FOURTH ORDER BOUNDARY VALUE PROBLEMS AND COMPARISON THEOREMS

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Introduction. This paper is primarily concerned with the existence of solutions of the fourth-order self-adjoint differential equation

$$[(r(x)y'')' + q(x)y']' - p(x)y = 0, \tag{1}$$

(where $r(x) > 0$, $q(x) \geqslant 0$, $p(x) \geqslant 0$ and all three coefficients are continuous on $[a, \infty)$) and one of the two-point boundary conditions:

$$y(a) = y'(a) = y(b) = y'(b) = 0, \tag{2}$$

or

$$y(a) = y'(a) = y_1(b) = y_2(b) = 0; \tag{3}$$

the subscript notation for any solution $y(x)$ denoting:

$$y_1(x) = r(x)y''(x) \quad \text{and} \quad y_2(x) = y_1'(x) + q(x)y'(x). \tag{4}$$

If $q(x)$ is differentiable then (1) is equivalent to the more familiar form: $(r(x)y'')'' + (q(x)y')' - p(x)y = 0$ and, in either case, solutions exist with the appropriate derivatives. In particular, attention is given to establishing criteria for the existence of a smallest number $b \in (a, \infty)$ for which (2), or (3), is satisfied by a non-trivial solution of (1).

DEFINITION 1. *The number $\eta_1(a)$ is the smallest number $b \in (a, \infty)$ such that* (2) *is satisfied non-trivially by a solution of* (1).

DEFINITION 2. *The number $\mu_1(a)$ is defined in the same way by* (3).

For $q \equiv 0$, $\eta_1(a)$ is the first conjugate point of $x = a$* as established by Leighton and Nehari **(6)** and for general q, $\eta_1(a)$ is recognized here to be the first conjugate point of a second-order vector-matrix equation in the usual sense. Also $\mu_1(a)$ reduces to the analogy to focal points for second-order scalar equations introduced by the author **(3)**. Other recent investigations for the case of $q \equiv 0$ and $p > 0$ have been made by Howard **(5)**, making extensive use of Rayleigh quotients, and by the author **(4)**, utilizing the simple vector-matrix formulation

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*The minimum $b > a$ such that a non-trivial solution of (1) has four zeros on $[a, b]$.

$$\begin{pmatrix} y \\ y_1 \end{pmatrix}'' = \begin{pmatrix} 0 & 1/r \\ p & 0 \end{pmatrix} \begin{pmatrix} y \\ y_1 \end{pmatrix}$$

and comparison with the special scalar equation

$$(y''/p(x))'' - p(x)y = 0, \; p(x) > 0.$$

It is also known (**6**, Theorems 12.1 and 12.2) that under certain conditions the middle term of (1) can be removed and the known theory of $q \equiv 0$ can be applied to the resulting equation. However, there are many cases where such a transformation cannot be made into an equation where $p(x)$ does not change sign, in order to apply the results of (**6**), or where $p(x)$ is positive, as required by (**3**; **4**; **5**). Also, in most of the above-mentioned studies, there is a strong dependence on the monotonicity of certain solutions and their derivatives. It is easy to see that for $q \geqslant 0$ such properties are not always available.

A systems formulation of (1), which is not as obvious but more useful than the above-mentioned one of the author (**4**), was introduced by Sternberg and Sternberg (**10**), prior to all of the papers previously mentioned here. They investigated fourth-order matrix equations of the form of (1), including the middle term and found what amounts to sufficient conditions on the coefficients to insure the existence of $\eta_1(a)$ (see concluding comment of § 2).

It is this paper which provides the systems formulation of (1) to be used throughout the present discussion. Following earlier fundamental results of Sternberg (**10**), they utilized one of his canonical systems, involving a variational multiplier and functional side conditions. Here, use will be made of his other (equivalent) canonical system and the need for monotonicity of solutions of (1) is avoided. An elementary self-contained derivation of this latter system will now be given.

Using any solution $y(x)$ of (1) and the notation (4), the vectors

$$\alpha(x) = \begin{pmatrix} y \\ y' \end{pmatrix} \quad \text{and} \quad \tilde{\alpha}(x) = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

satisfy the system*

$$(5) \quad \begin{cases} \alpha' = A\alpha + B\tilde{\alpha} \\ \tilde{\alpha}' = C\alpha + A\tilde{\alpha} \end{cases} \text{where } A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \frac{1}{r}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \text{ and } C = \begin{pmatrix} 0 & -q \\ p & 0 \end{pmatrix}.$$

This system can be simplified by the substitution

$$\alpha = D(x)\beta, \; \tilde{\alpha} = \tilde{D}(x)\tilde{\beta}$$

where $D(x)$ and $\tilde{D}(x)$ are both solutions of $D' = AD$ and the respective initial conditions:

$$D(a) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I, \qquad \tilde{D}(a) = C_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

*For a similar formulation, see Coddington and Levinson, *Theory of Ordinary Differential Equations*, p. 207, problem 19, replacing $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ by $\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}$.

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Then

$$D(x) = \begin{pmatrix} 1 & x-a \\ 0 & 1 \end{pmatrix}, \qquad \tilde{D}(x) = D(x)C_0$$

and (5) reduces to the second canonical system of Sternberg **(8**, (2.4)**)**:

$$(6) \qquad \begin{cases} \beta' = E(x)\tilde{\beta} \\ \tilde{\beta}' = -F(x)\beta \end{cases} \text{where } \begin{array}{l} E(x) = D^{-1}B\tilde{D} = \dfrac{1}{r(x)}E_1(x-a) \\ F(x) = -\tilde{D}^{-1}CD = p(x)F_1(x-a) + q(x)Q_1. \end{array}$$

and

$$E_1(x) = \begin{pmatrix} x^2 & -x \\ -x & 1 \end{pmatrix}, \quad F_1(x) = \begin{pmatrix} 1 & x \\ x & x^2 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The initial value $\tilde{D}(a) = C_0$ was chosen so that the resulting matrix coefficients E and F of (6) would be symmetric. Fortunately, they are also positive semi-definite, that is, $E(x) \geqslant 0$, $F(x) \geqslant 0$ and if $p(x)\,q(x) > 0$ then $F(x) > 0$, that is, positive definite. Since the determinant of $E(x) = \det E(x) \equiv 0$ then (6) cannot be expressed as a simple second-order vector-matrix equation in β (that is, $(R(x)\beta')' + P(x)\beta = 0$, $R > 0$, $P \geqslant 0$) but it is found that much of the known theory of such equations is also applicable to first-order systems of the type (6), particularly the use of matrix Riccati equations **(1**; **7)**.

The techniques and results of this paper are direct analogies of those applicable to the scalar system: $y' = e(x)z$, $z' = -f(x)y$, where $e(x) \geqslant 0$ and $f(x) \geqslant 0$, and the second-order scalar equation $(r(x)y')' + p(x)y = 0$, with non-negative coefficients **(2)**.

First of all, conditions for the existence of $\mu_1(a)$ will be established; secondly, $\mu_1(a)$ will be assumed to exist and further requirements on the coefficients of (1) will be added to insure the existence of $\eta_1(a)$; thirdly, these will be combined to give direct criteria for $\eta_1(a)$ and, finally, comparison theorems between two equations of type (1) will be established.

Throughout the paper, when proofs involve $E(x)$ and $F(x)$, $x = a$ will be taken to be $x = 0$ for simplicity, but no loss of generality. Also, subscript notation (4) and the inequality notation denoting definiteness of matrices will be adopted as standard notation.

1. Relations between system (4) and equation (1). Consider two fundamental solutions $u(x)$ and $v(x)$ of (1) satisfying, respectively, the initial conditions:

$$(7) \qquad u(a) = u'(a) = 0,\ u_1(a) = 1,\ u_2(a) = 0$$

and

$$(8) \qquad v(a) = v'(a) = v_1(a) = 0,\ v_2(a) = 1.$$

Any solution of (1) which has a double zero at $x = a$ must be a linear combination of $u(x)$ and $v(x)$ and the numbers $\mu_1(a)$ and $\eta_1(a)$ are the smallest zeros on (a, ∞) of, respectively, the sub-wronskians:

(9) $$\rho(x) = \rho[u; v] = u_1 v_2 - v_1 u_2$$

and

(10) $$\sigma(x) = \sigma[u; v] = uv' - vu'.$$

Recalling the transformation of the opening section of equation (1) into the system (6) let

(11) $$\mathfrak{Y}(x) = \begin{pmatrix} u & v \\ u' & v' \end{pmatrix}, \qquad \tilde{\mathfrak{Y}} = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$$

(12) $$Y(x) = D^{-1}(x)\mathfrak{Y}(x), \ \tilde{Y}(x) = \tilde{D}^{-1}(x)\tilde{\mathfrak{Y}}(x).$$

Then Y and $\tilde{Y}$ satisfy the matrix system

(13) $$\begin{aligned} Y' &= E(x)\tilde{Y}, \ Y(a) = 0 \\ \tilde{Y}' &= -F(x)Y, \ \tilde{Y}(a) = C_0. \end{aligned}$$

And, furthermore,

(14) $$\begin{aligned} \det Y(x) &= \det \mathfrak{Y}(x) = \sigma(x), \qquad \sigma(a) = 0, \\ \det \tilde{Y}(x) &= \det \tilde{\mathfrak{Y}}(x) = \rho(x), \qquad \rho(a) = 1. \end{aligned}$$

Therefore, the existence of $\eta_1(a)$ and $\mu_1(a)$ become problems of finding (matrix) singularities of components of solution pairs of (13).

2. Conditions for existence of $\mu_1(a)$. For $q \equiv 0$ conditions insuring the existence of $\mu_1(a)$ have already been given, the strongest of which is:

THEOREM 2.1 **(3)**. *If $q \equiv 0$ and*

$$\int^{\infty} p(x)(\overset{x}{I^2}p)^2 dx = \infty$$

then $\mu_1(b)$ exists for every $b \in [a, \infty)$, where $\overset{x}{I^2}p$ is the iterated antiderivative

$$\int^x \int^t p(s)ds\, dt.$$

For $p \equiv 0$ there is a striking relation between focal conditions for second-order equations and the existence of $\mu_1(a)$. Let $\bar{\mu}_1(a)$ be the smallest $b \in (a, \infty)$ for which a non-trivial solution of

(15) $$(r(x)y')' + q(x)y = 0 \quad (r \text{ and } q \text{ from } (1))$$

satisfies the focal conditions

(16) $$y(a) = y'(b) = 0.$$

Then it is a simple matter to check the following:

THEOREM 2.2. *If $p(x) \equiv 0$ then $\mu_1(a)$ exists if and only if $\bar{\mu}_1(a)$ exists and, furthermore, $\mu_1(a) = \bar{\mu}_1(a)$.*

Comparison theorems of § 4 will extend these theorems to the cases where $q(x) \geqslant 0$ and $p(x) \geqslant 0$, respectively, but first, consider the complete equation (1) and the matrix system (13).

Since $\rho(a) = 1$ there is a largest number $b, a < b \leqslant \infty$, such that $\tilde{Y}(x)$ is non-singular on $[a, b)$. On this interval let $K(x) = Y(x)\tilde{Y}^{-1}(x)$ then $K(x)$ satisfies the matrix Riccati equation and initial condition:

$$K' = E(x) + KF(x)K, \qquad K(a) = 0, \tag{17}$$

and $K(x)$ is symmetric on $[a, b)$. (For these and other properties of $K(x)$ see **(1; 7)**.)

Let $\xi = (\xi_i)$ be a non-zero constant (column) vector; then by

$$\xi^*K(x)\xi \geqslant \int_a^x \xi^*E(t)\xi dt = \int_a^x (t\xi_1 - \xi_2)^2 \frac{1}{r(t)} dt \qquad (a = 0), \tag{17}$$

$K(x)$ is positive-definite on (a, b), and $\det K(x) = \sigma(x)/\rho(x) > 0$, giving $\sigma(x) > 0$. Hence, $Y(x)$ is non-singular, $\eta_1(a)$ does not exist on (a, b) and since $\mu_1(a) \neq \eta_1(a)$ (see Lemma 2.1), then:

THEOREM 2.3. If $\eta_1(a)$ exists then $\mu_1(a)$ exists and $a < \mu_1(a) < \eta_1(a)$.*

The definiteness of $K(x)$ established above yield further information which will be used in succeeding sections. Recall from (12) that

$$K = Y\tilde{Y}^{-1} = D^{-1}\mathfrak{Y}\tilde{\mathfrak{Y}}^{-1}DC_0, \qquad C_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

A simple computation verifies that $DC_0 = C_0D^{*-1}$ and, hence,

$$DKD^* = \mathfrak{Y}\tilde{\mathfrak{Y}}^{-1}C_0 \tag{18}$$

where (again using $a = 0$)

$$DKD^* = \begin{pmatrix} k_{11} + 2xk_{12} + x^2k_{22} & k_{12} + xk_{22} \\ k_{12} + xk_{22} & k_{22} \end{pmatrix}, \quad \mathfrak{Y}\tilde{\mathfrak{Y}}^{-1}C_0 = \frac{1}{\rho}\begin{pmatrix} r\sigma' & \tau \\ \tau & \tau' \end{pmatrix}$$

and, as in **(3)**, $\tau(x) = u'v_1 - v'u_1 (\equiv uv_2 - vu_2)$. Since $K(x) > 0$ on (a, b) then $\sigma' > 0$ and $\tau' = \rho(x)k_{22}(x) > 0$. This together with the initial-value $\tau(a) = 0$ yields

LEMMA 2.1. *If $\rho(x) > 0$ on $[a, b), a < b \leqslant \infty$, then $\sigma(x) > 0$, $\sigma'(x) > 0$, $\tau'(x) > 0$ and $\tau(x) > 0$ on $(a, b]$ and, furthermore,*

$$\tau(x)/\rho(x) = K_{12}(x) + (x - a)K_{22}(x) > 0 \text{ on } (a, b).$$

Finally, the symmetry and definiteness of $K(x)$ yields that $\operatorname{tr} K(x) > 0$, $(\operatorname{tr} K(x))^2 > 4 \det K(x) = \sigma(x)/\rho(x)$ on (a, μ_1) and, hence, that:

LEMMA 2.2. *If $\mu_1(a)$ exists then* $\operatorname{tr} K(x) \to \infty$ *as $x \to \mu_1(a)$ on $(a, \mu_1(a)$.*

*Also true regardless of the signs or changes of sign of $q(x)$ and $p(x)$, as can be seen by a more careful examination of Lemma 2.1 and its consequences.

Now, suppose that $\mu_1(a)$ does not exist, that is, $b = \infty$ in the above discussion. Then $K(x)$ is defined and non-singular on (a, ∞) and by (17)

$$K^{-1}K'K^{-1} \geqslant F \text{ (the difference being positive semi-definite).}$$

If $x_0 \in (a, \infty)$ and $\xi = (\xi_i)$ is a constant non-zero vector then on $[x_0, \infty)$

$$\xi^*K^{-1}(x_0)\xi \geqslant \xi^*K^{-1}(x_0)\xi - \xi^*K^{-1}(x)\xi \geqslant \int_{x_0}^{x} (\xi_1 + t\xi_2)^2 p(t)dt + \int_{x_0}^{x} \xi_2^2 q(t)dt$$

and the following is an immediate consequence:

THEOREM 2.4. *If either*

$$\int^{\infty} x^2 p(x)dx = \infty \quad \text{or} \quad \int^{\infty} q(x)dx = \infty$$

then $\mu_1(b)$ exists for every $b \in [a, \infty)$.

Note that for $q(x) \equiv 0$ this is a corollary of Theorem 2.1. Also, for $q \equiv 0$ it is known **(3; 5)** that the existence of $\mu_1(a)$ combined with $\int^{\infty} 1/r = \infty$ give the existence of $\eta_1(a)$. This will be established in the next section for for $q \geqslant 0$. Sternberg and Sternberg **(10)** proved this result for the special case of $r(x)$ bounded, but for matrix equations.

3. Conditions for the existence of $\eta_1(a)$.

Assume that $\mu_1(a)$ exists but that $\eta_1(a)$ does not, that is, det $Y(x) = \sigma(x) > 0$ on (a, ∞). Let $H = -\tilde{Y}Y^{-1}$ then $H(x)$ is defined on (a, ∞), is symmetric, and

$$H' = F(x) + HE(x)H. \tag{19}$$

Also on (a, μ_1), $H(x) = -K^{-1}(x)$ and is negative definite. Since $H(\mu_1)$ is singular then by (19) the maximum eigenvalue of $H(x)$ (written max. e.v. $H(x)$) is a non-decreasing function on (a, ∞) and max. e.v. $H(\mu_1) = 0$. Suppose that max. e.v. $H(x) \equiv 0$ on (μ_1, ∞), then det $H(x) \equiv 0$ and $\rho(x) \equiv 0$ on that interval. This is possible if $p \equiv q \equiv 0$ on (μ_1, ∞) and it will now be shown that this is also necessary. Since $H'(x) \geqslant F(x) \equiv 0$ and there is a non-zero vector $\xi = (\xi_i)$ such that $H(\mu_1)\xi = 0$ then on (μ_1, ∞)

$$0 \geqslant \xi^*H(x)\xi \geqslant \int_{\mu_1}^{x} (\xi_1 + t\xi_2)^2 p(t)dt + \int_{\mu_1}^{x} (\xi_2^2 q(t)dt \geqslant 0$$

implying $p(x) \equiv 0$ and either $q(x) \equiv 0$ on (μ_1, ∞) or $\xi_2 = 0$. If $\xi_2 = 0$ then

$$\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xi_1 \neq 0$$

and $H(\mu_1)\xi = 0$ gives $h_{11}(\mu_1) = 0$ and $h_{12}(\mu_1) = 0$. By (19) $h_{11}(x)$ is non-decreasing, $h_{11}(x) \leqslant$ max. e.v. $H(x) = 0$, and

$$h'_{11} = \frac{1}{r(x)} (xh_{11} - h_{12})^2 \text{ on } [a, \infty) \qquad (a = 0).$$

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Using the fact that $H = -K^{-1}$ on (a, μ_1) then by Lemma 2.1

$$xh_{11} - h_{12} = \frac{xk_{22} + k_{12}}{\det K} = \frac{\tau/\rho}{\sigma/\rho} \to \frac{\tau(\mu_1)}{\sigma(\mu_1)} > 0 \text{ as } x \to \mu_1.$$

Therefore $h_{11}'(\mu_1) > 0$, which contradicts that $h_{11} \equiv 0$ on (μ_1, ∞). Hence, $q(x) \equiv p(x) \equiv 0$ on $[\mu_1, \infty)$, if max. e.v. $H(x) \equiv 0$ on $[\mu_1, \infty)$. (Note that if $p(x) \equiv q(x) \equiv 0$ on $[\mu_1, \infty)$ then $\rho(x) \equiv 0, \sigma(x) \equiv \sigma(\mu_1) > 0$ and $\det H(x) \equiv 0$ on $[\mu_1, \infty)$.) Suppose that $p(x) + q(x) \not\equiv 0$ on (X, ∞) for every $X \in (a, \infty)$. Then there is a number $x_1 \in (\mu_1, \infty)$ such that max. e.v. $H(x) > 0$ on $[x_1, \infty)$.

Suppose that min. e.v. $H(x) < 0$ on $[x_1, \infty)$, as it is on (a, μ_1). Then $\det H(x) < 0$ and $H(x)$ is non-singular on $[x_1, \infty)$. From (19), $H^{-1}H'H^{-1} \geqslant E(x)$ and if $\xi = (\xi_i)$ is any constant vector of length $|\xi| = 1$, then on $[x_1, \infty)$

$$\xi^*H^{-1}(x_1)\xi - \xi^*H^{-1}(x)\xi \geqslant \xi^*\left(\int_{x_1}^{x} E\right)\xi \tag{20}$$

and

$$\xi^*H^{-1}(x)\xi \leqslant \text{max. e.v. } H^{-1}(x_1) - \text{min. e.v. } \int_{x_1}^{x} E. \tag{21}$$

LEMMA 3.1. *If* $\int_1^\infty 1/r = \infty$ *then* min. e.v. $\int_{x_1}^x E \to \infty$ *as* $x \to \infty$.

Proof. Let $a = 0$ and compute

$$\det \int_{x_1}^{x} E = \int_{x_1}^{x} t^2/r \int_{x_1}^{x} 1/r - \left(\int_{x_1}^{x} t/r\right)^2 \geqslant 0 \quad \text{on} \quad [x_1, \infty).$$

Then $(\det \int_{x_1}^x E)' = L(x)/r(x)$, where $L(x) = \int_{x_1}^x t^2/r + x^2 \int_{x_1}^x 1/r - 2x \int_{x_1}^x t/r > 0$ on (x_1, ∞). Furthermore, $L'(x) = 2(x \int_{x_1}^x 1/r - \int_{x_1}^x t/r) > 0$ and $L''(x) = 2\int_{x_1}^x 1/r > 0$ and $\to \infty$ as $x \to \infty$. Successive applications of L'Hopital's Rule yield

$$\text{min. e.v.} \int_{x_1}^{x} E \geqslant \left(\det \int_{x_1}^{x} E \Big/ \operatorname{tr} \int_{x_1}^{x} E\right) = \int_{x_1}^{x} \frac{L(t)}{r(t)} dt \Big/ \int_{x_1}^{x} \frac{t^2+1}{r(t)} dt \to \infty$$

$$\text{as } x \to \infty$$

and the lemma is proved.

Add to the main argument that $\int^\infty 1/r = \infty$ and by Lemma 3.1 and (21) there is a number $x_2 \in (x_1, \infty)$ such that on $[x_2, \infty)$ both $H^{-1}(x)$ and $H(x)$ are negative definite. But this contradicts that max. e.v. $H(x) > 0$ and it follows that there exists a smallest number $\mu_2 \in [\mu_1, \infty)$ such that min. e.v. $H(\mu_2) = 0$. Suppose that min. e.v. $H(x) \equiv 0$ on $[\mu_2, \infty)$. Then $\det H \equiv 0$ and $H \geqslant 0$ on this interval. Let $\bar{x} \in (\mu_2, \infty)$ such that $p(x) + q(x) \not\equiv 0$ on $[\mu_2, \bar{x}]$. Then there is a (constant) vector $\bar{\xi}$ such that $H(\bar{x})\bar{\xi} = 0$. Therefore, by use of (19):

$$0 \geqslant \bar{\xi}^*H(\mu_2)\xi \geqslant \bar{\xi}^* \int_{\mu_2}^{\bar{x}} F\bar{\xi} \geqslant 0,$$

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which implies that

$$\int_{\mu_2}^{\bar{x}} (\bar{\xi}_1 + t\bar{\xi}_2)^2 p(t)dt + \bar{\xi}_2^2 \int_{\mu_2}^{\bar{x}} q(t)dt = 0, \quad p(\mathrm{t}) \equiv 0 \text{ on } [\mu_2, \bar{\mathrm{x}}] \text{and } \bar{\xi}_2 = 0.$$

Therefore $h_{11}(\bar{x}) = 0$ and this, together with $h_{11}' \geqslant 0$ and $h_{11}(x) \leqslant 0$ on $[\mu_2, \bar{x}]$, gives $h_{11} \equiv 0$ and $h_{12}(x) \equiv 0$, since $\det H(x) \equiv 0$, on this interval. Equation (19) then becomes

$$h_{22}' = q(x) + \frac{1}{r(x)} h_{22}^2 \text{ on } [\mu_2, \bar{x}].$$

Since $q(x) \not\equiv 0$ on $[\mu_2, \bar{x}]$ then there is a number $\bar{x}_1 \in (\mu_2, \bar{x})$ such that $h_{22}(x) > 0$ on $[\bar{x}_1, \bar{x}]$. Then $h_{22}{}^{-2}h_{22}{}' \geqslant 1/r(x)$ and

$$\int_{\bar{x}_1}^{\bar{x}} 1/r < \frac{1}{h_{22}(\bar{x}_1)}.$$

This is true for any $\bar{x} > \bar{x}_1$, which contradicts $\int^\infty 1/r = \infty$ and there exists a number $x_3 \in (\mu_2, \infty)$ such that min. e.v. $H(x) > 0$, that is, $H(x) > 0$ and $H^{-1}(x) > 0$, on $[x_3, \infty)$. This being the case, inequality (20) holds for x_1 replaced by x_3 and

$$\xi^* H^{-1}(x_3)\xi > \int_{x_3}^{x} \xi^* E(t)\xi dt = \int_{x_3}^{x} (t\xi_1 - \xi_2)^2 \frac{1}{r(t)} dt \qquad (a = 0)$$

which contradicts the assumption that $\int^\infty 1/r = \infty$. Since all consequences of the assumption that $\eta_1(a)$ does not exist are eliminated then:

THEOREM 3.1. *If $\mu_1(a)$ exists, $\int^\infty 1/r = \infty$, and $p(x)$ and $q(x)$ are not both identically zero for large x then $\eta_1(a)$ exists.* (See Theorem 4.2.)

THEOREM 3.2. *If $\eta_1(a)$ exists then* max. e.v. $H(x) \to \infty$, *as $x \to \eta_1(a)$ on* $(a, \eta_1(a))$.

Proof. Suppose that max. e.v. $H(x) \to \bar{h} < \infty$ as $x \to \eta_1(a)$. Then for any constant vector ξ, the non-decreasing scalar $\xi^* H(x)\xi$ has a limit as $x \to \eta_1(a)$. By taking, in turn,

$$\xi = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \text{ and } \begin{pmatrix}1\\1\end{pmatrix},$$

it follows readily that each component of $H(x)$ has a limit as $x \to \eta_1(a)$. Note that on (a, η_1): $Y' = E\tilde{Y} = (-EH)Y$, $\sigma(x) = -(\operatorname{tr} EH)\sigma(x)$ and $\sigma(x) = \sigma(x_1) \exp\{-\int_{x_1}^{x_2} \operatorname{tr} EH\}$, where $x \in (a, \eta_1)$. But $H(x)$ hasa limit as $x \to \eta_1(a)$ which implies that $\sigma(\eta_1) > 0$, a contradiction, thus proving the theorem.

Finally, if the zeros of $\rho(x)$ are isolated, let $\mu_i(a)$ be its ith zero on (a, ∞) and, in general, let $\mu_{i+1}(a)$ be the smallest zero of $\rho(x)$ on (μ_i, ∞) such that $\rho(x) \neq 0$ on (μ_i, μ_{i+1}).

Proof of the following theorems are omitted since they follow readily from the preceding analysis.

THEOREM 3.3. *If* $\mu_1(a)$ *and* $\mu_2(a)$ *both exist and* $\int^\infty x^2/r(x)dx = \infty$, *then* $\eta_1(a)$ *exists.*

THEOREM 3.4. *If* $\mu_1(a)$, $\mu_2(a)$, *and* $\mu_3(a)$ *all exist, then* $\eta_1(a)$ *exists.*

THEOREM 3.5. *If* $\eta_1(a)$ *does not exist and* $\int^\infty p = \infty$ *then* $\mu_1(a)$ *and* $\mu_2(a)$ *exist.*

THEOREM 3.6. *If*

$$\int^\infty \frac{x^2}{r(x)}\,dx = \infty$$

and $\int^\infty p = \infty$ *then* $\eta_1(c)$ *exists for all* $c \in [a, \infty)$.

By letting $p \equiv 0$ and $b = \eta_1(a)$, Theorems 2.4 and 3.1 yield, as an immediate corollary; the second-order result:

THEOREM 3.7. *If* $\int^\infty 1/r = \infty$ *and* $\int^\infty p = \infty$ *then either* (15) $(rz')' + qz = 0$ *or* $(rz')' + qz = 1$ *has a non-trivial solution* $z(x)$ *and a number* $b > a$ *such that*

$$z(a) = z(b) = 0 \quad \text{and} \quad \int_a^b z(x)dx = 0.$$

4. Comparison theorems for $\mu_1(a)$. Consider a second equation of type (1):

$$(1_0) \qquad [(r_0(x)y'')' + q_0(x)y']' - p_0(x)y = 0$$

and denote the various quantities defined for (1) in the preceding sections by the same symbols with an additional subscript or superscript "0," that is,

$$u_0,\ v_0,\ \sigma_0,\ \phi_0,\ Y_0,\ \tilde{Y}_0,\ \mu_1^0(a),\ \eta_1^0(a),\ K_0, \text{ etc.}$$

THEOREM 4.1. *If* $0 < r(x) \leqslant r_0(x)$, $0 \leqslant p_0(x) \leqslant p(x)$, $0 \leqslant q_0(x) \leqslant q(x)$ *and* $\mu_1{}^0(a)$ *exists, then* $\mu_1(a)$ *exists and* $\mu_1(a) \leqslant \mu_1{}^0(a)$. *Furthermore,* $\mu_1(a) = \mu_1{}^0(a)$ *if and only if*

$$(22) \qquad r_0(x) \equiv r(x),\ q_0(x) \equiv q(x),\ p_0(x) \equiv p(x)$$

on $(a, \mu_1{}^0(a)]$.

Proof. Suppose that $\mu_1{}^0(a)$ exists but that $\mu_1(a)$ does not on $[a, \mu_1{}^0]$. Then on $(a, \mu_1{}^0)$: $K(x)$ and $K_0(x) = Y_0\tilde{Y}_0{}^{-1}$ are defined, symmetric, and

$$K' = E + KFK, \qquad K_0' = E_0 + K_0F_0K_0.$$

Subtraction of these equations and adding and subtracting terms yields

$$(23) \quad (K - K_0)' - (K_0F_0)(K - K_0) - (K - K_0)(F_0K_0) \\ = E - E_0 + K(F - F_0)K + (K - K_0)F_0(K - K_0),$$

where

$$(24) \qquad E(x) - E_0(x) = \left[\frac{1}{r(x)} - \frac{1}{r_0(x)}\right]E_1(x-a),$$

$$F(x) - F_0(x) = [p(x) - p_0(x)]F_1(x-a) + [q(x) - q_0(x)]Q_1.$$

Therefore, each of the three terms on the right-hand side of (23) is positive semi-definite and the fact that $K - K_0 \geqslant 0$ may be obtained by "solving" (23) as a linear non-homogeneous matrix differential equation, as in **(1)** using the techniques of **(7)**. On $[a, \mu_1{}^0)$ let $K - K_0 = J(x)L(x)J^*(x)$, where $J' = (K_0F_0)J$ and $J(a)$ is non-singular, and substitution into (23) gives $L(a) = 0$ and

$$(25) \qquad L'(x) = J^{-1}\{(E - E_0) + K(F - F_0)K + (R - R_0)F_0(K - K_0)\}J^{*-1}.$$

One appropriate solution for $J(x)$ is $J = \tilde{Y}_0{}^{*-1}$ and, hence,

$$(25') \qquad L(x) = \int_a^x \overset{*}{\tilde{Y}_0}\{(E - E_0) + K(F - F_0)K + (K - K_0)F_0(K - K_0)\}\tilde{Y}_0,$$

which is positive semi-definite on $(a, \mu_1{}^0)$. It follows that $K \geqslant K_0$ and the fact that $\operatorname{tr} K(x) \geqslant \operatorname{tr} K_0(x)$, together with Lemma 2.1, contradicts the continuity of $K(x)$ on the closed interval $[a, \mu_1{}^0]$ and the first part of Theorem 4.1 is proved and $\mu_1(a) \leqslant \mu_1{}^0(a)$.

The proof of the remaining part is more difficult. Suppose that $\mu_1(a) = \mu_1{}^0(a)$, then $\tilde{Y}(x)$ and $\tilde{Y}_0(x)$ have their first singularity simultaneously, as do $H = -\tilde{Y}Y^{-1}$ and $H_0 = -\tilde{Y}_0Y_0{}^{-1}$. Note that by Theorem 2.3, $\eta_1(a)$ and $\eta_1{}^0(a)$ do not appear on $[a, \mu_1]$ and $H(x)$ and $H_0(x)$ are defined on $(a, \mu_1]$, and on (a, μ_1), $H = -K^{-1}$, $H_0 = -K_0{}^{-1}$. Hence $H(x)$ and $H_0(x)$ are both negative semi-definite on (a, μ_1).

Throughout the remainder of this paper there will be a repeated need to relate the definiteness of $H - H_0$ to that of $K - K_0$ The following general (any order) matrix theorem is useful.

LEMMA M. *If A and B are two non-singular matrices, $A - B$ is positive definite (semi-definite) and B is positive definite then $B^{-1} - A^{-1}$ is positive definite (semi-definite).*

The proof follows readily from the fact (see Bocher, *Introduction to Higher Algebra* [New York, 1907], p. 171) that A and B may be diagonalized simultaneously, and, in fact, all eigenvalues of B may be transformed to the value "1."

Returning to the main argument, $K - K_0 \geqslant 0$ implies $H - H_0 \geqslant 0$ and on $(a, \mu_1]$; as in the discussion for K and K_0.

$$H' = F + HEH, \qquad H_0' = F_0 + H_0E_0H_0,$$

and

$$(26) \qquad (H - H_0)' - (H_0E_0)(H - H_0) - (H - H_0)(E_0H_0) \\ = F - F_0 + H(E - E_0)H + (H - H_0)E_0(H - H_0).$$

Let $H - H_0 = J_1(x)M(x)J^*{}_1(x)$, where $J_1' = (H_0E_0)J_1$. Then $J_1 = Y_0{}^{*-1}$ is a multiplying factor and

$$(27) \qquad M'(x) = Y_0^*\{(F \quad F_0) + H(E - E_0)H + (H - H_0)E_0(H - H_0)\} \\ Y_0 \geqslant 0 \text{ on } (a, \mu_1].$$

Since $H_0(\mu_1)$ is singular, there is a non-zero vector ξ_0 such that $H_0(\mu_1)\xi_0 = 0$ and by letting $x \to \mu_1$,

$$0 \leqslant \xi_0^*[H(\mu_1) - H_0(\mu_1)]\xi_0 = \xi_0^* H(\mu_1)\xi_0 \leqslant 0$$

which implies that min. e.v. $(H - H_0)(\mu_1) = 0$ and min. e.v. $M(\mu_1) = 0$. Therefore, there is a non-zero vector ξ such that $\xi^*M(\mu_1)\xi = 0$ and since $H - H_0 \geqslant 0$ then so is $M(x)$ and $\xi^*M(x)\xi \geqslant 0$ on $(a, \mu_1]$. But by (27) $(\xi^*M(x)\xi)' \geqslant 0$ which implies that $\xi^*M(x)\xi = \xi^*M'(x)\xi = 0$ on $(a, \mu_1]$ and min. e.v. $M(x) =$ Min. e.v. $(H - H_0) =$ min. e.v. $(K - K_0) =$ min. e.v. $L(x)$ $= 0$ on (a, μ_1).

Suppose there is a subinterval $I = [x_1, x_2]$ of (a, μ_1) such that $r(x) < r_0(x)$ on I. Let ξ be a non-zero vector such that $\xi^*M(x_2)\xi = 0$. Since the integrand of (25) is the sum of the three exhibited positive semi-definite terms then

$$\int_a^{x_2} \xi^*\tilde{Y}_0^*(E - E_0)\tilde{Y}_0\xi = 0 \text{ and } \eta^*(E - E_0)\eta = 0 \text{ on } (a, x_2),$$

$$\text{where } \eta(x) = \tilde{Y}_0(x)\xi \neq 0.$$

Let $a = 0$, then $\eta^*E_1(x)\eta = (x\eta_1 - \eta_2)^2 = 0$ and $\eta_2 = x\eta_1$ on I. Recalling (13) and that $K_0 = Y_0\tilde{Y}_0{}^{-1}$, $\eta(x)$ satisfies

$$\eta' = -(F_0K_0)\eta \text{ on } (a, \mu_1) \text{ and } \eta(x) = \binom{1}{x}\eta_1(x),\ \eta_1 \neq 0 \qquad (a = 0).$$

That these are incompatible is seen by substituting the latter into the former and by multiplying on the left by the (row) vector $(-x, 1)$, which yields on I: (letting $K_0 = (k_{ij}{}^0)$)

$$1 = -(-x, 1)(F_0, K_0)\binom{1}{x} = -q_0(k_{12}^0 + xk_{22}^0) \qquad (a = 0).$$

But $q_0 \geqslant 0$ and by Lemma 2.1, $k_{12}{}^0 + xk_{22}{}^0 > 0$ on $(a = 0, \mu_1)$ and the assumption that $r < r_0$ on I is contradicted giving $r \equiv r_0$ on (a, μ_1).

To continue the proof, instead of using the more complicated second term of the integrand of (25), use the first term of the right-hand side of (26) or (27) involving $F - F_0 = (q - q_0)Q_1 + (p - p_0)F_1(x)$. An argument, paralleling the preceding one but involving (27), yields that if $\eta(x) = Y_0(x)\xi$ and $q > q_0$ on a subinterval I of (a, μ_1) then $\eta_2(x) \equiv 0$, or if $p > p_0$ on I then $\eta_1(x) + (x - a)\eta_2(x) \equiv 0$ on I, neither of which is compatible with $\eta' = (E_0K_0{}^{-1})\eta$. Therefore, $\mu_1 = \mu_1{}^0$ implies that (22) is true and the converse obviously holds, completing the proof of Theorem 4.1.

If equation (1_0) is equation (1) with $q \equiv 0$ or with $p \equiv 0$, that is,

$$(ry'')'' - py = 0 \text{ or } [(ry'')' + qy']' = 0.$$

Then the existence of $\mu_1(a)$ for either of the special cases implies the same for the complete equation (1). Therefore, using the results of §§ 2 and 3:

THEOREM 4.2. *If*

$$\int^{\infty} p(x)(\overset{x}{I^2}p)^2 dx = \infty$$

or $\int^{\infty} q(x)dx = \infty$ *then* $\mu_1(b)$ *exists for every* $b \in [a, \infty)$ *and if, in addition,* $\int^{\infty} 1/r = \infty$ *then* $\eta_1(b)$ *exists for every* $b \in [a, \infty)$.

THEOREM 4.3. *If* $\bar{\mu}_1(a)$ *exists for the second-order equation* (15) $(ry')' + qy = 0$ *then* $\mu_1(a)$ *exists for the fourth-order equation* (1) *and* $\mu_1(a) \leqslant \bar{\mu}_1(a)$.

5. Comparison theorems for $\eta_1(a)$. The discussion of this section is an extension of that of the preceding section and consider again the equations (1) and (1_0) and the same ordering of coefficients extended to the first conjugate point:

THEOREM 5.1. *If* $\eta_1^0(a)$ *exists and* $r(x) \leqslant r_0(x)$, $p_0(x) \leqslant p(x)$, $q_0(x) \leqslant q(x)$ *on* $[a, \eta_1^0]$ *then* $\eta_1(a)$ *exists and* $\eta_1(a) \leqslant \eta_1^0(a)$.

Proof. Suppose $\eta_1^0(a)$ exists and $\eta_1(a)$ does not exist on $(a, \eta_1^0]$. In the last section it was established that $H - H_0 \geqslant 0$ on (a, b) for some $b \in (a, \mu_1^0)$ and by (26)–(27) that semi-definiteness extends to the whole interval (a, η_1^0). But by Theorem 3.2:

$$\text{max. e.v. } H(x) \geqslant \text{max. e.v. } H_0(x) \to \infty, \text{ as } x \to \eta_1^0,$$

which contradicts that $H(x) = -\tilde{Y}Y^{-1}$ exists and is continuous on $(a, \eta_1^0]$, thus completing the proof of the theorem.

An immediate corollary is:

THEOREM 5.2. *If* $\eta_1^0(a)$ *exists for either*

$$(ry'')'' - py = 0 \text{ or } [(ry'')' + qy']' = 0$$

then $\eta_1(a)$ *exists for the complete equation* (1) *and* $\eta_1(a) \leqslant \eta_1^0(a)$.

This comparison theorem leads to the following:

THEOREM 5.3. *If the second-order equation* (15) $(r(x)y')' + q(x)y = 0$ *has a non-trivial solution with four zeros on* $[a, \infty)$ *then* $\eta_1(a)$ *exists for the fourth-order equation* (1).

Proof. First suppose that $p(x) \equiv 0$ on $[a, \infty)$. For this special case $\rho(x) \equiv u_1(x)$ and if $z(x) = u'(x)$ then since $u_2'(x) \equiv 0$ and $u_2(a) = 0$:

$$(rz')' + qz = 0,\ z(a) = 0,\ (rz')(a) = 1.$$

Therefore, $z(x)$ has at least four zeros on $[a, \infty)$, the first at $x = a$, and $z'(x)$ has at least three zeros on (a, ∞). Note that $\rho(x) \equiv u_1(x)$, which has the zeros of $z'(x)$ and these guarantee $\mu_1(a)$, $\mu_2(a)$, and $\mu_3(a)$. Theorem 3.4 now gives the existence of $\eta_1(a)$ when $p \equiv 0$. The comparison Theorem 5.2 eliminates the restriction on $p(x)$.

It should be clear that many of the methods of this paper are applicable to self-adjoint equations of higher order and to the matrix equations of Sternberg.

6. Non-existence of $\eta_1(a)$ and disconjugacy. In the last section of their paper **(6)**, Leighton and Nehari gave two changes of variable each of which removes the middle term of

$$[(r(x)y'')' + q(x)y']' - p(x)y = 0. \tag{1}$$

The transformation of the independent variable is repeated here in a slightly more general form.

LEMMA 6.1. *Let I be any subinterval of $[a, \infty)$, $\alpha(x)$ be a function of x such that α and $r\alpha'' \in C'$ with $\alpha' > 0$ on I. Then the change of variable*

$$t = \alpha(x), \; x = \alpha^{-1}(t) = x(t)$$

transforms I into a t-interval I' and (1) *into*

$$[(R(t)\ddot{Y})^{\cdot} + Q(t)\dot{Y}]^{\cdot} - P(t)Y = 0 \text{ on } I', \tag{26}$$

where $Y(t) = y[x(t)],\ R(t) = r[x(t)]\ (\alpha'[x(t)])^3,$

$$Q(t) = \alpha_2[x(t)], \alpha_2 = (r\alpha'')' + q\alpha \text{ and } P(t) = p[x(t)]/\alpha'[x(t)].$$

Note that if $\alpha_2 = 0$ on I then the middle term vanishes and the above lemma reduces to that of Leighton and Nehari:

LEMMA 6.2 **(6**, Theorem 12.1**)**. *If the second-order equation*

$$(r(x)y')' + q(x)y = 0 \text{ on } I = [b, c] \subset [a, \infty) \tag{15}$$

has a positive solution $z(x)$ then the substitution $t = \alpha(x) = \int_b^x z(s)ds$ transforms I into $I' = [0, T]$ and (1) *into*

$$[R(t)\ddot{Y}]^{\cdot\cdot} - P(t)Y = 0 \text{ on } I' \tag{27}$$

where $R(t) = r[x(t)]z^3[x(t)]$ *and* $P(t) = p[x(t)]/z[x(t)]$.

Proof of Lemma 6.1. As in **(6)**, successive differentiations of $Y(t) = y[x(t)]$ yield $\dot{Y} = y'/\alpha'$ and $\ddot{Y} = (\alpha'y'' - y'\alpha'')/(\alpha')^3$, let $Y_1(t) = R(t)\ddot{Y}(t)$ and $Y_2(t) = \dot{Y}_1 + Q\dot{Y}$, then $\dot{Y}_2 = PY$ (with R, Q, and P as defined in the lemma) and the lemma is proved.

THEOREM 6.1. *If* (15) $(ry')' + qy = 0$ *is disconjugate (no non-trivial solution has more than one zero) on $[a, b)$, $a < b \leqslant \infty$ and $\eta_1(a)$ does not exist on (a, b) then equation* (1) *is disconjugate (no non-trivial solution has more than three zeros on $[a, b)$.*

Proof. Let $c \in (a, b)$, then, by a well-known property of disconjugate equations, there exists a solution $z(x)$ of (15) which is positive on the closed finite interval $[a, c]$. As in Lemma 6.2, the substitution $t = \alpha(x) = \int_a^x z$

transforms $[a, c]$ into $[0, T]$ and (1) into (27). Also, $\eta_1(0)$ does not exist with respect to equations (27) on $(0, T]$ and, by **(6)**, no solution of (27) has more than three zeros on $[0, T]$. Therefore, no solution of (1) has more than three zeros on $[a, c]$ and, hence, on $[a, b)$.

THEOREM 6.2. *If $\eta_1(a)$ does not exist, $p(x) + q(x) \not\equiv 0$ for large x, and $\int^\infty 1/r = \infty$ then equation* (1) *is disconjugate.*

Proof. By Theorem 3.1, the non-existence of $\eta_1(a)$ implies the non-existence of $\mu_1(a)$ and Theorem 4.1 implies that $\mu_1(a)$ does not exist with respect to (1) with $p \equiv 0$. Hence, by Theorem 2.2, the second-order equation (15) is a disconjugate and Theorem 6.1 guarantees that equation (1) is disconjugate and the proof is complete.

If $\int^\infty 1/r < \infty$ or $(ry')' + qy = 0$ is oscillatory, it is not known to the author whether the non-existence of $\eta_1(a)$ implies disconjugacy of (1). Of course, by **(6**, Part II**)**, if $p(x) < 0$ and $q \equiv 0$ then neither $\mu_1(a)$ nor $\eta_1(a)$ exists, but it should be recalled that in many cases the fundamental solution $v(x)$ has a simple zero following its initial triple zero.

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