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On representations of selfmappings by contractions

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It is shown that any selfmapping (X, f) can be equivariantly and naturally embedded in the selfmapping of the form $(X_1, f_1) \cup (X_2, f_2) \times (Y, g)$ where f_1 and f_2 are contractive relative to suitably chosen metrics and g is a bijection with all points periodic.

1. Introduction

By a selfmapping we understand a pair (X, f) consisting of an abstract set X and a mapping $f : X \to X$. If (Y, g) is another selfmapping we say that (X, f) can be represented in (Y, g) if (X, f) can be equivariantly embedded in (Y, g); that is, if there is an injection $i : X \to Y$ such that $i \circ g = f \circ i$. The cartesian product $(X, f) \times (Y, g)$ and the disjoint union $(X, f) \cup (Y, g)$ (provided $X \cap Y = \emptyset$) are defined in the standard way. If (X, f) is a selfmapping we denote by $X^p \subset X$ the subset of X consisting of all periodic points; that is, $X^p = \{x \in X \mid f^n(x) = x \text{ for some } n \geq 1\}$. By X^* we denote the set of those $x \in X$ for which a certain iterate $f^t(x)$ belongs to X^p; that is, $X^* = \{x \in X \mid f^t(x) \in X^p \text{ for some } t \geq 0\}$ where we put $f^0(x) = x$. For $x \in X^*$ we define $t = t(x)$ (the tail-number of the element x) as the least integer t for which $f^t(x) \in X^p$, and by $n(x)$ we denote the corresponding period of $f^t(x)$. There are two special cases

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important for our investigation.

(a) $X^p = \{a\}$ is a singleton, a being the unique fixed point of (X, f). In this case the Bessaga Theorem [1] asserts that given any constant $\alpha \in (0, 1)$ there exists a metric $\rho(x, y)$ on X relative to which f is an α-contraction; that is,

$$\forall x, y \in X : \rho\big(f(x), f(y)\big) \leq \alpha\rho(x, y) .$$

(b) $X^p = \emptyset$; that is, there are no periodic points in X . In this case, taking any element a such that $a \notin X$ and defining $f(a) = a$, we extend f from X to $X \cup \{a\}$ bringing thus the case (b) under the previous one.

In order to formulate clearly our statements, we need the following.

DEFINITION. A selfmapping (X, f) is said to be an abstract contraction if there exists a metric on X relative to which f is contractive.

If (X, f) is a selfmapping and $Y \subset X$ an invariant subset of X , that is, $f(Y) \subset Y$, there is the unique factor selfmapping $(X/Y, \hat{f})$ obtained from (X, f) by identifying Y to a point, $\hat{f}$ being the mapping which makes the following diagram commutative:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j} \\ X/Y & \xrightarrow{\hat{f}} & X/Y \end{array}$$

where j is the natural projection.

The question arises whether, given a selfmapping (X, f) , one can find an invariant subset $Y \subset X$ such that (X, f) can be represented in the product $(X/Y, \hat{f}) \times (Y, g)$ where both factors have more simple structure. The purpose of this note is to show that the answer is basically affirmative, given by our

THEOREM. *If* (X, f) *is such that* $X^* = X$, *then* (X, f) *can be represented in the product* $\big(X/X^p, \hat{f}\big) \times \big(X^p, g\big)$ *where the first factor is an abstract contraction and* $g = f|X^p$ *is a bijection having all points periodic.*

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Finally, we shall apply our recent results [3] to the case when the cardinality $|X|$ of X does not exceed the power of continuum c. Introducing the concept of abstract separable contraction requiring that the metric involved in the above definition be separable we may sharpen our result for the case $|X| \le c$ requiring that the first factor be an abstract separable contraction.

2. Proof of the theorem

LEMMA 2.1. *Let* (X, f) *be a selfmapping and let* $r : X \to Y \subset X$ *be a retraction of* X *onto an* f*-invariant subset* Y *; that is,* $f(Y) \subset Y$ *and* $r|Y$ *equals the identity on* Y *, and finally assume that* r *commutes with* f *. Then* (X, f) *can be represented in the product* $(X/Y, \hat{f}) \times (Y, g)$ *where* $g = f|Y$ *.*

Proof. Denoting by j the natural projection $j : X \to X/Y$ we define the embedding $i : X \to X/Y \times Y$ by $i(x) = \big(j(x), r(x)\big)$ for $x \in X$. We obtain on one hand $i\big(f(x)\big) = \big(j\big(f(x)\big), r\big(f(x)\big)\big)$, and on the other $(\hat{f}\times g)i(x) = \big(\hat{f}\big(j(x)\big), g\big(r(x)\big)\big)$. The equality of both expressions follows from the commutativity of f and r which shows that i is equivariant. Since the one-to-one property of i is obvious, our lemma follows.

LEMMA 2.2. *Let* (X, f) *be a selfmapping and assume* $X^* = X$ *. Defining* $r : X \to X$ *by* $r(x) = f^{kn}(x)$ *where* $n = n(x)$ *and* k *is any integer such that* $kn \ge t(x)$ *, the mapping* r *is a retraction of* X *onto* X^p *and the hypotheses of Lemma* 2.1 *are satisfied for* r *by putting* $Y = X^p$ *.*

Proof. First we show that the definition of $r(x)$ is consistent; that is, independent of the number k used. Suppose we have k_1 and k_2 satisfying $k_1n \ge t(x)$ and $k_2n \ge t(x)$ and say $k_2 > k_1$. Then we have $f^{k_2n}(x) = f^{(k_2-k_1)n}\Big(f^{k_1n}(x)\Big)$ and since $f^{k_1n}(x)$ is periodic with the period n the equality $f^{k_2n}(x) = f^{k_1n}(x)$ follows. All that remains to show is the commutativity of r and f. We have

$$f\big(r(x)\big) = f\big(f^{kn}(x)\big) = f^{kn}\big(f(x)\big) = r\big(f(x)\big) ,$$

since $n(f(x)) = n(x) = n$ and $kn \geq t(x) \geq t(f(x))$, which accomplishes our lemma.

Our theorem now follows as a corollary of Lemmas 1 and 2 since $(X/X^p, \hat{f})$ has evidently the only fixed point and no other periodic points (case a), and $g = f|X^p$ is one-to-one and onto since for each $x \in X$ we have $f^n(x) = x$ where $n = n(x)$.

Going over to the general case $(X^* \subset X)$ we can formulate our result in the following form.

COROLLARY 2.1. *Any selfmapping* (X, f) *can be represented in the disjoint union*

$$(X_1, f_1) \cup (X_2, f_2) \times (Y, g) \tag{1}$$

where (X_1, f_1) *and* (X_2, f_2) *are abstract contractions and* g *is a bijection.*

Proof. Denoting by X_1 the complement of X^* in X , by f_1 the restriction of f to X_1 and putting $(X_2, f_2) = (X^*/X^p, \hat{f})$ and $Y = X^p$, the result follows from the fact that X^* and X_1 are disjoint and invariant and (X_1, f_1) is an abstract contraction (case b).

Finally we shall treat the case when $|X| \leq c$. In this case our results in [3] imply that the abstract contractions in the expression (1) can be assumed separable. Using the results of de Groot and de Vries [2] one can also metrize the factor (Y, g) in the expression (1) in such a way that Y becomes a totally bounded metric space and g a homeomorphism.

References

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