Preface

Since the first systematic exposition of the theory of involutive systems of vector fields ([**T5**]) was published almost 15 years ago, the subject has undergone considerable development and many new applications have been found. Systems of vector fields arise as a local basis of an involutive sub-bundle $\mathcal{V}$ of the complexified tangent bundle $\mathbb{C}T\mathcal{M}$. *Involutivity* of $\mathcal{V}$ means that the commutation bracket of two smooth sections of $\mathcal{V}$ must also be a section of $\mathcal{V}$. Examples of involutive structures $(\mathcal{M}, \mathcal{V})$ include foliations, complex structures, and CR structures. In these examples, $\mathcal{V} \cap \overline{\mathcal{V}}$ has constant rank. However, in recent work on integral geometry, natural examples of involutive structures have arisen for which the rank of $\mathcal{V} \cap \overline{\mathcal{V}}$ changes from point to point ([**BE**], [**BEGM**], and [**EG1**]). In the works [**BE**] and [**BEGM**], the cohomology of involutive structures is a key ingredient. Examples of involutive structures where the rank of $\mathcal{V} \cap \overline{\mathcal{V}}$ is not constant also arise naturally, for instance, on the tangent bundle of symmetric spaces (see [**Sz**] and the references therein) or in the study of the generalized similarity principle for the equation

$$Lu = Au + B\overline{u}$$

where L is a planar complex vector field not necessarily elliptic, which is intimately linked to the study of infinitesimal deformations of surfaces in $\mathbb{R}^3$ with non-negative curvature (see [**Me3**], [**Me4**], and the references therein).

This book introduces the reader to a number of results on systems of vector fields with complex-valued coefficients defined on a smooth manifold $\mathcal{M}$. Most of the time, it will be assumed that the involutive structure $(\mathcal{M}, \mathcal{V})$ is *locally integrable*. The latter means that the orthogonal of $\mathcal{V}$, which is a sub-bundle T' of the complexified cotangent bundle $\mathbb{C}T^*\mathcal{M}$, is locally generated by exact differentials. When $(\mathcal{M}, \mathcal{V})$ is locally integrable, each point has a neighborhood U such that if $\{L_1, \ldots, L_n\}$ are n smooth vector fields that form a basis of $\mathcal{V}$ over U, then we can find $m = \dim \mathcal{M} - n$ smooth, complex-valued functions $Z_1, \ldots, Z_m$ which are solutions of the equations

$$L_j h = 0, \quad 1 \leq j \leq n \tag{1}$$

and whose differentials are linearly independent over $\mathbb{C}$ at each point of U. The m functions $Z = (Z_1, \ldots, Z_m)$ are sometimes referred to as a complete set of *first integrals* in the neighborhood U.

In 1981, in **[BT1]**, Baouendi and Treves proved that in a locally integrable structure, each solution of (1) can be locally approximated by a sequence $P_k(Z)$ where the P_k are holomorphic polynomials of m variables and $Z = (Z_1, \ldots, Z_m)$ is a complete set of first integrals. This approximation theorem has enabled several researchers to use the methods of complex analysis, harmonic analysis, and partial differential equations to study many problems on locally integrable structures. These problems include: the local and microlocal regularity of the solutions of (1); the determination of sets of uniqueness for solutions of (1); the solvability of the differential complex associated with the structure $(\mathcal{M}, \mathcal{V})$; and many other properties of the solutions of (1).

This book attempts to present a systematic treatment of some of these results in a way that is accessible to graduate students with a background in real analysis, one complex variable, and basic introductions to several complex variables and linear PDEs including the theory of distributions.

Chapter I introduces the basic concepts in the theory of involutive and locally integrable structures. Special classes of involutive structures such as complex structures, CR structures, elliptic structures, and real analytic structures are identified and examples are provided. Useful local representations both for general involutive and locally integrable structures are also discussed. A proof of the Newlander–Nirenberg theorem is presented in the appendix to Chapter I. Chapter II is devoted to the approximation theorem of Baouendi and Treves. It is shown that the approximation is valid in many function spaces used in analysis: the Lebesgue spaces L^p, $1 \leq p < \infty$; Sobolev spaces; Hölder spaces; and localizable Hardy spaces h^p, $0 < p < \infty$. Applications to uniqueness in the Cauchy problem and extendability of CR functions are also included. Chapter III presents a variety of results on unique continuation for solutions and approximate solutions in a locally integrable structure $(\mathcal{M}, \mathcal{V})$. The orbits of Sussmann associated with the real parts $\Re L$ of the smooth sections of $\mathcal{V}$ play a crucial role in many problems, including the study of unique continuation and the chapter includes a discussion of some of the properties of these orbits. Chapter IV provides a detailed treatment of locally solvable vector fields. In the first part of the chapter, where the focus is on planar vector fields, the solvability condition $(\mathcal{P})$ of Nirenberg and Treves is discussed and a priori estimates are proved in L^p and in a mixed norm that involves the Hardy space $h^1(\mathbb{R})$. A duality argument is then used to derive local solvability results in L^p, $1 < p < \infty$ and in $L^\infty[\mathbb{R}; \mathrm{bmo}(\mathbb{R})]$. The chapter also includes sections on the sufficiency and necessity of condition $(\mathcal{P})$ for local solvability in higher dimensions. The first part of Chapter V introduces certain submanifolds in an involutive structure $(\mathcal{M}, \mathcal{V})$ which are important in the study of solutions. These submanifolds are generalizations of the totally real and generic CR submanifolds encountered in CR manifolds. The second part of the chapter introduces the FBI transform first in $\mathbb{R}^n$ and then in a locally integrable structure. The FBI transform is then applied to derive edge-of-the-wedge type results. It is also applied to study the microlocal singularities of the solutions of a first-order nonlinear PDE and a generalization of the F. and M. Riesz theorem. Chapter VI studies some boundary properties of the solutions of locally integrable vector fields. These properties include the existence of a trace at the boundary, pointwise convergence of solutions to their boundary values, and the validity of Hardy space-like properties. Chapter VII describes the differential complex attached to a general involutive structure. An invariant definition of this complex is followed by a useful representation in appropriate coordinates. An approximate Poincaré Lemma for locally integrable structures is also proved in the chapter. Chapter VIII deals with the local solvability theory of the undetermined systems of partial differential equations naturally associated with a locally integrable structure, that is, the cohomology theory of its differential complex. Necessary and sufficient conditions are studied in some detail when the structure is analytic, or elliptic, or has corank one. Concerning the latter class, a thorough exposition of the geometric characterization of local solvability in degree one for real analytic structures is presented.

Finally we conclude with an epilogue which summarizes some of the results obtained in recent years on diverse areas such as the similarity principle, Mizohata structures, and hyperfunction solutions in hypoanalytic manifolds. Two applications of the similarity principle are described. The first application concerns uniqueness in the Cauchy problem for a class of semilinear equations. The second application involves the theory of bending of surfaces.

There are numerous interesting results on complex vector fields and involutive structures that have been obtained since the publication of [**BT1**] and which are not covered in this book. The authors have selected the material with which they have had first-hand experience. In the notes at the end of each chapter, we indicate some related works and provide additional references.

The reader is referred to [**BER**] for a further reference on CR manifolds and to [**T5**] for additional topics on involutive structures.

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