcal{G}(A)$ and $\mathcal{H}(A)$.

In our case, the HS morphism corresponding to $\bar{m}$ is

$$(E := \mathbf{t}_M^{-1}(m(P \times_M P)) \cap \mathbf{s}_M^{-1}(P), m^{-1} \circ \mathbf{t}_M, \mathbf{s}_M).$$

The section $\sigma : M \to E$ is given by $x \mapsto 1_{0_x}$. Therefore, if we choose two small enough open neighborhoods $V \subset U$ of M in P, the bibundle representing the multiplication m_V is a section σ' over $V \times_M V$ of the map $m^{-1} \circ \mathbf{t}_M$ in E.

As the foliation $\mathcal{F}$ intersects each transversal slice only once, we can choose an open neighborhood O of M inside $P_0 A$ so that the leaves of the restricted foliation $\mathcal{F}|_O$ intersect U only once. We denote the homotopy induced by $\mathcal{F}|_O$ as $\sim_O$ and the holonomy induced by $\mathcal{F}_O$ by $\sim_O^{\mathrm{hol}}$. Then there is a unique element $a \in U$ such that $a \sim_O a_1 \odot a_2$. There exists a unique arrow $g : a_1 \odot a_2 \curvearrowleft a$ in $\mathrm{Mon}(P_0 A)$ near the identity arrows at 1_{0_x} as the leaf of $\mathcal{F}|_O$ is locally contractible.

Then we can choose the section σ' near σ to be

$$\sigma' : (a_1, a_2) \mapsto g.$$

So the multiplication m_V on U is

$$m_V(a_1, a_2) = a(\sim_O a_1 \odot a_2).$$

As the leaves of $\mathcal{F}$ intersect U only once, a has to be the unique element in U such that $a \sim_O^{\mathrm{hol}} a_1 \odot a_2$. It is not hard to verify that both Weinstein groupoids give the same local Lie groupoid structure on U.

Moreover, $U = O/\sim_O$ is exactly the local groupoid constructed in [CF03, § 5], which has Lie algebroid A. Therefore, $\mathcal{G}(A)$ and $\mathcal{H}(A)$ have the same local Lie groupoid and their Lie algebroids are both A. □

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