THE DIVISIBILITY OF DIVISOR FUNCTIONS

by R. A. RANKIN

(Received 1st October, 1960)

1. Introduction. For any positive integers n and ν let

$$\sigma_\nu(n) = \sum_{d \mid n} d^\nu,$$

where d runs through all the positive divisors of n. For each positive integer k and real $x > 1$, denote by $N(\nu, k; x)$ the number of positive integers $n \leqq x$ for which $\sigma_\nu(n)$ is not divisible by k. Then Watson [**6**] has shown that, when ν is odd,

$$N(\nu, k; x) = O\left\{\frac{x}{(\log x)^{1/\phi(k)}}\right\} \tag{1}$$

as $x \to \infty$; it is assumed here and throughout that ν and k are fixed and independent of x. It follows, in particular, that $\sigma_\nu(n)$ is almost always divisible by k. A brief account of the ideas used by Watson will be found in § 10.6 of Hardy's book† on Ramanujan [**2**].

Watson obtained his estimate for $N(\nu, k; x)$ by showing that it is majorized by the partial sum $B(x)$ of a Dirichlet series $\mathfrak{f}(s)$ for which

$$B(x) \sim \frac{Cx}{(\log x)^{1/\phi(k)}}, \tag{2}$$

where C is a certain constant. The purpose of the present paper is to show that, when k is an odd prime, it is possible to improve on (1) by replacing it by an asymptotic equation of the type (2); in the proof, the auxiliary function $\mathfrak{f}(s)$ is replaced by a Dirichlet series that is more closely connected with $N(\nu, k; x)$. More precisely, we prove

THEOREM 1. *Let q be an odd prime and write*

$$h = \frac{q-1}{(\nu, q-1)}. \tag{3}$$

Then, as $x \to \infty$,

$$N(\nu, q; x) \sim \frac{Ax}{(\log x)^{1/h}} \quad (h \text{ even}), \tag{4}$$

and

$$N(\nu, q; x) \sim Bx \qquad (h \text{ odd}). \tag{5}$$

The constants A and B depend only on ν and q and are evaluated in § 2.

Note that, in particular, h is even when ν is odd and h is odd when $\nu = q-1$.

Some results for a composite modulus k are given in § 4.

† In (10.6.3) the exponent $1-\kappa$ should be replaced by κ.

https://doi.org/10.1017/S2040618500034274 Published online by Cambridge University Press

2. The associated Dirichlet series. We define the arithmetical function $a(n)$ $(n \geqq 1)$ by

$$a(n) = 0 \quad \text{if} \quad q \mid \sigma_\nu(n), \quad a(n) = 1 \quad \text{if} \quad q \nmid \sigma_\nu(n).$$

Since $\sigma_\nu(mn) = \sigma_\nu(m)\sigma_\nu(n)$, when $(m, n) = 1$, it follows that

$$a(mn) = a(m)a(n).$$

Further, since, for any prime p and positive integer m,

$$\sigma_\nu(p^m) = \frac{p^{\nu(m+1)} - 1}{p^\nu - 1}$$

it is clear that $a(p^m) = 0$ if either

$$p^\nu \equiv 1 \pmod q \quad \text{and} \quad m+1 \equiv 0 \pmod q, \tag{6}$$

or

$$p^\nu \not\equiv 1 \pmod q \quad \text{and} \quad p^{\nu(m+1)} \equiv 1 \pmod q. \tag{ }$$

If neither (6) nor (7) holds, then $a(p^m) = 1$; in particular, $a(q^m) = 1$.

If $p \neq q$ and g is a primitive root modulo q, we put

$$p \equiv g^{c_p} \pmod q, \quad \text{where} \quad 0 \leqq c_p < q-1,$$

and define the integer μ_p by

$$\mu_p = q \quad \text{when} \quad h \mid c_p, \quad \text{and} \quad \mu_p = h/(h, c_p) \quad \text{when} \quad h \nmid c_p. \tag{8}$$

Here h is defined by (3). It follows that μ_p is the order of p^ν modulo q, except when $p^\nu \equiv 1 \pmod q$, in which case $\mu_p = q$. Conditions (6) and (7) can then be combined into the single congruence

$$m+1 \equiv 0 \pmod{\mu_p}. \tag{9}$$

With the function $a(n)$ we associate the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a(n)n^{-s}, \tag{10}$$

which is clearly holomorphic in the half plane $\sigma = \operatorname{Re} s > 1$. Also, for $\sigma > 1$,

$$\begin{aligned} f(s) &= \prod_p \sum_{m=0}^{\infty} a(p^m)p^{-ms} \\ &= (1-q^{-s})^{-1} \prod_{p \neq q} \left\{ \sum_{m=0}^{\infty} p^{-ms} - p^s \sum_{r=1}^{\infty} p^{-r\mu_p s} \right\} \\ &= (1-q^{-s})^{-1} \prod_{p \neq q} \frac{1-p^{-(\mu_p-1)s}}{(1-p^{-s})(1-p^{-\mu_p s})} \\ &= \zeta(s) \prod_{p \neq q} \frac{1-p^{-(\mu_p-1)s}}{1-p^{-\mu_p s}}. \end{aligned} \tag{11}$$

https://doi.org/10.1017/S2040618500034274 Published online by Cambridge University Press

From (8) it is clear that, when h is odd, μ_p cannot be even, so that $\mu_p \geqq 3$, and we have

THEOREM 2. *If h is odd, $f(s)$ is holomorphic for $\sigma > \frac{1}{2}$, with the exception of a simple pole at $s = 1$ of residue*

$$B = \prod_{p \neq q} \frac{1-p^{1-\mu_p}}{1-p^{-\mu_p}}.$$

From Theorem 2 the asymptotic equation (5) follows by the Wiener-Ikehara theorem, since $a(n) \geqq 0$ for all $n \geqq 1$.

If h is even, $\mu_p = 2$ whenever c_p is an odd multiple of $\frac{1}{2}h$ and further analysis is required. Let $\chi(n)$ be the Dirichlet character modulo q that is defined by

$$\chi(n) = e(\alpha/h) \quad \text{for} \quad n \equiv g^{\alpha} \pmod{q},$$

where $e(z)$ denotes $\exp(2\pi iz)$. Also write

$$F(s) = \prod_{r=1}^{h} \{L(s, \chi^r)/L(s, \chi^{2r})\}, \tag{12}$$

where $L(s, \chi^r)$ is the Dirichlet L-series associated with the character χ^r.

Let p be a prime not equal to q for which $h \nmid c_p$, so that $\mu_p = h/(h, c_p)$. It follows that

$$(c_p, \tfrac{1}{2}h) = \tfrac{1}{2}(c_p, h) \quad \text{or} \quad (c_p, h)$$

according as μ_p is odd or even, and so

$$\begin{aligned}\prod_{r=1}^{h} [\{1-\chi^r(p)p^{-s}\}/\{1-\chi^{2r}(p)p^{-s}\}] &= \prod_{r=1}^{h} [\{1-p^{-s}e(c_p r/h)\}\{1-p^{-s}e(c_p r/\tfrac{1}{2}h)\}^{-1}] \\ &= \begin{cases} 1 & (\mu_p \text{ odd}), \\ \{(1-p^{-\mu_p s})(1-p^{-\frac{1}{2}\mu_p s})^{-2}\}^{h/\mu_p} & (\mu_p \text{ even}). \end{cases}\end{aligned}$$

Hence

$$F(s) = \prod_{\substack{p \neq q \\ \mu_p \text{ even}}} \left\{\frac{(1-p^{-\frac{1}{2}\mu_p s})^2}{1-p^{-\mu_p s}}\right\}^{h/\mu_p}, \tag{13}$$

and so, by (11),

$$f(s) = \zeta(s)\{F(s)\}^{1/h}\psi(s), \tag{14}$$

where $\psi(s)$ is holomorphic for $\sigma > \frac{1}{2}$ and bounded for $\sigma \geqq \frac{1}{2}+\delta$ $(\delta > 0)$. For $\psi(s)$ can be expressed as an infinite product of powers of factors $1-p^{-\mu s}$ in which $\mu \geqq 2$.

To deduce (4) from this we need some further information about Dirichlet series and L-functions, which we state in the following two lemmas.

LEMMA 1. *Suppose that the function $g(s)$ defined by the Dirichlet series*

$$g(s) = \sum_{n=1}^{\infty} b_n n^{-s} \tag{15}$$

has the following four properties:

(i) $b_n \geqq 0$ for $n \geqq 1$.

(ii) *The series* (15) *defines $g(s)$ as a holomorphic function in the half-plane $\sigma > 1$.*

(iii) $g(s) = \{\zeta(s)\}^{1-1/h}G(s)$,

https://doi.org/10.1017/S2040618500034274 Published online by Cambridge University Press

where $G(s)$ is holomorphic in the domain $R(a, \beta)$ defined by

$$\sigma > 1 - \frac{a}{(\log|t|)^\beta} \quad (|t| \geqq 3),$$

$$\sigma > 1 - \frac{a}{(\log 3)^\beta} \quad (|t| \leqq 3).$$

Here $t = \operatorname{Im} s$ and $a > 0$, $\beta > 0$.

(iv) *In $R(a, \beta)$,*

$$G(s) = O\{(\log(|t|+3))^\gamma\}$$

for some $\gamma > 0$, as $|t| \to \infty$.

Then, as $x \to \infty$,

$$\sum_{n \leqq x} b_n \sim \frac{G(1)x}{\Gamma(1-1/h)(\log x)^{1/h}}.$$

The proof of the lemma is omitted. All that is necessary for the proof is contained in Watson [**6**], Hilfssätze 7 and 8; or see [**3**] or [**2**, pp. 62, 168].

Lemma 2. *If $X(n)$ is not the principal character modulo q, then there exist positive constants a, β, γ_1, γ_2, C_1, C_2 such that, for all s in the region $R(a, \beta)$ of Lemma 1,*

$$C_1\{\log(|t|+3)\}^{-\gamma_1} \leqq |L(s, X)| \leqq C_2\{\log(|t|+3)\}^{\gamma_2}.$$

This is proved for $|t| \geqq 3$ in Landau's Handbuch [**4**, pp. 462–467] with $\beta = 7$, $\gamma_1 = 5$, $\gamma_2 = 1$; the constants a, C_1, C_2 can then be chosen to make the result valid also for $|t| \leqq 3$.

If we apply Lemma 2 to the L-series in (12) that are not associated with the principal character and use (14), we deduce that $f(s)$ satisfies the conditions imposed on $g(s)$ in Lemma 1 for certain positive constants a, β and γ. For the contribution of those L-series associated with the principal character modulo q to the right-hand side of (12) is

$$(1-q^{-s})^{-1}\{\zeta(s)\}^{-1}.$$

It follows that (4) holds with

$$A = \frac{G(1)}{\Gamma(1-1/h)}. \tag{16}$$

Now, by (11), (13) and (14),

$$\psi(1) = \prod_{\substack{p \neq q \\ \mu_p > 2}} \frac{1-p^{1-\mu_p}}{1-p^{-\mu_p}} \prod_{\substack{p \neq q \\ \mu_p > 2 \\ \mu_p \text{ even}}} \left\{\frac{1+p^{-\frac{1}{2}\mu_p}}{1-p^{-\frac{1}{2}\mu_p}}\right\}^{1/\mu_p} \prod_{\substack{p \neq q \\ \mu_p = 2}} (1-p^{-2})^{-\frac{1}{2}}, \tag{17}$$

and, by (12) and (14),

$$G(1) = \psi(1)\{\lim_{s \to 1} \zeta(s)F(s)\}^{1/h} = \frac{\psi(1)}{(1-q^{-1})^{1/h}} \left\{\prod_{r=1}^{h-1} L(1, \chi^r) \Big/ \prod_{r=1}^{\frac{1}{2}h-1} L^2(1, \chi^{2r})\right\}^{1/h}. \tag{18}$$

https://doi.org/10.1017/S2040618500034274 Published online by Cambridge University Press

From (16), (17) and (18) the constant A in (4) can be calculated. For example, if $q = 5$ and v is odd,

$$A = \frac{1}{\Gamma(\frac{3}{4})}\left(\frac{5L_2L_3}{L_4}\right)^{\frac{1}{4}} \prod_{p\equiv 1 \pmod 5} \frac{1-p^{-4}}{1-p^{-5}} \prod_{p\equiv 2,\,3 \pmod 5} \frac{1-p^{-3}}{1-p^{-4}}\left(\frac{1+p^{-2}}{1-p^{-2}}\right)^{\frac{1}{4}} \prod_{p\equiv 4 \pmod 5} (1-p^{-2})^{-\frac{1}{4}},$$

where L_2, L_3 and L_4 are the values at $s = 1$ of the Dirichlet L-series associated with the characters

$$(1, i, -i, -1), \quad (1, -i, i, -1) \quad \text{and} \quad (1, -1, -1, 1).$$

This completes the proof of Theorem 1.

3. Ramanujan's function $\tau(n)$. From (1) and the fact that

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691}$$

Watson [6] deduced that (as Ramanujan conjectured) $\tau(n)$ is divisible by 691 for almost all n. See also [5] and [1] for similar results to other moduli. In his paper, Watson quotes a statement of Ramanujan in an unpublished manuscript that, on the other hand, for $n \leqq 5000$, $\tau(n)$ is only divisible by 691 if $n = 1381$. This can be established by means of the criterion (9) and one can show further than the only values of n less than 10,000 for which 691 divides $\tau(n)$ are the primes

$$n = 1381, \quad 5527 \quad \text{and} \quad 8291.$$

The first such composite values of n, of the form p^λ, are $(4583)^2$ and $(89)^4$.

4. Composite moduli. The problem of obtaining a sharper estimate than (1) for $N(v, k; x)$, when k is composite, remains to be considered. It does not seem to have been observed previously that the order of magnitude of the term on the right of (1) can be reduced in certain cases.

Suppose that

$$k = q_1^{r_1} q_2^{r_2} \cdots q_m^{r_m},$$

where $q_1, q_2, \ldots, q_m$ are different primes and $r_1, r_2, \ldots, r_m$ are positive integers. Write

$$\lambda = \max_{j \leqq m} \phi(q_j^{r_j}).$$

Since $q_j^{r_j} \mid \sigma_v(n)$ for $j = 1, 2, \ldots, m$, if $k \mid \sigma_v(n)$, it follows that

$$N(v, k; x) \leqq \sum_{j=1}^{m} N(v, q_j^{r_j}; x). \tag{19}$$

If we apply (1) to each term on the right of (19), we obtain

$$N(v, k; x) = O\left\{\frac{x}{(\log x)^{1/\lambda}}\right\}, \tag{20}$$

when v is odd. This is an improvement on (1) in all cases except when k is a power of a prime or twice such a power; for, except in these cases, $\lambda < \phi(k)$.

If k is odd and squarefree, we can, in some cases, obtain an asymptotic equation. We suppose that

$$h_j = \frac{q_j - 1}{(v, q_j - 1)}$$

is even for each $j \leqq m$ and that it takes its maximum value for exactly one value of j, so that, by renumbering the prime factors q_j if necessary, we may suppose that $h_1 > h_j$ for $j > 1$. This occurs, in particular, when v is prime to $\phi(k)$. Then, since $q_1 \mid \sigma_v(n)$ if $k \mid \sigma_v(n)$,

$$N(v, q_1 ; x) \leqq N(v, k ; x). \tag{21}$$

It follows from (19), (21) and (3) that

$$N(v, k ; x) \sim \frac{A_1 x}{(\log x)^{1/h_1}}, \tag{22}$$

where A_1 is the constant associated with v and the prime q_1.

Note also that, when $k = 2$, $f(s) = (1 + 2^{-s})\zeta(2s)$, so that

$$N(v, 2 ; x) \sim (1 + 2^{-\frac{1}{2}}) x^{\frac{1}{2}}.$$

The problem of improving (1) in other cases, such as when $k = q^m$ (q prime, $m \geqq 2$), remains open.

REFERENCES

1. S. Chowla, On a theorem of Walfisz, *J. London Math. Soc.* **22** (1947), 136–140.

2. G. H. Hardy, *Ramanujan* (Cambridge, 1940).

3. E. Landau, Über die Einteilung der positiven ganzen Zahlen in vier Klassen nach der Mindestzahl der zu ihrer additiven Zusammensetzung erforderlichen Quadrate, *Arch. Math. Phys.* (3) **13** (1908), 305–312.

4. E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, Band 1 (Leipzig, 1909).

5. A. Walfisz, Über einige Ramanujansche Sätze, *Trav. Inst. Math. Tbilissi* **5** (1938), 145–152.

6. G. N. Watson, Uber Ramanujansche Kongruenzeigenschaften der Zerfällungsanzahlen (I), *Math. Z.* **39** (1935), 712–731.

THE UNIVERSITY
GLASGOW

https://doi.org/10.1017/S2040618500034274 Published online by Cambridge University Press