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COMMUTATIVE ALGEBRAS IN DRINFELD CATEGORIES OF ABELIAN LIE ALGEBRAS

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Abstract We describe (braided-) commutative algebras with non-degenerate multiplicative form in certain braided monoidal categories, corresponding to abelian metric Lie algebras (so-called Drinfeld categories). We also describe local modules over these algebras and classify commutative algebras with a finite number of simple local modules.

Keywords: monoidal category; Drinfeld category; metric Lie algebra; local module

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1. Introduction

Motivated by applications for representation theory of vertex operator algebras, we systematically study commutative algebras and their local modules in Drinfeld categories of abelian metric Lie algebras.

In [**2**], Drinfeld associated to a non-degenerate invariant bilinear form (*metric*) on a Lie algebra an infinitesimal deformation of the canonical tensor structure on its representation category. This infinitesimal deformation is no longer a symmetric monoidal category. The deformed commutativity constraint is only a braiding. In [**2**], it was also explained how these (*Drinfeld*) categories are related to representation categories of the quantizations of Lie algebras.

It is well known that a metric on a Lie algebra gives rise to an *affinization* of (a central extension of the Lie algebra of Laurent polynomials with coefficients in) the original Lie algebra. The category of representations of the affinization has a tensor structure given by the so-called *fusion product*. It was explained in [**5**, **6**] how the induction functor links the Drinfeld category of a simple Lie algebra (with the Cartan–Killing metric) with the representation category of its affinization. In this case the infinitesimal deformation form [**2**]

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becomes a global one (on a certain subcategory of the Drinfeld category). Another class of metric Lie algebras for which the infinitesimal deformation form [**2**] becomes global is the class of abelian metric Lie algebras. Corresponding Drinfeld categories are related to categories of modules over Heisenberg vertex operator algebras [**8**].

It is known [**7**] that commutative algebras in the representation category of a vertex operator algebra correspond to extensions of this vertex operator algebra. Moreover, the category of representations of an extended algebra coincides with the category of so-called *local* modules over the corresponding commutative algebra.

Here we study commutative algebras and their local modules in Drinfeld categories of abelian Lie algebras. After recalling basic facts about commutative algebras in braided monoidal categories (§ 2) and Drinfeld categories (§ 3) we classify commutative algebras which possess a non-degenerate bilinear form, compatible with the multiplication and have trivial invariants (§ 4). We prove (Theorem 4.5) that such algebras correspond to subgroups in the abelian metric Lie algebra, such that the restriction of the metric is integer valued and even. Then we turn to local modules over commutative algebras (§ 5). We show that the category of local modules has a grading compatible with the tensor product (Proposition 5.3) and study the trivial component of this grading (Proposition 5.4). We also construct invertible modules sitting in every non-trivial graded component (Proposition 5.7). All this, together with some technical tools developed in the appendix, allows us to classify commutative algebras with a finite number of simple local modules (§ 6).

If not stated otherwise, all linear algebra constructions will be assumed linear over the ground field k, an algebraically closed field of characteristic zero.

2. Commutative algebras in braided categories and their local modules

In this preliminary section we recall basic facts about commutative algebras in braided monoidal categories and their modules. We denote by I the identity morphism and by 1 the unit object (in a monoidal category).

An (associative, unital) *algebra* in a monoidal category $\mathcal{C}$ is a triple (A, μ, ι) consisting of an object $A \in \mathcal{C}$ together with a *multiplication* $\mu\colon A \otimes A \to A$ and a *unit* map $\iota\colon 1 \to A$, satisfying *associativity*

$$\mu(\mu \otimes I) = \mu(I \otimes \mu)$$

and *unit*

$$\mu(\iota \otimes I) = I = \mu(I \otimes \iota)$$

axioms. Here we omit associativity and unit constraints of the monoidal category $\mathcal{C}$. Where it will not cause confusion we shall be talking about an algebra A, suppressing its multiplication and unit maps.

A left *module* over an algebra A is a pair (M, ν), where M is an object of $\mathcal{C}$ and $\nu\colon A \otimes M \to M$ is a morphism (*action map*), such that

$$\nu(\mu \otimes I) = \nu(I \otimes \nu), \qquad \nu(\iota \otimes I) = I.$$

A *homomorphism* of left A-modules $M \to N$ is a morphism $f\colon M \to N$ in $\mathcal{C}$ such that

$$\nu_N(I \otimes f) = f\nu_M.$$

Left modules over an algebra $A \in \mathcal{C}$ together with module homomorphisms form a category ${}_A\mathcal{C}$.

Now let $\mathcal{C}$ be a braided monoidal category with the braiding $c_{X,Y}\colon X \otimes Y \to Y \otimes X$ (see [**4**] for a definition). An algebra A in $\mathcal{C}$ is *commutative* if $\mu c_{A,A} = \mu$.

Let $\mathcal{C}$ be a cocomplete monoidal category with a colimit-preserving tensor product. It was shown in [**10**] that the category ${}_A\mathcal{C}$ of left modules over a commutative algebra A is monoidal with respect to the tensor product $M \otimes_A N$ over A, which can be defined by a coequalizer

$$M \otimes_A N \longleftarrow M \otimes N \;\overset{1\nu_N}{\underset{(\nu_M 1)(c_{M,A}1)}{\longleftleftarrows}}\; M \otimes A \otimes N$$

A left module (M, ν) over a commutative algebra A is *local* if and only if the diagram

$$\begin{array}{ccc} A \otimes M & \xrightarrow{\ \nu\ } & M \\ {\scriptstyle c_{A,M}}\downarrow & & \uparrow{\scriptstyle \nu} \\ M \otimes A & \xrightarrow{\ c_{M,A}\ } & A \otimes M \end{array}$$

commutes. Denote by ${}_A\mathcal{C}^{\mathrm{loc}}$ the full subcategory of ${}_A\mathcal{C}$ consisting of local modules. The following result was established in [**10**] (see also [**7**]).

Proposition 2.1. *The category ${}_A\mathcal{C}^{\mathrm{loc}}$ is a full monoidal subcategory of ${}_A\mathcal{C}$. Moreover, the braiding in $\mathcal{C}$ induces a braiding in ${}_A\mathcal{C}^{\mathrm{loc}}$.*

We call a commutative algebra A in a balanced category $\mathcal{C}$ *balanced* if $\theta_A = 1_A$, where θ is the balancing twist in $\mathcal{C}$. It was proved in [**7**] (see also [**3**]) that the balancing twist θ_M of a local module M over a balanced algebra A is a homomorphism of A-modules, i.e. the category ${}_A\mathcal{C}^{\mathrm{loc}}$ of local modules over a balanced algebra is naturally balanced.

3. Drinfeld categories

In this second preliminary section we recall the construction of our categories of interest, the so-called Drinfeld categories, associated to metric (and, more generally, Casimir) Lie algebras.

A Lie algebra $\mathfrak{g}$ is called *Casimir* if it is equipped with a $\mathfrak{g}$-invariant symmetric bi-tensor $\Omega \in \mathfrak{g}^{\otimes 2}$:

$$[x \otimes 1 + 1 \otimes x, \Omega] = 0.$$

A finite-dimensional Lie algebra $\mathfrak{g}$ is *metric* if it is equipped with a non-degenerate bilinear form $(\cdot,\cdot)\colon \mathfrak{g} \otimes \mathfrak{g} \to k$, which is $\mathfrak{g}$-invariant

$$([x, y], z) + (y, [x, z]) = 0, \quad x, y, z \in \mathfrak{g},$$

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and symmetric

$$(x,y)=(y,x),\quad x,y\in\mathfrak{g}.$$

We shall denote by $\Omega\in\mathfrak{g}\otimes\mathfrak{g}$ its *Casimir* element, i.e. a unique element with the property

$$\sum_i \Omega_i^1(\Omega_i^2,x)=x=\sum_i(x,\Omega_i^1)\Omega_i^2,$$

where $\sum_i \Omega_i^1\otimes\Omega_i^2=\Omega$. In particular, Ω can be written as $\sum_i u_i\otimes u_i$, where $\{u_i\}$ is an orthonormal basis of $\mathfrak{g}$, i.e. $(u_i,u_j)=\delta_{i,j}$. Note that a metric Lie algebra is a Casimir Lie algebra.

For example, a simple Lie algebra has a unique (up to a scalar metric) structure, given by the Cartan–Killing form (the trace form in the adjoint representation). Another example comes from an abelian Lie algebra with a non-degenerate symmetric bilinear from (which in this case is automatically invariant). It can be shown that a general metric Lie algebra is an orthogonal sum of a semisimple Lie algebra and a solvable Lie algebra, and that a solvable metric Lie algebra is an iterated double extension of a one-dimensional metric Lie algebra (see [**9**] for details).

It was shown in [**2**] that over the formal power series $k[\![h]\!]$ the category of representations $\mathcal{R}\mathrm{ep}(\mathfrak{g})$ of a Casimir Lie algebra $\mathfrak{g}$ can be equipped with a structure of braided monoidal category, where the tensor product is the original tensor product of representations ($\mathfrak{g}$-modules) and the braiding given by

$$c_{M,N}(m\otimes n)=\mathrm{e}^{\pi \mathrm{i} h\Omega}(n\otimes m),\quad m\in M,\ n\in N,\ M,N\in\mathcal{R}\mathrm{ep}(\mathfrak{g}). \tag{3.1}$$

It turns out that, for the coherence axioms for braiding to work, one also needs to deform the associativity constraint:

$$\alpha_{L,M,N}(l\otimes(m\otimes n))=\Phi((l\otimes m)\otimes n),\quad l\in L,\ m\in M,\ n\in N,\ L,M,N\in\mathcal{R}\mathrm{ep}(\mathfrak{g}). \tag{3.2}$$

A solution for Φ (the so-called *Drinfeld associator*) was found in [**2**], which is a formal (non-commutative) power series in $h\Omega_{12}$ and $h\Omega_{23}$. Here $\Omega_{12}=\Omega\otimes 1$ and $\Omega_{23}=1\otimes\Omega$ are elements of $U(\mathfrak{g})^{\otimes 3}$. Moreover, it was shown in [**2**] that $\Phi(h\Omega_{12},h\Omega_{23})$ is the exponent of a formal power Lie series in $h\Omega_{12}$ and $h\Omega_{23}$. In particular, it depends only on iterated commutators of $h\Omega_{12}$ and $h\Omega_{23}$.

To be precise, it is not enough to work with $k[\![h]\!]$-modules with $\mathfrak{g}$-action. To make (3.1) and (3.2) work we need to consider the category $\mathcal{R}\mathrm{ep}_h(\mathfrak{g})$ of modules over $U(\mathfrak{g})[\![h]\!]$. We denote by $\mathcal{C}_h(\mathfrak{g},\Omega)$ the category $\mathcal{R}\mathrm{ep}_h(\mathfrak{g})$ with the braided monoidal structure given by (3.1), (3.2). We shall call it the *Drinfeld category* corresponding to a Casimir Lie algebra $\mathfrak{g}$. Note that the Drinfeld category $\mathcal{C}_h(\mathfrak{g},\Omega)$ comes equipped with a balancing structure, with the balancing twist defined by

$$\theta_M(m)=\mathrm{e}^{\pi\mathrm{i}h\omega}m,\quad m\in M,\ M\in\mathcal{R}\mathrm{ep}(\mathfrak{g}).$$

Here $\omega=\mu(\Omega)\in U(\mathfrak{g})$ is the quadratic Casimir (the image of Ω under the multiplication map $\mu\colon U(\mathfrak{g})\otimes U(g)\to U(\mathfrak{g})$ of the universal enveloping algebra of $\mathfrak{g}$). Indeed, the identity

$$\Delta(\omega)-\omega\otimes 1-1\otimes\omega=2\Omega$$

implies the balancing axiom.

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Often (after suitably redefining $\mathcal{R}\mathrm{ep}(\mathfrak{g})$) the formal parameter h in Drinfeld category $\mathcal{C}_h(\mathfrak{g}, \Omega)$ can be specialized to a number $c \in k$. For example, if $\mathfrak{g}$ is a simple Lie algebra, then, restricting to the category $\mathcal{R}\mathrm{ep}_{\mathrm{fd}}(\mathfrak{g})$ of finite-dimensional $\mathfrak{g}$-modules, one can set h to be any non-rational number $c \in k$ and define the braided monoidal category $\mathcal{C}_c(\mathfrak{g}, \Omega)$. Moreover, the category $\mathcal{C}_c^{\mathrm{fd}}(\mathfrak{g}, \Omega)$ is equivalent to the category of finite-dimensional representation of the quantum universal enveloping algebra $U_q(\mathfrak{g})$, where $q = \mathrm{e}^{\pi \mathrm{i}/c}$ [**5**, **6**].

Another series of examples is provided by nilpotent metric Lie algebras. In that case, the Drinfeld associator is the exponent of a Lie polynomial in $h\Omega_{12}$ and $h\Omega_{23}$. In addition to the phase in (3.1), the action of this exponent is well defined on finite-dimensional $\mathfrak{g}$-modules, and hence on modules, with every cyclic submodule being finite dimensional. The following provides an intrinsic characterization of such $\mathfrak{g}$-modules.

We call a module V over a Lie algebra $\mathfrak{g}$ *locally finite* if, for any $x \in \mathfrak{g}$ and for any $v \in V$, there is a polynomial $f(t)$ such that $f(x)v = 0$.

Lemma 3.1. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. A $\mathfrak{g}$-module M is locally finite if and only if for any $m \in M$ the cyclic submodule $U(\mathfrak{g})m$ is finite dimensional.*

Proof. If $U(\mathfrak{g})m$ is finite dimensional, then for any $x \in g$ a certain linear combination of $x^i m$ is zero.

Conversely, choose a basis $\{x_1, \dots, x_n\}$ in $\mathfrak{g}$. By the Poincaré–Birkhoff–Witt Theorem $U(\mathfrak{g})m$ is spanned by $\{x_1^{i_1} \cdots x_n^{i_n} m \mid i_1, \dots, i_n \in \mathbb{Z}_{\geqslant 0}\}$. We shall show that i_j, $j = 1, \dots, n$, are all bounded. By local finiteness of a $\mathfrak{g}$-module M, the span of $\{x_n^{i_n} m\}$ is finite dimensional, i.e. i_n is bounded for, say, $i_n \leqslant s_n$. For any $i_n = 1, \dots, s_n$ the span $\{x_{n-1}^{i_{n-1}} x_n^{i_n} m\}$ is again finite dimensional. By continuing this argument, we get that $\{x_1^{i_1} \cdots x_n^{i_n} m \mid i_1, \dots, i_n \in \mathbb{Z}_{\geqslant 0}\}$ is finite dimensional. □

In particular, the tensor product of locally finite modules is locally finite. For a nilpotent Lie algebra $\mathfrak{g}$ we shall denote by $\mathcal{C}_c(\mathfrak{g}, \Omega)$ the category of locally finite $\mathfrak{g}$-modules with the braided monoidal structure given by (3.1), (3.2), where h is replaced by $c \in k^\times$. Note that $\mathcal{C}_c(\mathfrak{g}, \Omega) = \mathcal{C}_1(\mathfrak{g}, c\Omega)$. In particular, for abelian $\mathfrak{g}$ the category $\mathcal{C}_c(\mathfrak{g}, \Omega)$ does not depend on c (since all metric structures on an abelian Lie algebra are equivalent). Thus, for abelian $\mathfrak{g}$ we shall denote $\mathcal{C}_c(\mathfrak{g}, \Omega)$ simply by $\mathcal{C}(\mathfrak{g}, \Omega)$.

4. Commutative algebras in Drinfeld categories of abelian Lie algebras

Let $\mathfrak{h}$ be an abelian metric Lie algebra with the non-degenerate form $(\cdot, \cdot)$ and the Casimir Ω. Let $\mathcal{C}(\mathfrak{h}, \Omega)$ be the corresponding Drinfeld category. Since for abelian Lie algebra $\mathfrak{h}$ the Drinfeld associator (3.2) is trivial (due to the fact that Φ is a formal series in commutators), $\mathcal{C}(\mathfrak{h}, \Omega)$ is the category of locally finite $\mathfrak{h}$-modules with ordinary tensor product and associativity constraint and with the braiding defined by

$$c_{M,N}(m \otimes n) = \mathrm{e}^{\pi \mathrm{i} \Omega}(n \otimes m), \quad m \in M,\ n \in N. \tag{4.1}$$

Lemma 4.1. *Any locally finite* $\mathfrak{h}$*-module* V *can be written as a sum*

$$V = \bigoplus_{x\in\mathfrak{h}} V_x,$$

where $V_x = \{v \in V \mid (y-(y,x)1)^{n_y}v = 0$ *for all* $y \in \mathfrak{h}\}$ *is a generalized eigenspace with the character* $(x,\cdot) \in \mathfrak{h}^*$*.*

Proof. Indeed, for any $v \in V$, the space $\tilde{V} = U(\mathfrak{h})v$ is finite dimensional by Lemma 3.1. Hence, commuting operators $x, x \in \mathfrak{h}$ on $\tilde{V}$ have simultaneous generalized eigenspace decomposition: $\tilde{V} \subset \bigoplus_{x\in\mathfrak{h}} V_x \subset V$. In particular, $v \in \bigoplus_{x\in\mathfrak{h}} V_x$, which implies the statement. □

We call the subset

$$\mathfrak{l}(V) = \{x \in \mathfrak{h} \mid V_x \neq 0\} \subset \mathfrak{h}$$

the *support* of V.

Lemma 4.2. *For* $V, U \in \mathcal{C}(\mathfrak{h}, \Omega)$,

$$(V \otimes U)_x = \bigoplus_{y+z=x} V_y \otimes U_z.$$

Proof. By Lemma 3.1 $V \otimes U$ is locally finite. By Lemma 4.1 we have

$$V \otimes U = \bigoplus_{x\in\mathfrak{h}} (V \otimes U)_x.$$

On the other hand, $V = \bigoplus_{y\in\mathfrak{h}} V_y$, $U = \bigoplus_{z\in\mathfrak{h}} U_z$, and hence

$$V \otimes U = \bigoplus_{y,z\in\mathfrak{h}} V_y \otimes U_z.$$

Taking into account the fact that, for any $v \in V$, $u \in U$, $x, h \in \mathfrak{h}$ and any $y, z \in \mathfrak{h}$ such that $x = y + z$, we have

$$(h-(x,h)1)^N(v \otimes u) = \sum_{k=0}^{N} \binom{N}{k} (h-(y,h)1)^k v \otimes (h-(z,h)1)^{N-k} u,$$

we conclude that $V_y \otimes U_z \subset (V \otimes U)_x$. Since the subspaces $(V \otimes U)_x$ and $(V \otimes U)_{x'}$ do not intersect if $x \neq x'$, the statement follows. □

An algebra A in the category $\mathcal{C}(\mathfrak{h}, \Omega)$ is an associative algebra with an $\mathfrak{h}$-action by derivations. It follows from Lemma 4.2 that the decomposition

$$A = \bigoplus_{x\in\mathfrak{l}(A)} A_x \tag{4.2}$$

into the sum of generalized eigenspaces is an algebra grading, i.e. $A_xA_y \subset A_{x+y}$. A bilinear form $\beta\colon A \otimes A \to k$ on an algebra A is called *multiplicative* if

$$\beta(ab, c) = \beta(a, bc) \quad \text{for all } a, b, c \in A.$$

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Proposition 4.3. *Let A be an algebra in the category $\mathcal{C}(\mathfrak{h},\Omega)$ with a non-degenerate multiplicative bilinear form and with the trivial subalgebra of invariants $A^{\mathfrak{h}} = k$. Then $\mathfrak{l} = \mathfrak{l}(A)$ is a subgroup of $\mathfrak{h}$ and A is isomorphic to a skew group algebra $k[\mathfrak{l},\alpha]$ for some 2-cocycle $\alpha\colon \mathfrak{l}\times\mathfrak{l}\to k^{\times}$, i.e. A is spanned by e_x, $x\in\mathfrak{l}$, with multiplication*

$$e_x e_y = \alpha(x,y)e_{x+y}. \tag{4.3}$$

The $\mathfrak{h}$-action has the form

$$y(e_x) = (x,y)e_x \quad \text{for all } x\in\mathfrak{l},\ y\in\mathfrak{h}. \tag{4.4}$$

Proof. We shall show first that the presence of a non-degenerate multiplicative bilinear form together with the condition $A^{\mathfrak{h}} = k$ force homogeneous elements with respect to the grading (4.2) to be invertible. Indeed, multiplicativity of a non-degenerate form implies that it is compatible with the grading, in particular for any $x\in\mathfrak{l}$ the restriction $\beta\colon A_x\otimes A_{-x}\to k$ is non-degenerate. On the other hand, multiplicativity of β and the condition $A^{\mathfrak{h}} = k$ imply that, for $a\in A_x$ and $b\in A_{-x}$,

$$ab = \beta(ab,1)1 = \beta(a,b)1.$$

By non-degeneracy of β for any non-zero a there exists b such that $\beta(a,b)=1$. Hence, $b = a^{-1}$. Now we can show that generalized eigenspaces A_x are at most one dimensional. Indeed, for non-zero $a,b\in A_x$ we have that $b^{-1}a = \lambda 1$ for some $\lambda\in k$. Thus, $a=\lambda b$.

By choosing non-zero elements $e_x\in A_x$ for $x\in\mathfrak{l}$, we see that the multiplication should have the form (4.3) for some non-zero $\alpha\colon\mathfrak{l}\times\mathfrak{l}\to k^{\times}$. Associativity of the multiplication implies that α is a normalized 2-cocycle.

Finally, being a unique (up to a scalar) generalized eigenvector with a given character, e_x has to be a genuine eigenvector, i.e. the $\mathfrak{h}$-action on it has to have the form (4.4). □

An algebra A in $\mathcal{C}(\mathfrak{h},\Omega)$ is *commutative* if

$$\mu(\mathrm{e}^{\pi\mathrm{i}\Omega}(b\otimes a)) = ab \quad \text{for all } a,b\in A. \tag{4.5}$$

First we calculate the effect of Casimir on the tensor product of eigenvectors.

Lemma 4.4. *Let $a,b\in A$ are such that*

$$z(a) = (x,z)a, \quad z(b) = (y,z)b \quad \text{for all } z\in\mathfrak{h}.$$

Then

$$\Omega(b\otimes a) = (x,y)b\otimes a, \qquad \omega(a) = (x,x)a.$$

Proof. Writing $\Omega = \sum_i x_i\otimes x_i$ in the orthonormal basis for the form $(x_i,x_j) = \delta_{i,j}$, we get

$$\Omega(b\otimes a) = \sum_i x_i(b)\otimes x_i(a) = \sum_i (y,x_i)(x,x_i)(b\otimes a) = (x,y)(b\otimes a)$$

and

$$\omega(a) = \sum_i x_i x_i(a) = \sum_i (x,x_i)(x,x_i)a = (x,x)a.$$

□

It is well known that up to an isomorphism the skew group algebra $k[\mathfrak{l},\alpha]$ depends only on the cohomology class of the cocycle α.

Here we assume that our ground field k is the field of complex numbers $\mathbb{C}$.

Theorem 4.5. *A commutative algebra A in the braided category $\mathcal{C}(\mathfrak{h},\Omega)$, which has a non-degenerate multiplicative bilinear form and has the trivial subalgebra of invariants $A^{\mathfrak{h}}=k$, has the form $k[\mathfrak{l},\alpha]$, where $\mathfrak{l}$ is a subgroup of $\mathfrak{h}$ such that the restriction of the form on $\mathfrak{l}$ is integer and even:*

$$(x,y)\in\mathbb{Z},\quad (x,x)\in 2\mathbb{Z},\quad x,y\in\mathfrak{l}.$$

The cohomology class of the cocycle α (and hence the isomorphism class of $k[\mathfrak{l},\alpha]$) is uniquely defined by the condition

$$\frac{\alpha(x,y)}{\alpha(y,x)}=\mathrm{e}^{\pi\mathrm{i}(x,y)}. \tag{4.6}$$

Proof. Setting $a=e_x$ and $b=e_y$ in (4.5) and using Lemma 4.4, we get

$$\alpha(x,y)e_{x+y}=e_xe_y=\mathrm{e}^{\pi\mathrm{i}(x,y)}e_ye_x=\alpha(y,x)\mathrm{e}^{\pi\mathrm{i}(x,y)}e_{x+y},$$

which gives (4.6). By setting $x=y$ we have that $\mathrm{e}^{\pi\mathrm{i}(x,x)}=1$ for all $x\in\mathfrak{l}$, which means that (x,x) must be an even integer. This implies that (x,y) is an integer for any $x,y\in\mathfrak{l}$, since

$$(x,y)=\tfrac{1}{2}((x+y,x+y)-(x,x)-(y,y)).$$

By the exact sequence of universal coefficients,

$$0\longrightarrow \mathrm{Ext}(\mathfrak{l},k^{\times})\longrightarrow H^2(\mathfrak{l},k^{\times})\longrightarrow \mathrm{Hom}(\Lambda^2_{\mathbb{Z}}\mathfrak{l},k^{\times})\longrightarrow 0.$$

The group $k^{\times}=\mathbb{C}^{\times}=\mathbb{C}/\mathbb{Z}$ is divisible; hence, $\mathrm{Ext}(\mathfrak{l},k^{\times})=0$ [**1**] and the cohomology class of α is uniquely defined by its skew-symmetrization. □

Corollary 4.6. *Isomorphism classes of commutative algebras A in the braided category $\mathcal{C}(\mathfrak{h},\Omega)$, which have a non-degenerate multiplicative bilinear form and have trivial subalgebra of invariants $A^{\mathfrak{h}}=k$, are in one-to-one correspondence with subgroups $\mathfrak{l}\subset\mathfrak{h}$ such that the restriction of the form on $\mathfrak{l}$ is integer and even:*

$$(x,y)\in\mathbb{Z},\quad (x,x)\in 2\mathbb{Z},\quad x,y\in\mathfrak{l}. \tag{4.7}$$

5. Local modules in Drinfeld categories of abelian Lie algebras

Let A be a commutative algebra in $\mathcal{C}(\mathfrak{h},\Omega)$. A left A-module in $\mathcal{C}(\mathfrak{h},\Omega)$ is *local* if

$$\mu(\mathrm{e}^{2\pi\mathrm{i}\Omega}(a\otimes m))=am\quad\text{for all } a\in A,\ m\in M. \tag{5.1}$$

The following is a generalization of Lemma 4.4.

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Lemma 5.1. *Let A be an algebra and M be a left A-module in $\mathcal{C}(\mathfrak{h}, \Omega)$. Let $a \in A$ be such that*

$$z(a) = (x, z)a \quad \text{for all } z \in \mathfrak{h}$$

for some $x \in \mathfrak{h}$. Then

$$\Omega(a \otimes m) = a \otimes x(m) \quad \text{for all } m \in M.$$

Proof. Writing $\Omega = \sum_i x_i \otimes x_i$ in the orthonormal basis for the form $(x_i, x_j) = \delta_{i,j}$, we get

$$\Omega(a \otimes m) = \sum_i x_i(a) \otimes x_i(m) = a \otimes \left(\sum_i (x, x_i) x_i(a) \right) = a \otimes x(m).$$

□

Let $\mathfrak{l} \subset \mathfrak{h}$ and α be as in Theorem 4.5.

Lemma 5.2. *A module M over the commutative algebra $k[\mathfrak{l}, \alpha]$ in the braided category $\mathcal{C}(\mathfrak{h}, \Omega)$ is local if and only if M is semisimple as an $\mathfrak{l}$-module:*

$$M = \bigoplus_{\chi \in \mathfrak{l}^\vee} M_\chi, \qquad M_\chi = \{m \in M \mid x(m) = \chi(x)m \text{ for all } x \in \mathfrak{l}\} \tag{5.2}$$

and for $M_\chi \neq 0$ the character $\chi\colon \mathfrak{l} \to k$ has integer values: $\chi(\mathfrak{l}) \subset \mathbb{Z}$. The algebra action permutes the components as follows:

$$e_x M_\chi = M_{\chi + (x, \cdot)}, \quad x \in \mathfrak{l}, \ \chi \in \mathfrak{l}^\vee.$$

Proof. Setting $a = e_x$ in (5.1) and using Lemma 5.1, we get $e_x m = e_x \mathrm{e}^{2\pi \mathrm{i} x}(m)$, which means that for $x \in \mathfrak{l}$ the operator $\mathrm{e}^{2\pi \mathrm{i} x}$ is the identity on M. Thus, M is semisimple as an $\mathfrak{l}$-module with integer valued eigencharacters.

For $m \in M_\chi$ and $y \in \mathfrak{l}$, we have

$$y(e_x m) = y(e_x)m + e_x y(m) = (x, y)e_x m + \chi(y)e_x m.$$

□

We define the *support* of a local $k[\mathfrak{l}, \alpha]$-module M as

$$\operatorname{supp}(M) = \{\chi \in \operatorname{Hom}(\mathfrak{l}, \mathbb{Z}) \mid M_\chi \neq 0\}.$$

By Lemma 5.2, the support of a local A-module is a union of $\mathfrak{l}$-orbits in $\operatorname{Hom}(\mathfrak{l}, \mathbb{Z})$ with respect to the $\mathfrak{l}$-action on $\operatorname{Hom}(\mathfrak{l}, \mathbb{Z})$ given by the homomorphism

$$\mathfrak{l} \to \operatorname{Hom}(\mathfrak{l}, \mathbb{Z}), \quad x \mapsto (x, \cdot).$$

Now we shall present the group of $\mathfrak{l}$-orbits in $\operatorname{Hom}(\mathfrak{l}, \mathbb{Z})$ in a slightly different way. Define

$$\mathfrak{l}^\# = \{x \in \mathfrak{h} \mid (x, \mathfrak{l}) \subset \mathbb{Z}\}.$$

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The commutative diagram

$$\begin{array}{ccccc} \mathfrak{h} & = & \mathfrak{h}^* & \twoheadrightarrow & \mathrm{Hom}(\mathfrak{l}, k) \\ \uparrow & & & & \uparrow \\ \mathfrak{l}^{\#} & & \longrightarrow & & \mathrm{Hom}(\mathfrak{l}, \mathbb{Z}) \end{array}$$

implies that the homomorphism $\mathfrak{l}^{\#} \to \mathrm{Hom}(\mathfrak{l}, \mathbb{Z})$ is surjective. Clearly, its kernel is $\mathfrak{l}^{\perp} = \{x \in \mathfrak{h} \mid (x, \mathfrak{l}) = 0\}$. Thus, $\mathrm{Hom}(\mathfrak{l}, \mathbb{Z})$ can be identified with $\mathfrak{l}^{\#}/\mathfrak{l}^{\perp}$. If the form has integer values on $\mathfrak{l}$, the group $\mathfrak{l}$ maps into $\mathfrak{l}^{\#}$ and the cokernel of the map $\mathfrak{l} \to \mathrm{Hom}(\mathfrak{l}, \mathbb{Z})$ is $\mathfrak{l}^{\#}/(\mathfrak{l} + \mathfrak{l}^{\perp})$.

Proposition 5.3. *Let A be $k[\mathfrak{l}, \alpha]$. The category ${}_A\mathcal{C}(\mathfrak{h}, \Omega)^{\mathrm{loc}}$ has a monoidal grading by the group $\mathfrak{l}^{\#}/(\mathfrak{l} + \mathfrak{l}^{\perp})$:*

$$ {}_A\mathcal{C}(\mathfrak{h}, \Omega)^{\mathrm{loc}} = \bigoplus_{X \in \mathfrak{l}^{\#}/(\mathfrak{l}+\mathfrak{l}^{\perp})} {}_A\mathcal{C}(\mathfrak{h}, \Omega)^{\mathrm{loc}}_X, \tag{5.3}$$

where the category ${}_A\mathcal{C}(\mathfrak{h}, \Omega)^{\mathrm{loc}}_X$ is the full subcategory of ${}_A\mathcal{C}(\mathfrak{h}, \Omega)^{\mathrm{loc}}$, consisting of modules M with support X (or rather the corresponding $\mathfrak{l}$-orbit in $\mathrm{Hom}(\mathfrak{l}, \mathbb{Z})$).

Proof. Clearly, the support of an indecomposable local $k[\mathfrak{l}, \alpha]$-module is an orbit, which provides the decomposition (5.3). To establish monoidality of the grading we need to show that the tensor product $M \otimes_A N$ of local modules $M \in {}_A\mathcal{C}(\mathfrak{h}, \Omega)^{\mathrm{loc}}_X$, $N \in {}_A\mathcal{C}(\mathfrak{h}, \Omega)^{\mathrm{loc}}_Y$ belongs to ${}_A\mathcal{C}(\mathfrak{h}, \Omega)^{\mathrm{loc}}_{X+Y}$. This follows from the inclusion of eigenspaces $M_\chi \otimes N_\xi \subset (M \otimes N)_{\chi+\xi}$. □

Note that the kernel of the restriction $(\cdot,\cdot)|_{\mathfrak{l}^{\perp}}$ coincides with $\mathfrak{l}^{\perp} \cap \mathfrak{l}^{\perp\perp}$. Note also that the double orthogonal $\mathfrak{l}^{\perp\perp}$ coincides with the vector subspace of $\mathfrak{h}$ spanned by $\mathfrak{l}$. Thus, the induced form on $\bar{\mathfrak{l}} = \mathfrak{l}^{\perp}/(\mathfrak{l}^{\perp} \cap \mathfrak{l}^{\perp\perp})$ is non-degenerate. Let $\bar{\Omega}$ be the Casimir of this form on $\bar{\mathfrak{l}}$.

The next proposition gives a 'bound from below' for the degree-zero component of the category of local modules.

Proposition 5.4. *There is a braided monoidal functor from the category ${}_A\mathcal{C}(\mathfrak{h}, \Omega)^{\mathrm{loc}}_0$ to the Drinfeld category $\mathcal{C}(\bar{\mathfrak{l}}, \bar{\Omega})$ which has a section, i.e. a functor $\mathcal{C}(\bar{\mathfrak{l}}, \bar{\Omega}) \to {}_A\mathcal{C}(\mathfrak{h}, \Omega)^{\mathrm{loc}}_0$ such that the composition $\mathcal{C}(\bar{\mathfrak{l}}, \bar{\Omega}) \to \mathcal{C}(\bar{\mathfrak{l}}, \bar{\Omega})$ is the identity.*

Proof. We shall prove the proposition by constructing two braided equivalences:

$$ {}_A\mathcal{C}(\mathfrak{h}, \Omega)^{\mathrm{loc}}_0 \to {}_{A_0}\mathcal{C}(\mathfrak{h}/\mathfrak{l}^{\perp\perp}, \Omega') \to \mathcal{C}(\bar{\mathfrak{l}}, \bar{\Omega}).$$

First we shall prove that the category ${}_A\mathcal{C}(\mathfrak{h}, \Omega)^{\mathrm{loc}}_0$ is equivalent, as a braided monoidal category, to the category ${}_{A_0}\mathcal{C}(\mathfrak{h}/\mathfrak{l}^{\perp\perp}, \Omega')$ of modules over the group algebra $A_0 = k[\mathfrak{l} \cap \mathfrak{l}^{\perp}]$ in the Drinfeld category $\mathcal{C}(\mathfrak{h}/\mathfrak{l}^{\perp\perp}, \Omega')$. Here $\Omega' = (f \otimes f)(\Omega)$ is the image of the Casimir $\Omega \in \mathfrak{h}^{\otimes 2}$ under the epimorphism $f \colon \mathfrak{h} \to \mathfrak{h}/\mathfrak{l}^{\perp\perp}$.

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By the definition of the subcategory ${}_A\mathcal{C}(\mathfrak{h},\Omega)_0^{\mathrm{loc}}$, a local A-module M belongs to ${}_A\mathcal{C}(\mathfrak{h},\Omega)_0^{\mathrm{loc}}$ if and only if its eigenspace decomposition, as an $\mathfrak{l}$-module, has the form

$$M = \bigoplus_{x\in\mathfrak{l}/\mathfrak{l}\cap\mathfrak{l}^\perp} M_x, \quad M_x = \{m \in M \mid y(m) = (x,y)m \text{ for all } y \in \mathfrak{l}\}. \tag{5.4}$$

Note that the A-action is given by $e_x M_y = M_{x+y}$.

Define a functor ${}_A\mathcal{C}(\mathfrak{h},\Omega)_0^{\mathrm{loc}} \to {}_{A_0}\mathcal{C}(\mathfrak{h}/\mathfrak{l}^{\perp\perp},\Omega')$ by an assignment $M \mapsto M_0$, where

$$M_0 = \{m \in M \mid z(m) = 0 \text{ for all } z \in \mathfrak{l}\} = M^{\mathfrak{l}}$$

is the space of $\mathfrak{l}$-invariants (which coincides with the space of $\mathfrak{l}^{\perp\perp}$-invariants). The A-action on M gives an A_0-action on M_0. Note that $A_0 = A^{\mathfrak{l}}$ coincides with the subalgebra of A, spanned by e_t with $t \in \mathfrak{l}\cap\mathfrak{l}^\perp$. Since the form is a zero on $\mathfrak{l}\cap\mathfrak{l}^\perp$, A_0 is isomorphic to the (untwisted) group algebra $k[\mathfrak{l}\cap\mathfrak{l}^\perp]$.

A quasi-inverse functor ${}_{A_0}\mathcal{C}(\mathfrak{h}/\mathfrak{l}^{\perp\perp},\Omega') \to {}_A\mathcal{C}(\mathfrak{h},\Omega)_0^{\mathrm{loc}}$ sends N into $A\otimes_{A_0} N$ with the $\mathfrak{h}$-action $z(a\otimes n) = z(a)\otimes n + a\otimes z(n)$ and the A-action $a(b\otimes n) = ab\otimes n$. The A-module $A\otimes_{A_0} N$ is clearly local, since the $\mathfrak{l}$-action on N is trivial. The monoidal structure of this functor is given by

$$(A\otimes_{A_0} N)\otimes_A (A\otimes_{A_0} L) \to A\otimes_{A_0} (N\otimes L), \qquad (a\otimes n)\otimes_A (b\otimes l) \mapsto ab\otimes n\otimes l.$$

Since $N^{\mathfrak{l}} = 0$ and $A^{\mathfrak{l}} = A_0$, the natural inclusion

$$N \to (A\otimes_{A_0} N)^{\mathfrak{l}}, \quad n \mapsto 1\otimes n$$

is an isomorphism. The map $A\otimes_{A_0} M^{\mathfrak{l}} \to M$ (induced by the A-action) is an isomorphism because of the special shapes of the eigenspace decomposition (5.4) and the A-action. Indeed, it follows from the comparison of decompositions for $A\otimes_{A_0} M^{\mathfrak{l}}$ and M:

$$(A\otimes_{A_0} M^{\mathfrak{l}})_x = e_x\otimes M^{\mathfrak{l}} \to e_x M^{\mathfrak{l}} = M_x.$$

The fact that the functor ${}_A\mathcal{C}(\mathfrak{h},\Omega)_0^{\mathrm{loc}} \to {}_{A_0}\mathcal{C}(\mathfrak{h}/\mathfrak{l}^{\perp\perp},\Omega')$ is braided can be checked directly. Indeed, it transforms the braiding $c_{M,N}(m\otimes m) = \mathrm{e}^{\pi\mathrm{i}\Omega}(n\otimes m)$ in ${}_A\mathcal{C}(\mathfrak{h},\Omega)_0^{\mathrm{loc}}$ into

$$\bar{c}_{M^{\mathfrak{l}},N^{\mathfrak{l}}}(m\otimes n) = \mathrm{e}^{\pi\mathrm{i}(f\otimes f)(\Omega)}(n\otimes m),$$

where as before $f\colon \mathfrak{h}\to\mathfrak{h}/\mathfrak{l}^{\perp\perp}$ is the quotient map.

Now we construct a functor

$${}_{A_0}\mathcal{C}(\mathfrak{h}/\mathfrak{l}^{\perp\perp},\Omega') \to \mathcal{C}(\bar{\mathfrak{l}},\bar{\Omega}) \tag{5.5}$$

by sending a $k[\mathfrak{l}\cap\mathfrak{l}^\perp]$-module N into the space $N_{\mathfrak{l}\cap\mathfrak{l}^\perp} = N/\sum_{t\in\mathfrak{l}\cap\mathfrak{l}^\perp}(1-e_t)(N)$ of co-invariants. Since $(\mathfrak{l}\cap\mathfrak{l}^\perp)^\perp = \mathfrak{l}^\perp+\mathfrak{l}^{\perp\perp}$ acts trivially on A_0, the $(\mathfrak{l}^\perp+\mathfrak{l}^{\perp\perp})/(\mathfrak{l}^\perp\cap\mathfrak{l}^{\perp\perp})$-action on N descends to $N_{\mathfrak{l}\cap\mathfrak{l}^\perp}$. The natural map $(N\otimes_{A_0} P)_{\mathfrak{l}\cap\mathfrak{l}^\perp} \to N_{\mathfrak{l}\cap\mathfrak{l}^\perp}\otimes P_{\mathfrak{l}\cap\mathfrak{l}^\perp}$ defines a monoidal structure on the functor (5.5).

To construct a section of the functor (5.5), we choose a section $\sigma\colon \mathfrak{h}/\mathfrak{l}^{\perp\perp}\to\bar{\mathfrak{l}}$ of the embedding of abelian Lie algebras $\bar{\mathfrak{l}} = (\mathfrak{l}^{\perp\perp}+\mathfrak{l}^\perp)/\mathfrak{l}^{\perp\perp} \to \mathfrak{h}/\mathfrak{l}^{\perp\perp}$. This will allow us to extend an $\bar{\mathfrak{l}}$-module structure on Q to a $\mathfrak{h}/\mathfrak{l}^{\perp\perp}$-module structure. Then the free A_0-module $A_0\otimes Q$ (with the diagonal $\mathfrak{h}/\mathfrak{l}^{\perp\perp}$-action) will have a property $(A_0\otimes Q)_{\mathfrak{l}\cap\mathfrak{l}^\perp} = Q$. □

As can be seen from the next example, in some cases the functors from Proposition 5.4 are equivalences.

Example 5.5. Let $\mathfrak{h} = \langle x, y\rangle$ be a two-dimensional abelian Lie algebra with the form $(x, x) = (y, y) = 1$, $(x, y) = 0$. The canonical element has a form $\Omega = x \otimes x + y \otimes y$. Let $\mathfrak{l} = \mathbb{Z}y$. Then the restriction of the form $(\cdot\,,\cdot)$ on $\mathfrak{l}$ is integer valued. We have $\mathfrak{l}^\perp = \langle x\rangle$, $\mathfrak{l}^{\perp\perp} = \langle y\rangle$ and $\mathfrak{l}^\# = \mathfrak{l} + \mathfrak{l}^\perp$. In particular, $\mathfrak{l}^\#/(\mathfrak{l} + \mathfrak{l}^\perp) = 0$, so the category ${}_A\mathcal{C}(\mathfrak{h}, \Omega)^{\mathrm{loc}}$ is trivially graded:

$$ {}_A\mathcal{C}(\mathfrak{h}, \Omega)^{\mathrm{loc}} = {}_A\mathcal{C}(\mathfrak{h}, \Omega)^{\mathrm{loc}}_0. $$

The algebra $A_0 = k[\mathfrak{l} \cap \mathfrak{l}^\perp]$ is trivial and the category ${}_A\mathcal{C}(\mathfrak{h}, \Omega)^{\mathrm{loc}}_0$ coincides with a Drinfeld category $\mathcal{C}(\mathfrak{l}^\perp, \bar{\Omega})$, where $\bar{\Omega} = x \otimes x$.

The next example shows that in general the functors from Proposition 5.4 are far from being equivalences.

Example 5.6. Again let $\mathfrak{h} = \langle x, y\rangle$ be a two-dimensional abelian Lie algebra but now with the form $(x, x) = (y, y) = 0$, $(x, y) = 1$. The canonical element has a form $\Omega = x \otimes y + y \otimes x$. Let $\mathfrak{l} = \mathbb{Z}y$. Then the restriction of the form $(\cdot, \cdot)$ on $\mathfrak{l}$ is zero and in particular integer valued. We have $\mathfrak{l}^\perp = \mathfrak{l}^{\perp\perp} = \langle y\rangle$ and $\mathfrak{l}^\# = \mathbb{Z}x + \mathfrak{l}^\perp$. In particular, $\mathfrak{l}^\#/(\mathfrak{l} + \mathfrak{l}^\perp) = \mathbb{Z}$ so the category ${}_A\mathcal{C}(\mathfrak{h}, \Omega)^{\mathrm{loc}}$ is $\mathbb{Z}$-graded:

$$ {}_A\mathcal{C}(\mathfrak{h}, \Omega)^{\mathrm{loc}} = \bigoplus_{i\in\mathbb{Z}} {}_A\mathcal{C}(\mathfrak{h}, \Omega)^{\mathrm{loc}}_i. $$

The algebra $A_0 = k[\mathfrak{l} \cap \mathfrak{l}^\perp]$ coincides with $A = k[\mathfrak{l}]$ and the category ${}_A\mathcal{C}(\mathfrak{h}, \Omega)^{\mathrm{loc}}_0$ is equivalent to ${}_A\mathcal{C}(\mathfrak{h}/\mathfrak{l}^{\perp\perp}, 0)$. Any A-module in $\mathcal{C}(\mathfrak{h}/\mathfrak{l}^{\perp\perp}, 0)$ is free, so the category ${}_A\mathcal{C}(\mathfrak{h}/\mathfrak{l}^{\perp\perp}, 0)$ can be identified with the category $k(k/\mathbb{Z}) \otimes k[\![x]\!]$-mod of finite-dimensional modules over the tensor product of the dual group algebra $k(k/\mathbb{Z})$ (of the abelian group $k/\mathbb{Z}$) with the algebra of formal series $k[\![x]\!]$. The first gives the grading by eigenvalues, while the second gives the (nilpotent) action of $x \in \mathfrak{h}$ on each of the generalized eigenspaces. Note that in this case the corresponding Drinfeld category $\mathcal{C}(\mathfrak{l}^\perp/(\mathfrak{l}^\perp \cap \mathfrak{l}^{\perp\perp}), \bar{\Omega})$ is trivial.

Define a bimultiplicative function $c\colon \mathfrak{l}^\# \times \mathfrak{l}^\# \to k^\times$ by $c(x, y) = \mathrm{e}^{\pi \mathrm{i}(x,y)}$. Let $\mathcal{G}(\mathfrak{l}^\#)$ be the category of finite-dimensional $\mathfrak{l}^\#$-graded vector spaces. Define a braiding on $\mathcal{G}(\mathfrak{l}^\#)$ by means of c and denote this braided monoidal category by $\mathcal{G}(\mathfrak{l}^\#, c)$ (see the appendix).

Proposition 5.7. *There is a braided monoidal functor $\mathcal{G}(\mathfrak{l}^\#, c) \to {}_A\mathcal{C}(\mathfrak{h}, \Omega)^{\mathrm{loc}}$ such that the image of $[x]$ belongs to ${}_A\mathcal{C}(\mathfrak{h}, \Omega)^{\mathrm{loc}}_X$, where X is the coset of x.*

Proof. For $x \in \mathfrak{h}$ define an $\mathfrak{h}$-module $A(x)$ as a span of $\{e^x_y, y \in \mathfrak{l}\}$ with the $\mathfrak{h}$-action:

$$ z(e^x_y) = (z, x + y)e^x_y. $$

Define an A-action on $A(x)$ by $e_u e^x_y = \alpha(u, y)e^x_{u+y}$, which turns $A(x)$ into an A-module in $\mathcal{C}(\mathfrak{h}, \Omega)$. Indeed,

$$ z(e_u)e^x_y + e_u z(e^x_y) = (u, z)e_u e^x_y + (x + y, z)e_u e^x_y $$

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coincides with

$$z(e_u e^x_y) = \alpha(u,y) z(e^u_{u+y}) = \alpha(u,y)(u+x+y,z)e^x_{u+y}.$$

Clearly, the module $A(x)$ has the following eigenspace decomposition:

$$A(x) = \bigoplus_{y\in\mathfrak{l}} A(x)_{(x+y,\cdot)}.$$

In particular, for $x \in \mathfrak{l}^{\#}$ the A-module $A(x)$ is local and belongs to the subcategory ${}_A\mathcal{C}(\mathfrak{h},\Omega)^{\mathrm{loc}}_X$, where X is the $\mathfrak{l}+\mathfrak{l}^{\perp}$-coset of x.

Now we need to calculate tensor products $A(x)\otimes_A A(y)$ for $x,y\in\mathfrak{h}$. The map

$$\phi_{x,y}\colon A(x)\otimes A(y)\to A(x+y), \quad e^x_u\otimes e^y_v \mapsto \alpha(u,v)\mathrm{e}^{-\pi\mathrm{i}(x,v)}e^{x+y}_{u+v}$$

is obviously $\mathfrak{h}$-linear. It has a property

$$\phi_{x,y}(e_w e^x_u\otimes e^y_v) = \phi_{x,y}(\mathrm{e}^{\pi\mathrm{i}(w,x+u)}(e^x_u\otimes e_w e^y_v)).$$

Indeed, by the 2-cocycle property of α and since $\mathrm{e}^{\pi\mathrm{i}(u,w)} = \alpha(u,w)\alpha(w,u)^{-1}$,

$$\phi_{x,y}(e_w e^x_u\otimes e^y_v) = \alpha(w,u)\phi_{x,y}(e^x_{w+u}\otimes e^y_v) = \alpha(w,u)\alpha(w+u,v)\mathrm{e}^{-\pi\mathrm{i}(x,v)}e^{x+y}_{w+u+v}$$

coincides with

$$\begin{aligned}\phi_{x,y}(\mathrm{e}^{\pi\mathrm{i}(w,x+u)})(e^x_u\otimes e_w e^y_v) &= \mathrm{e}^{\pi\mathrm{i}(w,x+u)}\phi_{x,y}(e^x_u\otimes\alpha(w,v)e^y_{w+v})\\ &= \alpha(w,v)\alpha(u,w+v)\mathrm{e}^{\pi\mathrm{i}(w,x+u)}\mathrm{e}^{-\pi\mathrm{i}(x,w+v)}e^{x+y}_{w+u+v}.\end{aligned}$$

Thus, the map $\phi_{x,y}$ factors through the map $A(x)\otimes_A A(y)\to A(x+y)$. Using the relation $e_w e^x_u\otimes e^y_v = \mathrm{e}^{\pi\mathrm{i}(w,x+u)}(e^x_u\otimes e_w e^y_v)$, valid in $A(x)\otimes_A A(y)$, we can see that the map $\phi_{x,y}\colon A(x)\otimes_A A(y)\to A(x+y)$ is an isomorphism. For $x,y,z\in\mathfrak{h}$, maps ϕ fit into a commutative diagram

$$\begin{array}{ccc} A(x)\otimes_A A(y)\otimes_A A(z) & \xrightarrow{\phi_{x,y}1} & A(x+y)\otimes_A A(z)\\ \downarrow{\scriptstyle 1\phi_{y,z}} & & \downarrow{\scriptstyle \phi_{x+y,z}}\\ A(x)\otimes_A A(y+z) & \xrightarrow{\phi_{x,y+z}} & A(x+y+z)\end{array}$$

Indeed, the top-right composition acts as

$$\begin{aligned}e^x_u\otimes e^y_v\otimes e^z_w &\mapsto \alpha(u,v)\mathrm{e}^{-\pi\mathrm{i}(x,v)}e^{x+y}_{u+v}\otimes e^z_w\\ &\mapsto \alpha(u,v)\alpha(u+v,w)\mathrm{e}^{-\pi\mathrm{i}((x,v)+(x+y,w))}e^{x+y+z}_{u+v+w},\end{aligned}$$

while the bottom-left composite has the form

$$\begin{aligned}e^x_u\otimes e^y_v\otimes e^z_w &\mapsto \alpha(v,w)\mathrm{e}^{\pi\mathrm{i}(y,w)}e^x_u\otimes e^{y+z}_{u+w}\\ &\mapsto \alpha(v,w)\alpha(u,v+w)\mathrm{e}^{\pi\mathrm{i}((y,w)+(x,v+w))}e^{x+y+z}_{u+v+w}.\end{aligned}$$

Hence, we have a monoidal functor $\mathcal{G}(\mathfrak{l}^{\#}) \to {}_A\mathcal{C}(\mathfrak{h}, \Omega)^{\mathrm{loc}}$, which is in fact a braided functor. Indeed, the diagram

$$\begin{array}{ccc} A(x) \otimes_A A(y) & \xrightarrow{\phi_{x,y}} & A(x+y) \\ {\scriptstyle c_{A(x),A(y)}}\downarrow & & \downarrow{\scriptstyle c(x,y)1} \\ A(y) \otimes_A A(x) & \xrightarrow{\phi_{y,x}} & A(x+y) \end{array}$$

commutes; the top-right composition acts as

$$e_u^x \otimes e_v^y \mapsto \alpha(u,v)\mathrm{e}^{\pi\mathrm{i}(x,v)} e_{u+v}^{x+y} \mapsto \alpha(u,v)\mathrm{e}^{\pi\mathrm{i}((x,v)+(x,y))} e_{u+v}^{x+y},$$

which coincides with the action of the bottom-left composite

$$e_u^x \otimes e_v^y \mapsto \mathrm{e}^{\pi\mathrm{i}(x+u,y+v)} e_v^y \otimes e_u^x \mapsto \alpha(v,u)\mathrm{e}^{\pi\mathrm{i}((x+u,y+v)+(y,v))} e_{u+v}^{x+y}.$$

This gives us the desired braided monoidal functor $[x] \mapsto A(x)$. □

Remark 5.8. Note that for $x \in \mathfrak{l}$ the local A-module $A(x)$, defined in the proof of Proposition 5.7, is isomorphic to A. Indeed, define a map $\psi_x \colon A(x) \to A$ by $e_u^x \mapsto \alpha(u,x)e_{x+u}$. While $\mathfrak{h}$-linearity of ψ_x is obvious, A-linearity follows from the 2-cocycle property of α, which implies that

$$\psi_x(e_v e_u^x) = \psi_x(\alpha(v,u)e_{v+u}^x) = \alpha(v,u)\alpha(v+u,x)e_{x+v+u}$$

coincides with

$$e_v \psi_x(e_u^x) = e_v \alpha(u,x)e_{x+u} = \alpha(u,x)\alpha(v,u+x)e_{x+v+u}.$$

Moreover, for $x, y \in \mathfrak{l}$ the diagram

$$\begin{array}{ccc} A(x) \otimes_A A(y) & \xrightarrow{\phi_{x,y}} & A(x+y) \\ {\scriptstyle \psi_x\psi_y}\downarrow & & \downarrow{\scriptstyle \psi_{x+y}} \\ A \otimes_A A & \xrightarrow{\mu} & A \end{array}$$

commutes up to multiplication by $\alpha(x,y)$. Indeed, the top-right composition has the effect

$$e_u^x \otimes_A e_v^y \mapsto \alpha(u,v)\mathrm{e}^{\pi\mathrm{i}(x,v)} e_{u+v}^{x+y} \mapsto \alpha(u,v)\mathrm{e}^{\pi\mathrm{i}(x,v)}\alpha(u+v,x+y)e^{x+y+u+v},$$

while the bottom-left composition acts as

$$e_u^x \otimes_A e_v^y \mapsto \alpha(u,x)\alpha(v,y)e_{x+u} \otimes_A e_{y+v} \mapsto \alpha(u,x)\alpha(v,y)\alpha(x+u,y+v)e_{x+u+y+v}.$$

The ratio of the coefficients

$$\alpha(u,x)\alpha(v,y)\alpha(x+u,y+v)\alpha(u,v)^{-1}\alpha(u+v,x+y)^{-1}\mathrm{e}^{\pi\mathrm{i}(x,v)}$$

is equal to

$$\alpha(x,y)\mathrm{e}^{\pi\mathrm{i}((v,y)+(x+y,v)+(x,v))}d(\alpha)(u,x,y+v)d(\alpha)(x,y,v)^{-1}d(\alpha)(u,v,x+y)^{-1},$$

where

$$d(\alpha)(x,y,z)=\alpha(x,y)\alpha(x+y,z)\alpha(y,z)^{-1}\alpha(x,y+z)^{-1}$$

equals 1 by the 2-cocycle property of α.

6. Finite categories of local modules

Here we describe commutative algebras $A = k[\mathfrak{l},\alpha]$ which have only a finite number of simple local modules. By Proposition 5.4 the category of local modules ${}_A\mathcal{C}(\mathfrak{h},\Omega)^{\mathrm{loc}}$ contains the Drinfeld category $\mathcal{C}(\bar{\mathfrak{l}},\bar{\Omega})$, where $\bar{\mathfrak{l}} = \mathfrak{l}^{\perp}/(\mathfrak{l}^{\perp}\cap\mathfrak{l}^{\perp\perp})$. For a non-zero $\bar{\mathfrak{l}}$ the category $\mathcal{C}(\bar{\mathfrak{l}},\bar{\Omega})$ has a continuous family of simple objects. Thus, for A to have only a finite number of simple local modules, we need to assume that $\bar{\mathfrak{l}} = 0$, which motivates the following definition.

We call a subgroup $\mathfrak{l}\subset\mathfrak{h}$ *coisotropic* if $\mathfrak{l}^{\perp}\subset\mathfrak{l}^{\perp\perp}$.

Note that the group $\mathrm{Hom}(\mathfrak{l},\mathbb{Z})$ is torsion free. Since the vector space $\mathrm{Hom}(\mathfrak{l},\mathbb{Z})\otimes_{\mathbb{Z}}k = \mathrm{Hom}(\mathfrak{l},k)\subset\mathfrak{h}$ is finite dimensional, the group $\mathrm{Hom}(\mathfrak{l},\mathbb{Z})$ is a free abelian of finite rank. Hence, the group $\mathfrak{l}^{\#}/(\mathfrak{l}+\mathfrak{l}^{\perp})$ is finitely generated and as such is a sum of a finite abelian and a free abelian group of finite rank. The rank coincides with the dimension of the vector space $\mathfrak{l}^{\#}/(\mathfrak{l}+\mathfrak{l}^{\perp})\otimes_{\mathbb{Z}}k$. We can identify the vector space $\mathfrak{l}^{\#}/(\mathfrak{l}+\mathfrak{l}^{\perp})\otimes_{\mathbb{Z}}k$ with the cokernel of the map

$$\mathfrak{l}^{\perp\perp}\to\mathrm{Hom}(\mathfrak{l},k),\quad x\mapsto(x,\cdot). \tag{6.1}$$

Note that the dimension of the cokernel of (6.1) equals the dimension of its kernel $\mathfrak{l}^{\perp\perp}\cap\mathfrak{l}^{\perp}$. Thus, the group $\mathfrak{l}^{\#}/(\mathfrak{l}+\mathfrak{l}^{\perp})$ is finite if and only if $\mathfrak{l}^{\perp\perp}\cap\mathfrak{l}^{\perp}=0$. For a coisotropic $\mathfrak{l}$ it means that $\mathfrak{l}^{\perp}=0$. In particular, $\mathfrak{l}$, as a subgroup of $\mathrm{Hom}(\mathfrak{l},\mathbb{Z})$, is a free abelian group. The short exact sequence

$$0\to\mathfrak{l}^{\perp}\to\mathfrak{h}\to\mathrm{Hom}(\mathfrak{l},k)\to 0$$

shows that in the coisotropic case $\mathrm{Hom}(\mathfrak{l},k)=\mathfrak{h}$, i.e. $\mathfrak{l}$ is a lattice in $\mathfrak{h}$.

Now we are ready to state our main result. We shall show that for an even $\mathfrak{l}$ the category of local A-modules is equivalent to the category of vector spaces graded by $\mathfrak{l}^{\#}/(\mathfrak{l}+\mathfrak{l}^{\perp})$. To formulate the result we need to describe the quadratic function on $\mathfrak{l}^{\#}/(\mathfrak{l}+\mathfrak{l}^{\perp})$ controlling the associativity and braiding constraints.

We choose a section $\sigma\colon\mathfrak{l}^{\#}/(\mathfrak{l}+\mathfrak{l}^{\perp})\to\mathfrak{l}^{\#}$ and define a function

$$q\colon\mathfrak{l}^{\#}/(\mathfrak{l}+\mathfrak{l}^{\perp})\to k^{\times},\quad q(X)=\mathrm{e}^{\pi\mathrm{i}(\sigma(X),\sigma(X))}. \tag{6.2}$$

Note that q does not depend on the choice of the section.

Theorem 6.1. *The commutative algebra $A = k[\mathfrak{l},\alpha]$ has (up to isomorphism) a finite number of simple local modules if and only if $\mathfrak{l}\subset\mathfrak{h}$ is an even lattice. In that case the category ${}_A\mathcal{C}(\mathfrak{h},\Omega)^{\mathrm{loc}}$ is equivalent, as a braided monoidal category, to the category $\bar{\mathcal{G}}(\mathfrak{l}^{\#}/\mathfrak{l},q)$ of $\mathfrak{l}^{\#}/\mathfrak{l}$-graded vector spaces, with the associativity and the braiding defined by the quadratic function q from (6.2).*

Proof. By Proposition 5.3 the category ${}_A\mathcal{C}(\mathfrak{h},\Omega)^{\mathrm{loc}}$ is a braided monoidal category, graded by the group $\mathfrak{l}^{\#}/(\mathfrak{l}+\mathfrak{l}^{\perp})$. Using the first functor constructed in the proof of Proposition 5.4, we can see that for an even lattice $\mathfrak{l}\subset\mathfrak{h}$, the degree-0 component is trivial, i.e. ${}_A\mathcal{C}(\mathfrak{h},\Omega)_0^{\mathrm{loc}}$ is equivalent to the category of vector spaces $\mathcal{V}$ect. Indeed, the category ${}_A\mathcal{C}(\mathfrak{h},\Omega)_0^{\mathrm{loc}}$ is equivalent to ${}_{A_0}\mathcal{C}(\mathfrak{h}/\mathfrak{l}^{\perp\perp},\Omega')$ which is trivial for a lattice $\mathfrak{l}$.

By Proposition 5.7 we have a braided monoidal functor $\mathcal{G}(\mathfrak{l}^{\#},c)\to {}_A\mathcal{C}(\mathfrak{h},\Omega)^{\mathrm{loc}}$, compatible with the grading. Finally, the isomorphisms from Remark 5.8 allow us to apply Theorem A 3 (see the appendix). □

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Appendix A. Graded monoidal categories

Here we recall basic facts about gradings on monoidal categories in general and braided monoidal categories of group-graded vector spaces in particular.

Let H be a group. An *H-grading* on a monoidal category $\mathcal{C}$ is a decomposition $\mathcal{C}=\bigoplus_{h\in H}\mathcal{C}_h$ such that, for $X\in\mathcal{C}_f$ and $Y\in\mathcal{C}_g$, the tensor product $X\otimes Y$ belongs to $\mathcal{C}_{fg}$.

The simplest examples of graded monoidal categories are provided by the following construction. For a group H, denote by $\mathcal{G}(H)$ the category of finite-dimensional H-graded vector spaces. The tensor product of H-graded vector spaces can be equipped with the H-grading

$$(V\otimes U)_h=\bigoplus_{fg=h}V_f\otimes U_g.$$

This makes the category $\mathcal{G}(H)$ monoidal. Clearly, it is H-graded, with V_f belonging to $\mathcal{G}(H)_f\simeq\mathcal{V}$ect. We are also interested in braidings on the categories $\mathcal{G}(H)$. We shall denote by $[x]$ the one-dimensional vector concentrated in degree $x\in G$.

Now let H be an abelian group and let $c\colon H\times H\to k^{\times}$ be a bi-multiplicative function (multiplicative in each variable). It is straightforward to see that

$$c_{V,U}(v\otimes u)=c(f,g)(u\otimes v),\quad v\in V_f,\ u\in U_g,$$

defines a braiding on $\mathcal{G}(H)$. We shall denote by $\mathcal{G}(H,c)$ the corresponding braided monoidal category.

Note that $\mathcal{G}(H,c)$ is not the most general braided monoidal category structure on the category of graded vector spaces. In full generality, such structures were classified in [**4**]. Here we formulate the result. A general solution for the associativity constraint for the tensor product of H-graded vector spaces is given by a (normalized) 3-cocycle $a\colon H \times H \times H \to k^\times$:

$$a_{V,U,W}(v \otimes (u \otimes w)) = a(f,g,h)(v \otimes u) \otimes w, \quad v \in V_f,\ u \in U_g,\ w \in W_h.$$

A braiding compatible with the associativity given by a corresponds to a function $c\colon H \times H \to k^\times$:

$$c_{V,U}(v \otimes u) = c(f,g)u \otimes v, \quad v \in V_f,\ u \in U_g.$$

Hexagon coherence axioms for the braiding c are equivalent to the equations

$$a(g,h,f)c(f,gh)a(f,g,h) = c(f,h)a(g,f,h)c(f,g),$$
$$a(h,f,g)^{-1}c(fg,h)a(f,g,h)^{-1} = c(f,h)a(f,h,g)^{-1}c(g,h).$$

The pair (a,c) is called an *abelian* 3-cocycle. Up to braided monoidal equivalences (whose functor part is just the identity functor), the braided monoidal category corresponding to (a,c) depends only on the (abelian) cohomology class of (a,c), which is determined by the quadratic function $q(f) = c(f,f)$. Recall (from, for example, [**4**]) that a function $q\colon H \to k^\times$ is quadratic if and only if $q(f^{-1}) = q(f)$, for all $f \in H$, and the function $\sigma\colon H \times H \to k^\times$,

$$\sigma(f,g) = g(fg)q(f)^{-1}q(g)^{-1},$$

is multiplicative in each argument (a so-called *bi-character*). In other words, braided equivalence classes of braided monoidal structures on the category of H-graded vector spaces correspond to quadratic functions on H. We shall denote by $\bar{\mathcal{G}}(H,q)$ a representative of the class corresponding to the quadratic function q.

Now we are ready to study the situation in § 5. Let $F\colon \mathcal{G}(H,c) \to \mathcal{C}$ be a braided monoidal functor. Let $K \subset H$ be a subgroup and for each $x \in K$ let $\psi_x\colon F([x]) \to I$ be an isomorphism (we shall assume that ψ_0 is the identity, or rather unit isomorphism, of the functor F). Being an automorphism of the unit object of $\mathcal{C}$, the (anticlockwise) composition of arrows of the diagram

$$\begin{array}{ccc} F([x] \otimes [y]) & \xrightarrow{F_{[x],[y]}} & F([x]) \otimes F([y]) \\ \| & & \downarrow{\scriptstyle \psi_x\psi_y} \\ F([x+y]) & \xrightarrow{\psi_{x+y}} & I \end{array}$$

is a multiple of the identity. Let $\alpha(x,y)$ be the coefficient. Then we have the following properties for $\alpha\colon K \times K \to k^\times$.

Lemma A 1. *The function α is a normalized 2-cocycle such that*

$$\alpha(x,y)\alpha(y,x)^{-1} = c(x,y) \quad \textit{for all } x, y \in K.$$

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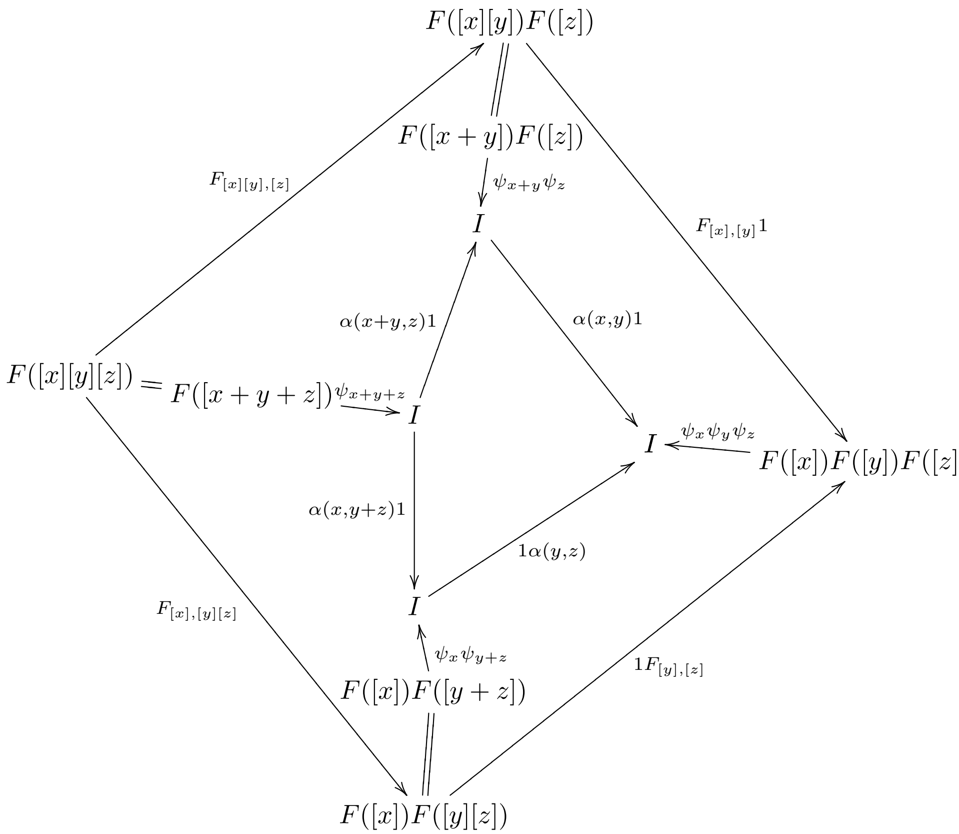
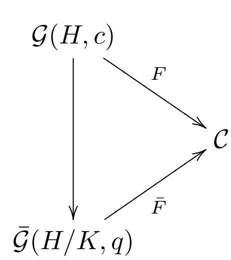


Figure 1.

Proof. The 2-cocycle property follows from the commutativity of the diagram in Figure 1. The property $\alpha(x,y)\alpha(y,x)^{-1} = c(x,y)$ follows from the commutativity of the diagram in Figure 2.

□

Now we construct a reduction $\bar{F}\colon \bar{\mathcal{G}}(H/K,q) \to \mathcal{C}$ of the functor F, i.e. a braided monoidal functor which fits in a commutative diagram of functors



(A 1)

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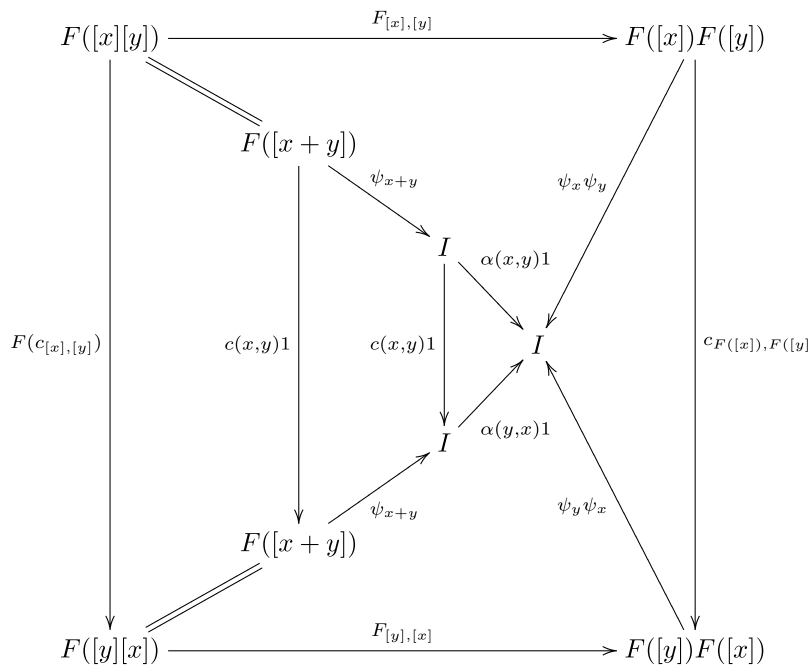

Figure 2.

Proposition A 2. *Let $F\colon \mathcal{G}(H,c)\to\mathcal{C}$ be a braided monoidal functor. Let $K\subset H$ be a subgroup and for each $x\in K$ let $\psi_x\colon F([x])\to I$ be an isomorphism (with $\psi_0=1$). Let $s\colon H/K\to H$ be a section of the quotient map. Then the function $q\colon H/K\to k^\times$, defined by $q(X)=c(s(X),s(X))$, is quadratic. Moreover, there is a braided monoidal functor $\bar{F}\colon\bar{\mathcal{G}}(H/K,q)\to\mathcal{C}$, making the diagram of functors (A 1) commutative.*

Proof. Define the functor $\bar{F}$ by $\bar{F}([X])=F([s(X)])$. Define the (pre)monoidal structure

$$\bar{F}_{[X],[Y]}\colon \bar{F}([X]\otimes[Y])\to\bar{F}([X])\otimes\bar{F}([Y])$$

by the diagram in Figure 3. Here $\theta\colon H/K\otimes H/K\to K$ is the 2-cocycle associated with the section s: $\theta(X,Y)=s(X+Y)-s(X)-s(Y)$.

Note that to construct a braided monoidal structure on $\mathcal{G}(H/K)$, fitting in the diagram (A 1), it is enough to have a pair of functions $a\colon H/K\times H/K\times H/K\to k^\times$ and $\bar{c}\colon H/K\times H/K\to k^\times$ such that the diagrams in Figure 4 commute. Indeed, pentagon and hexagon axioms for a, $\bar{c}$ will be fulfilled automatically.

By substituting our choice for $\bar{F}_{[X],[Y]}$ into Figures 3 and 4 we get the following results for a, $\bar{c}$:

$$a(X,Y,Z)=c(s(X),\theta(Y,Z))\alpha(\theta(X,Y+Z),\theta(Y,Z))\alpha(\theta(X+Y,Z),\theta(Y,Z))^{-1},$$
$$\bar{c}(X,Y)=c(s(X),s(Y)).$$

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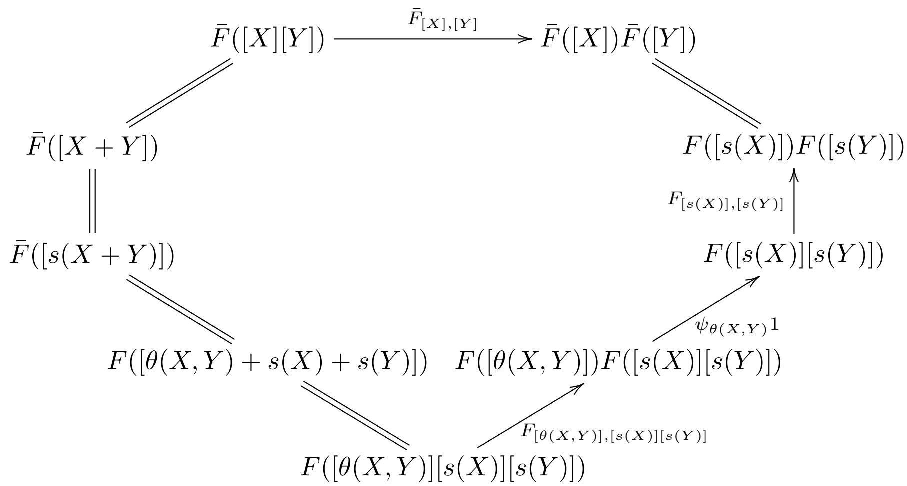

Figure 3.

In particular, the quadratic function associated to the pair a, $\bar{c}$ is

$$q(X) = c(s(X), s(X))$$

□

In the remainder of this appendix we shall characterize those graded categories which are tensor products of the trivial degree component and a category of graded vector spaces. A monoidal category is *tensor* if it is abelian with tensor product exact in each variable. We denote by $\mathcal{A} \boxtimes \mathcal{B}$ the Deligne tensor product of two tensor categories.

We call two objects X and Y in a braided monoidal category *mutually transparent* if and only if the double braiding is trivial on them:

$$c_{Y,X} c_{X,Y} = 1.$$

Two subcategories $\mathcal{A}$ and $\mathcal{B}$ of a braided monoidal category are *mutually transparent* if and only if, for any $X \in \mathcal{A}$ and $Y \in \mathcal{B}$, X and Y are mutually transparent objects.

Theorem A 3. *Let $\mathcal{C} = \bigoplus_{X \in G/K} \mathcal{C}_X$ be a G/K-graded braided tensor category. Suppose that there exists a braided monoidal functor $F\colon \mathcal{G}(G, c) \to \mathcal{C}$ such that $F([x]) \in \mathcal{C}_x$ (here we identify x with its coset modulo K) and such that $F([x])$ is transparent with $\mathcal{C}_e$. Suppose also that there are isomorphisms $\psi_x\colon F([x]) \to I$ with the identity object for every $x \in K$. Then we have an isomorphism of braided monoidal categories*

$$\mathcal{C} \simeq \mathcal{C}_e \boxtimes \bar{\mathcal{G}}(G/K, q),$$

where q is defined as in Proposition A 2.

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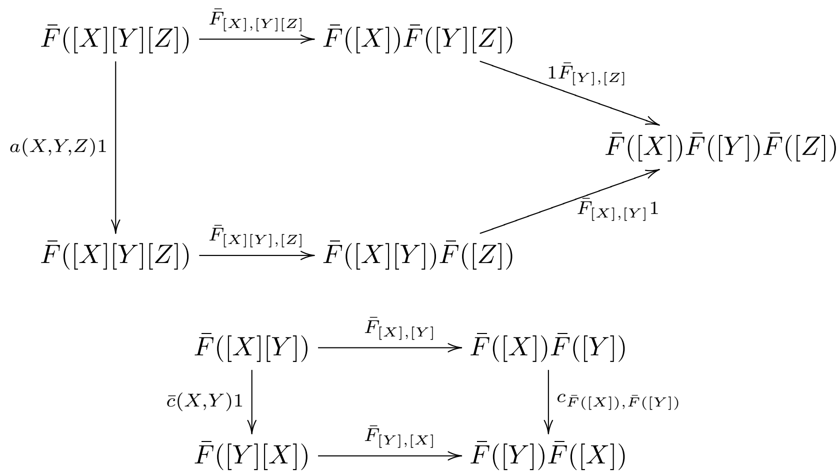

Figure 4.

Proof. By Proposition A 2 the monoidal functor $F\colon \mathcal{G}(G,c) \to \mathcal{C}$, together with isomorphisms ψ, induces a braided monoidal functor $\bar F\colon \mathcal{G}(G/K,c) \to \mathcal{C}$, which, together with the braided monoidal embedding $\mathcal{C}_e \to \mathcal{C}$, gives a grading-preserving braided monoidal functor $\mathcal{C}_e \boxtimes \bar{\mathcal{G}}(G/K,q) \to \mathcal{C}$. To see that it is an equivalence it is enough to note that for an arbitrary $x \in G/K$ any object X of $\mathcal{C}_x$ can be written as $X \otimes \bar F([x])^{-1} \otimes \bar F([x])$ and that $X \otimes \bar F([x])^{-1}$ belongs to $\mathcal{C}_e$. □

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