IMMERSED SURFACES AND PENCILS OF PLANES IN 3-SPACE

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1. Introduction. Let M be a compact connected boundaryless surface and $f: M \to \mathbb{R}^3$ a smooth immersion transverse to a straight line L. Thus there is an even number p of points $x \in M$ such that $f(x) \in L$. Under further transversality assumptions on f (see §3) there is a finite number q of points x of M such that the plane containing f(x) and L touches f(M) at f(x). These assumptions are mild in the sense that they hold for any f in an open dense subset of the space of smooth immersions under consideration. Suppose that the Gaussian curvature of f(M) is positive at q^+ of these points and negative at q^- , with $q = q^+ + q^-$. Then

$$p+q^+-q^-=e(M),$$
 (*)

where e(M) denotes the Euler number of M.

The proof is an application of the Poincaré-Hopf theorem (see [1], [2]). Our theorem may be interpreted as a development of the theory of horizon immersions for surfaces. In particular the main result of [3] is a consequence of our result.

We illustrate the relation (*) with a few examples. Take the standard embedding of the torus T into \mathbb{R}^3 , and let L be parallel to the axis of T. In Figure 1, we show three positions of L viewed from above.

I am grateful to Stewart Robertson for suggesting this problem and I thank him and David Chillingworth for many helpful discussions.

2. Notations. In what follows we shall be dealing with compact connected smooth $(=C^{\infty})$ boundaryless manifolds. All the maps are smooth unless otherwise stated.

Given manifolds M and N and a map $g: M \to N$ the derivative of g is denoted by $g_*: TM \to TN$, and $g_{*x}: T_xM \to T_{g(x)}N$ denotes the restriction of g_* to the tangent space to M at x. The critical set of g is denoted by C(g).

If $f: M \to \mathbb{R}^{n+1}$ (where dim M = n) is an immersion, then we denote by T_x the affine tangent n-plane to f(M) at f(x). Such an immersion induces a map $F: M \to \mathbb{R}_n^{n+1}$, where \mathbb{R}_n^{n+1} denotes the Grassmannian of affine n-planes in \mathbb{R}^{n+1} . By the Gaussian curvature of f(M) at f(m) we mean the Gaussian curvature of f(U) at f(m), where U is an oriented open neighbourhood of m and $f \mid U$ is an embedding.

We may assume that the line L in \mathbb{R}^3 is defined by $x_1 = x_2 = 0$. The pencil \tilde{L} of 2-planes containing L, as a submanifold of \mathbb{R}^3 , is diffeomorphic to the 1-dimensional real projective space P^1 . For P^1 we shall consider the two standard charts $\psi: V \to \mathbb{R}$, $\psi': V' \to \mathbb{R}$, where

$$V(\text{resp. } V') = \{\pi \varepsilon P^1 \mid \pi \text{ is defined by } (1, x)(\text{resp.}(x, 1))\}$$

and $\psi(\pi) = x$, $\psi'(\pi) = x$.

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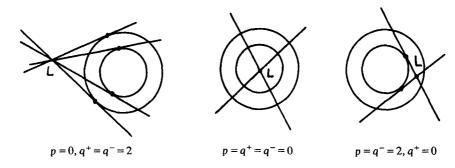


Figure 1

For any immersion $f: M \to \mathbb{R}^3$, with dim M = 2, we shall use three types of chart α , β , γ for M where

$$f \circ \alpha^{-1}(x, y) = (x, y, g(x, y)),$$

 $f \circ \beta^{-1}(x, y) = (x, h(x, y), y),$
 $f \circ \gamma^{-1}(x, y) = (k(x, y), x, y).$

Finally $I_2(F, \tilde{L})$ will denote the intersection number mod(2) of F with \tilde{L} .

- 3. The main result. The proof of the relation stated in §1 will occupy the whole of this section. We start with two elementary observations.
 - 3.1. Suppose $f^{-1}(L) \neq \emptyset$. Then $f \not h L$ at $m \in f^{-1}(L)$ iff T_m intersects L in a point.
 - 3.2. Suppose $F^{-1}(\tilde{L}) \neq \emptyset$. Then $F \not h \tilde{L}$ at $m \in F^{-1}(\tilde{L})$ iff
 - (a) $f(m) \notin L$,
 - (b) f(M) has non-zero Gaussian curvature at f(m).

Bearing in mind 3.2, we suppose, from now on, that $F \Phi \tilde{L}$. Thus the condition $f \Phi L$ follows automatically (see §1).

Let $\Delta = f^{-1}(L)$. Define a map $\phi: M \setminus \Delta \to P^1$ by associating to $m \in M \setminus \Delta$ the 2-plane containing f(m) and L. The map ϕ is smooth and $m \in C(\phi)$ iff $T_m \in \tilde{L}$. Since $F \not h \tilde{L}$, the set $C(\phi) \cup \Delta$ is finite.

Consider the vector fields $\operatorname{grad}(\psi \circ \phi)$ and $\operatorname{grad}(\psi' \circ \phi)$. We use these to obtain a well-defined vector field Z on $M \setminus \{\Delta \cup C(\phi)\}$ as follows:

$$Z(m) = \begin{cases} \operatorname{grad}(\psi \circ \phi)(m) / || \operatorname{grad}(\psi \circ \phi)(m)|| & \text{if } m \in \phi^{-1}(V), \\ -\operatorname{grad}(\psi' \circ \phi)(m) / || \operatorname{grad}(\psi' \circ \phi)(m)|| & \text{if } m \in \phi^{-1}(V'). \end{cases}$$

Take a smooth function $\gamma': M \to \mathbb{R}$, such that

$$\gamma'(x) = 0$$
 if $x \in \Delta \cup C(\phi)$,
 $\gamma'(x) > 0$ if $x \in M \setminus \{\Delta \cup C(\phi)\}$.

Using γ' we define another vector field Y by

$$Y(m) = \begin{cases} \gamma'(m)Z(m) & \text{if } m \in M \setminus \{\Delta \cup C(\phi)\}, \\ 0 & \text{if } m \in \Delta \cup C(\phi). \end{cases}$$

It is not hard to show that Y is continuous. If $\phi(m)$ is a regular value of ϕ , then Y(m) is orthogonal to the contour of ϕ passing through m.

As we see, we are now in a position which allows us to use the Poincaré-Hopf theorem. To calculate the indices at the zeros of Y we shall distinguish two cases:

- (a) m_0 is a zero of Y and $m_0 \in C(\phi)$;
- (b) m_0 is a zero of Y and $m_0 \in \Delta$.

Case (a). Due to the transversality condition we have imposed, both $\psi \circ \phi$ and $\psi' \circ \phi$ are Morse functions. Then the index of m_0 as a zero of Y can be calculated by looking at the index of m_0 as a critical point of either $\psi \circ \phi$ or $-(\psi' \circ \phi)$. It follows from standard calculations that the index is 1 or -1 according as the Gaussian curvature at $f(m_0)$ is positive or negative.

Case (b). Let $m_0 \in \Delta$. As $f(m_0) \in L$ and $f \not \Lambda L$ it is possible to find a chart $\alpha : U \to U'$ such that $f \mid U$ is an embedding, $f \circ \alpha^{-1}(x, y) = (x, y, g(x, y))$ and no other point of $\Delta \cup C(\phi)$, other than m_0 , is in U. The restriction $f \mid U$ is transversal to any $\pi \in \tilde{L}$. For any $m \in U \setminus \{m_0\}$, we have that Y(m) is orthogonal to the contour of $\phi \mid U \setminus \{m_0\}$ through m, and this contour is $(f \mid U)^{-1}(\pi) \setminus \{m_0\}$, where $\phi(m) = \pi$. Moreover $(f \mid U)^{-1}(\pi)$ is a 1-dimensional submanifold of M.

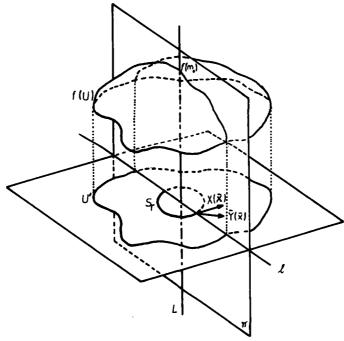


Figure 2

If π is as shown in Figure 2, $(f \mid U)^{-1}(\pi)$ is diffeomorphic to $U' \cap l$, where $l = (\mathbb{R}^2 \times \{0\}) \cap \pi$. Using the chart α we get the "pullback" \bar{Y} of Y, defined on U'. This vector field is never zero except at the origin. The important remark to make about \bar{Y} is that because Y(m) is orthogonal to $i_{*m}(T_m((f \mid U)^{-1}(\pi)))$ then

$$\bar{\mathbf{Y}}(\bar{\mathbf{x}}) \notin i_{\mathbf{x}\bar{\mathbf{y}}}(T_{\bar{\mathbf{y}}}(U' \cap l)),$$

for any m such that $\alpha(m) = \bar{x} \neq (0, 0)$. Here we have denoted the obvious inclusions by i, and $\bar{x} = (x, y)$.

Let S_r be a small sphere round the origin such that $S_r \subset U'$. We define a vector field X of unit norm on S_r and tangent to S_r such that at a particular point \bar{x} , $X(\bar{x})$ and $\bar{Y}(\bar{x})$ form an acute angle. Looking at the vector fields as maps into \mathbb{R}^2 ,

$$X: S_r \to S^1 \subset \mathbb{R}^2$$
, $\bar{Y} \mid S_r: S_r \to \mathbb{R}^2$,

we see that they are homotopic, a homotopy $H: S_r \times [0, 1] \to \mathbb{R}^2$ being defined by

$$H(\bar{x}, t) = tX(\bar{x}) + (1-t)\bar{Y}(\bar{x}).$$

The map H is never zero and therefore induces a homotopy between X and $\bar{Y}_1: S_r \to S^1$, where $\bar{Y}_1(\bar{x}) = \bar{Y}(\bar{x}) / \|\bar{Y}(\bar{x})\|$. Consequently the maps X and \bar{Y}_1 have the same degree and the degree of X is 1. Hence the index of Y at m_0 is 1.

Having calculated the indices, we can deduce the relation (*) in §1 immediately.

4. Consequences of the main result. The theorem we have just proved yields the following two corollaries.

COROLLARY 4.1. Assume $F \Phi \tilde{L}$. Then $I_2(F, \tilde{L}) \equiv e(M) \pmod{2}$.

Proof. By intersection theory, $I_2(f, L) = 0$. The result follows now from (*) in §1.

The next corollary is just the theorem on the existence of horizon maps for surfaces [3].

COROLLARY 4.2. If f is a horizon immersion then M is diffeomorphic to S^2 , $S^1 \times S^1$ or the Klein bottle. If M is diffeomorphic to S^2 then $\#f^{-1}(L) = 2$; otherwise, $\#f^{-1}(L) = 0$.

Proof. If f is a horizon immersion then $F \Phi \tilde{L}$ and $F^{-1}(\tilde{L}) = \emptyset$. Because $\#f^{-1}(L)$ is even it follows from (*) in §1 that e(M) is even and greater than or equal to zero.

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