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Spectral approximation theorems for bounded linear operators

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In this paper we present some approximation theorems for the eigenvalue problem of a compact linear operator defined on a Banach space. In particular we examine: criteria for the existence and convergence of approximate eigenvectors and generalized eigenvectors; relations between the dimensions of the eigenmanifolds and generalized eigenmanifolds of the operator and those of the approximate operators.

1. Introduction

Let X be a real or complex Banach space and $[X]$ the space of bounded linear operators on X into X. For A in $[X]$ let $\|A\|$ denote the usual operator norm $\|A\| = \sup_{\|x\|\le 1} \|Ax\|$, and $\eta(A)$ denote the null space of A. Let $\sigma(A)$ denote the spectrum of A, that is, the set of numbers λ for which $\lambda I - A$ fails to have an inverse in $[X]$.

In numerical solutions for the eigenvalue problem for an operator equation

$$Tx = \lambda x ,$$

often we are led to solve corresponding approximate equations

$$T_n x_n = \lambda_n x_n ,$$

where T, T_n belong to $[X]$ and $\|T_n - T\| \to 0$. It is of interest to know:

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(a) for an arbitrary eigenvalue λ of T, is there a sequence of eigenvalues λ_n of T_n such that $\lambda_n \to \lambda$?

(b) for an arbitrary eigenvector x of T, is there a sequence of eigenvectors x_n of T_n such that $x_n \to x$?

The first question was answered by Putnam [9] under very general conditions. As for the second question, Pol'skii [8] showed by way of an example that for x in $\eta(\mu-T)$ there need not exist x_n in $\eta(\mu_n-T_n)$ such that $x_n \to x$ even when T and T_n, $n \geq 1$, are compact. Andrew and Elton [2] established a necessary and sufficient condition for which (b) holds when X is a Hilbert space and T, T_n, $n \geq 1$, are compact. This paper offers improvements and generalizations of their main result. As a generalization it establishes, for an arbitrary but fixed generalized eigenvector of a compact operator T on a Banach space, a necessary and sufficient condition for the existence and convergence of generalized eigenvectors of the approximate operators T_n. Other results compare the dimensions of eigenmanifolds and generalized eigenmanifolds of T with those of T_n.

2. Eigenvectors and eigenmanifolds.

The following theorem is essential for obtaining the later results.

THEOREM 1. *Assume* $T, T_n \in [X]$ *and* $\|T_n-T\| \to 0$. *Let* μ_n *in* $\sigma(T_n)$ *be such that* $\mu_n \to \mu$. *Then* μ *belongs to* $\sigma(T)$. *Now assume* T *is compact,* $\mu \neq 0$, $x_n \in \eta(\mu_n-T_n)$ *and* $\|x_n\| = 1$. *Then there exist sequences* $\{T_{n_i}\}$, $\{x_{n_i}\}$ *and* x *in* X *such that* $x_{n_i} \to x \in \eta(\mu-T)$ *as* $i \to \infty$. *For* n *sufficiently large we have*

$$\dim\eta(\mu_n-T_n) \leq \dim\eta(\mu-T) .$$

Let $M \subset \eta(\mu-T)$ *and* $M_n \subset \eta(\mu_n-T_n)$ *be subspaces such that* $x_n \in M_n$, $x_n \to x$ *implies* $x \in M$. *Then* $\dim M_n \leq \dim M$ *eventually.*

Proof. The first part is well known and can be proved

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contrapositively as follows: if $\mu \notin \sigma(T)$ then $\lambda - T_n = I - (\mu-T)^{-1}(T_n-T+\mu-\lambda)$, and hence $(\lambda-T_n)^{-1} \in [X]$ whenever

$$\|T-T_n+\mu-\lambda\| \le \frac{1}{\|(\mu-T)^{-1}\|} .$$

To prove the second part, let us consider the sequence $\{Tx_n\}$. Now T is compact implies there exists a subsequence $\left\{Tx_{n_i}\right\}$ and a vector x in X such that $Tx_{n_i} \to \mu x$ as $i \to \infty$. Since $T_{n_i}x_{n_i} = \mu_{n_i}x_{n_i}$ we have

$$\left\|\mu_{n_i}x_{n_i}-\mu x\right\| \le \left\|T_{n_i}-T\right\|\left\|x_{n_i}\right\| + \left\|Tx_{n_i}-\mu x\right\| .$$

Hence $\left\|\mu_{n_i}x_{n_i}-\mu x\right\| \to 0$ as $i \to \infty$. Now $\mu_{n_i} \ne 0$ eventually and

$$\left\|x_{n_i}-x\right\| = \left\|\frac{1}{\mu_{n_i}}\left[\mu_{n_i}x_{n_i}-\mu x-x\left(\mu_{n_i}-\mu\right)\right]\right\|$$

implies $\left\|x_{n_i}-x\right\| \to 0$ as $i \to \infty$. It follows that

$$\|Tx-\mu x\| \le \left\|Tx-T_{n_i}x\right\| + \left\|T_{n_i}x-T_{n_i}x_{n_i}\right\| + \left\|T_{n_i}x_{n_i}-\mu_{n_i}x_{n_i}\right\| + \left\|\mu_{n_i}x_{n_i}-\mu x\right\| .$$

But each term on the right hand side of the inequality tends to zero, so $Tx = \mu x$ and x is an eigenvector of T corresponding to μ .

Note that special cases of M and M_n are $M = \eta(\mu-T)$, $M_n = \eta\left(\mu_n-T_n\right)$. It remains to prove that $\dim M_n \le \dim M$ for n sufficiently large. Suppose that $\dim M_n \ge m$ for all n in an infinite set J . Then there exists x_{nk} in M_n such that

$$\|x_{nk}\| = 1 , \quad \left\|x_{nk} - \sum_{j=1}^{k-1} c_j x_{nj}\right\| \ge 1$$

for n in J , $k = 1, \ldots, m$, and all choices of c_j . Hence by the hypotheses on M and M_n and the part of the theorem already proved there

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exists $\{T_{n_i}\}$, $\{x_{n_i k}\}$, and x_k, $k = 1, \ldots, m$, in M with $x_{n_i k} \to x_k$ as $i \to \infty$, n in J. Therefore $\|x_k\| = 1$ and $\left\|x_k - \sum_{i=1}^{k-1} c_i x_i\right\| \geq 1$ for $k = 1, \ldots, m$, and all choices of c_j, so that $\dim M \geq m$. Contrapositively, if $\dim M < m$ then $\dim M_n < m$ for n sufficiently large.

LEMMA 1. *Let M and M_n, $n = 1, 2, \ldots$, be subspaces of X, and $\dim M < \infty$. If for every x in M there exists x_n in M_n such that $\|x_n - x\| \to 0$ then there exists an integer N such that $\dim M_n \geq \dim M$ for all $n \geq N$.*

Proof. Without loss of generality assume $\dim M = m$. Let $\{x_i : i = 1, \ldots, m\}$ be a basis for M. Suppose for each $i = 1, \ldots, m$ there exists x_{ni} in M_n such that $\|x_{ni} - x_i\| \to 0$ as $n \to \infty$. Let $E^m = \{(c_1, \ldots, c_m) : c_i$ is a scalar for $1 \leq i \leq m\}$. Define the compact set $D \subset E^m$ by $D = \{(c_1, \ldots, c_m) : \max|c_i| = 1\}$. Define functions f and f_n on D:

$$f(c_1, \ldots, c_m) = \left\|\sum_{i=1}^{m} c_i x_i\right\| ,$$

and

$$f_n(c_1, \ldots, c_m) = \left\|\sum_{i=1}^{m} c_i x_{ni}\right\| .$$

Note that f is continuous and, by the triangle inequality, $f_n \to f$ uniformly on D. Now it follows from the linear independence of $\{x_i : i = 1, \ldots, m\}$ that $\min_D f > 0$. Therefore there exists an integer N such that $\{x_{ni} : i = 1, \ldots, m\}$ is linearly independent and $\dim M_n \geq \dim M$ for all $n \geq N$.

The next theorem gives a necessary and sufficient condition for the

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existence of x_n in $\eta(\mu_n - T_n)$, $n = 1, 2, \ldots$, such that x_n converges to an arbitrary but fixed element x in $\eta(\mu - T)$. Pol'skiĭ [8] showed by way of an example that when $\dim\eta(\mu - T) > 1$ there may be vectors in $\eta(\mu - T)$ which can not be obtained as the limit of any sequence of eigenvectors of T_n , even with T_n compact for $n = 1, 2, \ldots$.

THEOREM 2. *Let* $T, T_n \in [X]$, T *compact, and* $\|T_n - T\| \to 0$. *Let* $\mu \neq 0$ *be an eigenvalue of* T , *and let* μ_n *be eigenvalues of* T_n *such that* $\mu_n \to \mu$. *Then the following are equivalent:*

(a) $\dim\eta(\mu_n - T_n) = \dim\eta(\mu - T)$ *eventually;*

(b) for every x *in* $\eta(\mu - T)$, $\|x\| = 1$, *there is a sequence* $\{x_n\}$ *such that* $x_n \in \eta(\mu_n - T_n)$ *and* $x_n \to x$.

Proof. We note that, in the complex case, the existence of μ_n such that $\mu_n \to \mu$ was proved by Putnam [9].

To show *(a)* implies *(b)*, first note that T is compact implies that $\dim\eta(\mu - T) = m < \infty$. Suppose *(a)* does not imply *(b)*. Then there exist a vector x in $\eta(\mu - T)$, a strictly increasing sequence of positive integers S , and a number $d > 0$ such that $\|x_n - x\| > d$ for all n in S , and for all x_n in $\eta(\mu_n - T_n)$ such that $\|x_n\| = 1$. By *(a)* for each n sufficiently large, n in S , there exists ψ_{ni} , $\left\|\psi_{ni} - \sum_{j=1}^{i-1} c_j \psi_{nj}\right\| \geq 1$ for $1 \leq i \leq m$, and for any choices of c_j . By Theorem 1 there exists a subsequence of positive integers $S_0 \subset S$ and ψ_i in $\eta(\mu - T)$ with $\psi_{ni} \to \psi_i$ as $n \to \infty$, for $1 \leq i \leq m$, $n \in S_0$. It follows that $\|\psi_i\| = 1$ for $1 \leq i \leq m$ and

$$\left\|\psi_i - \sum_{j=1}^{i-1} c_j \psi_j\right\| \geq 1$$

for any choices of c_j . Therefore $\psi_1, \ldots, \psi_m$ are linearly independent, and $\eta(\mu - T) = \operatorname{span}\{\psi_i, \ldots, \psi_m\}$. Hence there exist a_i , $1 \leq i \leq m$,

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such that $x = \sum_{i=1}^{m} a_i\psi_i$. Let $x_n = \sum_{i=1}^{m} a_i\psi_{ni}$ for n in S_0 . Then $x_n \in \eta(\mu_n - T_n)$ and $\|x_n - x\| \to 0$ as $n \to \infty$, n in S_0 which is a contradiction.

(b) implies *(a)* follows from Theorem 1 and Lemma 1.

REMARKS. 1. Theorem 2 is a generalization and an improvement of a theorem proved by Andrew and Elton [2]. In addition to the hypothesis in Theorem 2, they assumed that X is a Hilbert space and the operators T_n , $n = 1, 2, \ldots$, are compact. As a consequence they obtained a dimensional inequality $\dim\eta(\mu_n - T_n) \geq \dim\eta(\mu - T)$ in *(a)* instead of the dimensional equality $\dim\eta(\mu_n - T_n) = \dim\eta(\mu - T)$ for n sufficiently large.

2. If in Theorem 2 we assume in addition that X is a complex Banach space, then results in [6] state that for x in $\eta(\mu - T)$, $\|x\| = 1$, there exist x_n in $\eta(\mu_n - T_n)$ such that $x_n \to x$, and some sort of error estimate is also given there. Andrew [1] proved the same result by assuming, in addition to the assumptions in Theorem 2, that X is a real or complex Hilbert space, T_n is compact for each n , and μ is a simple eigenvalue of T (that is, $\dim\eta(\mu - T) = 1$).

3. Generalized eigenvectors and generalized eigenmanifolds

Assume T is compact, and μ is a non-zero eigenvalue of T . Let $D(\mu, \varepsilon)$ be a disc (or interval in the real case) centered at μ with radius ε . Choose ε so small that $D(\mu, \varepsilon) \cap D(\mu', \varepsilon) = \emptyset$ for μ' any eigenvalue of T other than μ . For $k_n < \infty$, let μ_{nj} in $D(\mu, \varepsilon)$, for $j = 1, \ldots, k_n$, be eigenvalues of T_n . We note that for a fixed n there may be an infinite number of eigenvalues of T_n in $D(\mu, \varepsilon)$. It is shown in [6] that when X is a complex Banach space and for n sufficiently large, the set of eigenvalues of T_n in $D(\mu, \varepsilon)$ is a non-empty finite set, $\{\mu_{nj} : j = 1, \ldots, k_n\}$, such that

$$\max_{1 \leq j \leq k_n} |\mu - \mu_{nj}| \to 0 .$$

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The following theorem compares the dimensions of the generalized eigenmanifolds of T with those of T_n.

LEMMA 2. *Let* $T, T_n \in [X]$, $n = 1, 2, \ldots$. *Assume* $\|T_n - T\| \to 0$, T *compact, and* μ *a non-zero eigenvalue of* T. *Suppose for each* n, μ_{nk} *is an eigenvalue of* T_n *for* $k = 1, \ldots, k_n$, *and* $\max_{1 \le k \le k_n} |\mu_{nk} - \mu| \to 0$. *Choose any non-negative integers* γ *and* γ_{nk}, $k = 1, \ldots, k_n$, *such that* $\sum_{k=1}^{k_n} \gamma_{nk} \le \gamma$. *Then for all* n *sufficiently large* $\sum_{k=1}^{k_n} \dim \eta\left[\left(\mu_{nk} - T_n\right)^{\gamma_{nk}}\right] \le \dim \eta\left[(\mu - T)^{\gamma}\right]$.

Proof. Without loss of generality, $\sum_{k=1}^{k_n} \gamma_{nk} = \gamma$ for all n. It follows from [10, p. 317] that

$$\eta\left[\prod_{k=1}^{k_n} \left(\mu_{nk} - T_n\right)^{\gamma_{nk}}\right] = \bigoplus_{k=1}^{k_n} \eta\left[\left(\mu_{nk} - T_n\right)^{\gamma_{nk}}\right],$$

and

$$\dim \eta\left[\prod_{k=1}^{k_n} \left(\mu_{nk} - T_n\right)^{\gamma_{nk}}\right] = \sum_{k=1}^{k_n} \dim \eta\left[\left(\mu_{nk} - T_n\right)^{\gamma_{nk}}\right].$$

Define μ_n, $\tilde{T}_n$ and $\tilde{T}$ by

$$\prod_{k=1}^{k_n} \left(\mu_{nk} - T_n\right)^{\gamma_{nk}} = \mu_n - \tilde{T}_n, \quad \mu_n = \prod_{k=1}^{k_n} \left(\mu_{nk}\right)^{\gamma_{nk}}, \quad (\mu - T)^{\gamma} = \mu^{\gamma} - \tilde{T}.$$

Then $\tilde{T}_n \to \tilde{T}$ and $\mu_n \to \mu^{\gamma}$. Since $\tilde{T}$ is compact, Lemma 1 implies that $\dim \eta\left(\mu_n - \tilde{T}_n\right) \le \dim \eta\left(\mu^{\gamma} - \tilde{T}\right)$ eventually. The assertion follows.

An immediate consequence of Theorem 2 and Lemma 2 is the following generalized version of Theorem 2.

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THEOREM 3. *Let* $T, T_n \in [X]$, $n = 1, 2, \ldots$. *Assume* $\|T_n - T\| \to 0$, T *compact and* μ *a non-zero eigenvalue of* T. *For* $k = 1, \ldots, k_n$, *let* μ_{nk} *be eigenvalues of* T_n *such that* $\max_{1 \le k \le k_n} |\mu_{nk} - \mu| \to 0$. *Choose any non-negative integers* γ *and* γ_{nk}, $k = 1, \ldots, k_n$ *satisfying* $\sum_{k=1}^{k_n} \gamma_{nk} \le \gamma$. *Then the following are equivalent:*

(a) $\sum_{k=1}^{k_n} \dim \eta\left(\mu_{nk} - T_n\right)^{\gamma_{nk}} = \dim \eta(\mu - T)^{\gamma}$ *eventually;*

(b) *for every* x *in* $\eta(\mu - T)^{\gamma}$, $\|x\| = 1$, *there exists a sequence* $\{x_n\}$ *such that* x_n *in* $\eta\left[\prod_{k=1}^{k_n} \left(\mu_{nk} - T_n\right)^{\gamma_{nk}}\right]$ *and* $x_n \to x$.

Applying Theorem 3 to the case in which $\mu_n \to \mu$, $T_n x_n = \mu_n x_n$ and $Tx = \mu x$ with $\mu \ne 0$. We then obtain a necessary and sufficient condition for the existence of a sequence of generalized eigenvectors $\{x_n\}$ of $\{T_n\}$ converging to an arbitrary but fixed generalized eigenvector x of T.

COROLLARY. *Let* μ *and* μ_n, $n = 1, 2, \ldots$, *be eigenvalues of* T *and* T_n *respectively, such that* $\mu \ne 0$ *and* $\mu_n \to \mu$. *Then for any positive integer* γ *the following are equivalent:*

(a) $\dim \eta\left(\mu_n - T_n\right)^{\gamma} = \dim \eta(\mu - T)^{\gamma}$ *eventually;*

(b) *for every* x *in* $\eta(\mu - T)^{\gamma}$, $\|x\| = 1$, *there exists a sequence* $\{x_n\}$ *such that* $x_n \in \eta\left[\left(\mu_n - T_n\right)^{\gamma}\right]$ *and* $x_n \to x$.

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