Canad. Math. Bull. Vol. **48** (2), 2005 pp. 275–282

Krull Dimension of Injective Modules Over Commutative Noetherian Rings

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Abstract. Let R be a commutative Noetherian integral domain with field of fractions Q. Generalizing a forty-year-old theorem of E. Matlis, we prove that the R-module Q/R (or Q) has Krull dimension if and only if R is semilocal and one-dimensional. Moreover, if X is an injective module over a commutative Noetherian ring such that X has Krull dimension, then the Krull dimension of X is at most 1.

Miller and Turnidge [11] give an example of a ring R and a Noetherian injective left R-module X such that X is not Artinian. On the other hand, Fisher [5, Corollary 3.3] shows that if R is a commutative ring, then every Noetherian injective R-module is Artinian. A related result of Vinsonhaler [14, Theorem A] states that if R is any left Noetherian ring such that the injective hull of the left R-module R is a finitely generated left R-module, then the ring R is left Artinian. For a generalisation of Vinsonhaler's Theorem see [7].

Let R be a ring. Every Noetherian left R-module has Krull dimension (see [10, 6.2.3]). In view of the above comments, it is natural to investigate when injective modules have Krull dimension. It is well known that if U is a simple module over a commutative Noetherian ring R, then the injective hull E of U is an Artinian R-module, *i.e.,* E has Krull dimension 0 (see, for example, [13, Theorem 4.30]. We can easily give an instance of a particular injective module over a certain ring having Krull dimension 1. If R is a DVR with field of fractions $Q \neq R$ and if Ra is the unique maximal ideal of R, then the R-submodules of Q form a totally ordered chain:

$$0 = \bigcap_{n=1}^{\infty} Ra^n \subseteq \cdots \subseteq Ra^2 \subseteq Ra \subseteq R \subseteq R(1/a) \subseteq R(1/a^2) \subseteq \cdots \subseteq \bigcup_{n=1}^{\infty} R(1/a^n) = Q.$$

Note that Q is an injective R-module with Krull dimension 1.

We shall show that if an injective module over a commutative Noetherian ring has Krull dimension, then its Krull dimension cannot exceed 1 (Theorem 2.10).

1 Krull Dimension

Throughout this section R will denote a (not necessarily commutative) ring with identity and all modules will be unital left R-modules. An R-module M is called *(Goldie) finite dimensional* if M does not contain a direct sum of an infinite number of non-zero submodules. The module M is called *quotient finite dimensional* if M/N is finite dimensional for every submodule N of M. Camillo [2] proved that a

Received by the editors June 19, 2003; revised September 2, 2003.
AMS subject classification: 13E05, 16D50, 16P60.

module M is quotient finite dimensional if and only if every submodule N contains a finitely generated submodule L such that N/L has no maximal submodule. For more information on quotient finite dimensional modules see [1, 2, 4].

Given a module M with Krull dimension, the Krull dimension of M will be denoted by $k(M)$. For the definition and basic properties of Krull dimension see [10, Chapter 6]. It will be convenient to recall at this point some basic facts concerning Krull dimension. For the proof of the first result see [10, Chapter 6].

Lemma 1.1 *Let M be an R-module.*

(i) *If M is Noetherian, then M has Krull dimension.*
(ii) *If M has Krull dimension, then M is quotient finite dimensional.*
(iii) *If N is a submodule of M, then M has Krull dimension if and only N and M/N both have Krull dimension, and in this case $k(M) = \sup\{k(N), k(M/N)\}$.*
(iv) *If $M = M_1 \oplus \cdots \oplus M_n$ is a finite direct sum of submodules $M_i (1 \le i \le n)$, then M has Krull dimension if and only if M_i has Krull dimension for each $1 \le i \le n$, and in this case $k(M) = \sup\{k(M_i) : 1 \le i \le n\}$.*
(v) *If the (left) R-module R has Krull dimension and M is finitely generated, then M has Krull dimension and $k(M) \le k(R)$.*
(vi) *If M has Krull dimension, $\alpha \ge 0$ is an ordinal and*

$$N = \sum\{L : L \text{ is a submodule of } M \text{ such that } k(L) \le \alpha\}$$

then $k(N) \le \alpha$.

Note that Lemma 1.1(ii) shows that for any non-zero ring R, not every R-module has Krull dimension. If R is a non-zero ring then every free R-module F of infinite rank does not have Krull dimension and the injective hull of F is an injective module which does not have Krull dimension. The following result will be required later.

Lemma 1.2 *Let R be a ring such that the R-module R has Krull dimension. Then an R-module M has Krull dimension if and only if M is quotient finite dimensional.*

Proof The necessity follows by Lemma 1.1(ii). Conversely, suppose that M is quotient finite dimensional. Suppose that $k(R) = \alpha$ for some ordinal $\alpha \ge 0$. Let N be any submodule of M. By [10, 2.2.8], the R-module M/N contains a finitely generated essential submodule H. By Lemma 1.1(v), H has Krull dimension and $k(H) \le \alpha$. Finally, M has Krull dimension by [3, 6.3]. ■

Corollary 1.3 *Let R be a left Noetherian ring. Then an R-module M has Krull dimension if and only if M is quotient finite dimensional.*

Proof By Lemmas 1.1(i) and 1.2. ■

Not every quotient finite dimensional module has Krull dimension. To give an easy example, let p be any prime, let F be any field of characteristic p and let $R = F[G]$ be the group algebra over F of the Prüfer p-group G. If $a, b \in R$ then there

https://doi.org/10.4153/CMB-2005-026-3 Published online by Cambridge University Press

exists a finite (cyclic) subgroup H of G such that $a, b \in F[H]$. Because $F[H]$ is a local principal ideal ring (see [12, Lemmas 3.1.1 and 3.1.6]), it follows that $F[H]a \subseteq F[H]b$ or $F[H]b \subseteq F[H]a$. Thus the R-module R/A is uniform for every proper ideal A of R, *i.e.,* the R-module R is quotient finite dimensional. Let A denote the augmentation ideal of R. For each $x \in G$, there exists $y \in G$ such that $x = y^p$ and there exists a positive integer n such that $x^{p^n} = 1$. Hence

$$x - 1 = (y-1)^p \text{ and } (x-1)^{p^n} = 0.$$

It follows that A is a non-zero nil idempotent ideal of R. By [10, 6.3.7] the R-module R does not have Krull dimension.

2 Injective Modules

Throughout the remainder of this paper, *R will denote a commutative ring with identity and all modules will be unital R modules.*

Given a non-negative integer n, a prime ideal P of the ring R has *height n* if there exists a chain $P = P_0 \supset P_1 \supset \cdots \supset P_n$ of prime ideals $P_i (i \geq 0)$ of R but no longer such chain. The ring R is defined to have *dimension n* if R contains a prime ideal of height n but no prime ideal of height $n+1$. Given a positive integer n, a commutative Noetherian ring has dimension n if and only if $k(R) = n$ by [10, 6.4.8]. Rings R of dimension 1 are usually called *one-dimensional*! A ring R is *semilocal* if R contains only a finite number of maximal ideals.

Let R be an integral domain with field of fractions Q. For any ideal A of R we set $A^* = \{q \in Q : qA \subseteq R\}$. Note that A^* is an R-submodule of Q such that $A \subseteq A^*A \subseteq R \subseteq A^*$. An ideal A of R is called *invertible* provided $A^*A = R$.

Note the following well known fact.

Lemma 2.1 *Let R be any (commutative) ring and let P be a minimal prime ideal of R. Then for each $a \in P$ there exists $0 \neq b \in R$ such that $ab = 0$.*

Proof By [8, Theorem 84]. ■

Lemma 2.2 *Let R be an integral domain with field of fractions Q such that the R-module Q/R is finite dimensional. Then there exist a positive integer n and prime ideals $P_i (1 \leq i \leq n)$ of R such that for each height 1 prime ideal P of R there exists $1 \leq j \leq n$ such that $P \subseteq P_j$.*

Proof By [10, 2.2.8] there exist a positive integer n and independent uniform submodules $U_i (1 \leq i \leq n)$ of the R-module Q/R such that $U_1 \oplus \cdots \oplus U_n$ is an essential submodule of Q/R. For each $1 \leq i \leq n$, let $P_i = \{r \in R : rx = 0$ for some $0 \neq x \in U_i\}$. Then it is well known (and easy to check) that P_i is a prime ideal of R for each $1 \leq i \leq n$.

Let P be a height 1 prime ideal of R. Let $0 \neq a \in P$. Then P/Ra is a minimal prime ideal of the ring R/Ra. Let $p \in P$. By Lemma 2.1 there exists $b \in R \backslash Ra$ such that $pb \in Ra$. Let $y = (b/a) + R \in Q/R$. Then $y \neq 0$ and $py = 0$. There exist

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$r \in R$ and $u_i \in U_i$ $(1 \leq i \leq n)$ such that $0 \neq ry = u_1 + \cdots + u_n$. It follows that $pu_1 + \cdots + pu_n = rpy = 0$ and hence $pu_i = 0$ $(1 \leq i \leq n)$. We conclude that $p \in P_k$ for some $1 \leq k \leq n$. This proves that $P \subseteq P_1 \cup \cdots \cup P_n$. By [8, Theorem 81] $P \subseteq P_j$ for some $1 \leq j \leq n$. ■

Lemma 2.3 *Let R be a Noetherian integral domain with field of fractions Q such that the R-module Q/R is finite dimensional. Then R is a semilocal ring such that $R \subset P^*$ for every maximal ideal P of R.*

Proof Let $n, U_i (1 \leq i \leq n)$ and $P_i (1 \leq i \leq n)$ be as in Lemma 2.2 and its proof. For each $1 \leq i \leq n$, P_i is a finitely generated ideal and hence there exists $0 \neq v_i \in U_i$ such that $P_i v_i = 0$. Let H be a maximal ideal of R. Let $0 \neq a \in H$. Let G be a minimal prime ideal of the ideal Ra. By the Principal Ideal Theorem (see, for example, [8, Theorem 142]), G is a height 1 prime ideal of R. By Lemma 2.2, $G \subseteq P_j$ for some $1 \leq j \leq n$. In particular, $a \in P_j$. Hence $H \subseteq P_1 \cup \cdots \cup P_n$. By [8, Theorem 81], $H = P_k$ for some $1 \leq k \leq n$. There exists $q \in Q \backslash R$ such that $v_k = q + R$ and hence $q \in H^* \backslash R$. The result follows. ■

Theorem 2.4 *Let R be an integral domain with field of fractions Q. Then R is a semilocal principal ideal domain if and only if R is Noetherian and integrally closed and the R-module Q/R is finite dimensional. In this case Q/R is Artinian and the R-module Q has Krull dimension* 1.

Proof Suppose first that R is a semilocal principal ideal domain. Then R is Noetherian and, by [8, Theorem 50], integrally closed. By [9, Theorem 1] (although it is easy to prove this directly), the R-module Q/R is Artinian and hence finite dimensional. By Lemma 1.1(iii), the R-module Q has Krull dimension 1.

Conversely, suppose that R is Noetherian and integrally closed and that the R-module Q/R is finite dimensional. By Lemma 2.3 R is a semilocal ring. Let P be a maximal ideal of R. By Lemma 2.3 again, $R \subset P^*$ and hence, by [8, Theorem 12], $P^*P \neq P$. Hence $P^*P = R$ because P is a maximal ideal of R. Thus every maximal ideal of R is invertible. It follows that R is a Dedekind domain (see [8, p. 73 ex. 12). Finally R is a principal ideal domain by [15, p. 278 Theorem 16]. ■

Next we strengthen Lemma 2.1 for rings with Krull dimension.

Lemma 2.5 *Let R be a ring with Krull dimension and let P be a minimal prime ideal of R. Then $Pa = 0$ for some $0 \neq a \in R$.*

Proof By [10, 6.3.8], R contains only a finite number of minimal prime ideals $P = P_1, P_2, \ldots, P_n$, for some positive integer n, and $(P_1 \cdots P_n)^k = 0$ for some positive integer k. The result follows. ■

Next we prove the following variant of Lemma 2.3. It is based on an unpublished result of J. T. Stafford given in a seminar in the University of Glasgow in 1987.

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Lemma 2.6 *Let R be an integrally closed integral domain having Krull dimension and let Q be the field of fractions of R such that the R-module Q/R is finite dimensional. Then R has only a finite number of height 1 prime ideals.*

Proof Let P be a height 1 prime ideal of R. Let $0 \neq a \in P$. Then P/Ra is a minimal prime ideal of the ring R/Ra. By Lemma 2.5, there exists $b \in R \backslash Ra$ such that $bP \subseteq Ra$. Then the element b/a of Q satisfies $(b/a)P \subseteq R$, *i.e.,* $b/a \in P^*$, but $b/a \notin R$. Thus $R \subset P^*$ for any height 1 prime ideal P of R.

Let n be a positive integer and let $P_i (1 \leq i \leq n)$ be distinct height 1 prime ideals of R. Let $q_i \in P_i^*$ $(1 \leq i \leq n)$ such that $q_1 + \cdots + q_n = r \in R$. Then

$$q_1(P_1 \cdots P_n) = (r - q_2 - \cdots - q_n)P_1 \cdots P_n \subseteq P_1.$$

But $q_1 P_1 \subseteq R$ and $P_2 \cdots P_n \nsubseteq P_1$, so that $q_1 P_1 \subseteq P_1$. By [8, Theorem 12] R integrally closed gives that $q_1 \in R$. It follows that $q_i \in R(1 \leq i \leq n)$. This proves that $(P_1^*/R) + \cdots + (P_n^*/R)$ is a direct sum of non-zero submodules of the R-module Q/R. But the R-module Q/R is finite dimensional. It follows that R has only a finite number of height 1 prime ideals. ■

Compare the next result with Theorem 2.4. It incorporates [9, Theorem 1].

Theorem 2.7 *Let R be a Noetherian integral domain with field of fractions $Q \neq R$. Then the following statements are equivalent.*

(i) *R is semilocal and one-dimensional.*
(ii) *The R-module Q/R is Artinian.*
(iii) *The R-module Q/L is Artinian for every non-zero R-submodule L of Q.*
(iv) *The R-module Q has Krull dimension.*
(v) *The R-module Q/R has Krull dimension.*
(vi) *The R-module Q/R is quotient finite dimensional.*

In this case the R-module Q has Krull dimension 1.

Proof (i) $\Leftrightarrow$ (ii)$\Leftrightarrow$ (iii) By [9, Theorem 1].

(ii) $\Rightarrow$ (iv) $\Leftrightarrow$ (v) By Lemma 1.1(iii).

(v) $\Leftrightarrow$ (vi) By Corollary 1.3.

(iv) $\Rightarrow$ (i) Let $\hat{R}$ denote the integral closure of R in Q. By Lemma 1.1(iii), the R-module $\hat{R}$ has Krull dimension. Since every ideal of $\hat{R}$ is an R-submodule of $\hat{R}$, it follows that the ring $\hat{R}$ has Krull dimension (see [10, 6.1.5]). Similarly the $\hat{R}$-module Q has Krull dimension. By Lemma 2.6, the ring $\hat{R}$ contains only a finite number of height 1 prime ideals.

Let P be a height 1 prime ideal of the ring R. By [8, Theorem 44] there exists a height 1 prime ideal P' of $\hat{R}$ such that $P' \cap R = P$. Hence R has only a finite number of height 1 prime ideals, say $P_1, \ldots, P_n$. Let H be a maximal ideal of R. The proof of Lemma 2.3 gives that $H = P_i$ for some $1 \leq i \leq n$. Now (i) follows. ■

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Corollary 2.8 *Let R be an integral domain with field of fractions Q. Then R is semilocal, one-dimensional and Noetherian if and only if the R-module Q has Krull dimension* 1*.*

Proof The necessity follows by Theorem 2.7. Conversely, suppose that $k(Q) = 1$. By Lemma 1.1(iii), $k(R) = 1$. Let $0 \neq a \in R$. Then R/Ra is an Artinian ring by [10, 6.3.9] and hence a Noetherian ring by [13, Theorem 3.25 Corollary]. It follows that R is a Noetherian ring. By Theorem 2.7, R is semilocal and one-dimensional. ■

Theorem 2.9 *The following statements are equivalent for a Noetherian integral domain R.*

(i) *R is semilocal and one-dimensional.*
(ii) *There exists a non-zero torsion-free injective R-module having Krull dimension.*
(iii) *Every finite dimensional R-module has Krull dimension* 0 *or* 1*.*

Proof (i) ⇒ (iii) Let M be any finite dimensional R-module. If E denotes the injective hull of M, then E is finite dimensional and hence $E = E_1 \oplus \cdots \oplus E_n$ for some positive integer n and indecomposable submodules $E_i (1 \leq i \leq n)$ by [13, Propositions 2.23 and 2.28]. In view of Lemma 1.1(iii), (iv), we can suppose without loss of generality that M is an indecomposable injective R-module. By [13, Theorem 2.32], M contains an essential submodule N such that $N \cong R/P$ for some prime ideal P of R. If $P \neq 0$ then P is a maximal ideal of R, N is simple and M is Artinian by [13, Theorem 4.30]. If $P = 0$ then the R-module M is isomorphic to the R-module Q, where Q is the field of fractions of R. By Theorem 2.7, $k(M) = 1$.

(iii) ⇒ (ii) The R-module Q is non-zero torsion-free injective and has Krull dimension.

(ii) ⇒ (i) Let X be a non-zero torsion-free injective R-module having Krull dimension. By Lemma 1.1(ii), X is finite dimensional and, by [13, Proposition 2.6], X is a finite dimensional vector space over the field Q. Hence, as Q-modules, $X \cong Q^{(n)}$ for some positive integer n. It follows that $X \cong Q^{(n)}$ as R-modules and hence the R-module Q has Krull dimension by Lemma 1.1(iii). Now (i) follows by Theorem 2.7. ■

Theorem 2.10 *Let R be a Noetherian ring and let X be an injective module having Krull dimension. Then X has Krull dimension at most* 1*.*

Proof By Lemma 1.1(ii), X has finite uniform dimension. Hence $X = X_1 \oplus \cdots \oplus X_n$ for some positive integer n and indecomposable submodules $X_i (1 \leq i \leq n)$ by [13, Propositions 2.23 and 2.28]. In view of Lemma 1.1(iv), we can suppose without loss of generality that X is indecomposable. By [13, Theorem 2.32], there exists an essential submodule Y of X such that $Y \cong R/P$ for some prime ideal P of R. Let $Z = \{x \in X : Px = 0\}$. Then Z is a submodule of X and $PZ = 0$. Clearly Y is an essential submodule of Z. By [13, Proposition 2.27], Z is an injective (R/P)-module having Krull dimension. Moreover, because Y is a torsion-free (R/P)-module and an essential submodule of Z, we know that Z is a non-zero torsion-free

(R/P)-module. By Theorem 2.9, the Noetherian integral domain R/P is semilocal and one-dimensional.

Let T denote the sum of all R-submodules S of X such that $k(S) \leq 1$. By Theorem 2.9, $Z \subseteq T$ and, by Lemma 1.1(vi), $k(T) \leq 1$. Suppose that $X \neq T$. Let $x \in X \backslash T$. By [13, Proposition 4.23], $P^m x = 0$ for some positive integer m. There exists a positive integer k such that $P^{k-1}x \nsubseteq T$ but $P^k x \subseteq T$. Now $(P^{k-1}x+T)/T$ is a finitely generated (R/P)-module. By Lemma 1.1(v), $k((P^{k-1}x + T)/T) \leq k(R/P) = 1$ and, by Lemma 1.1(iii), $k(P^{k-1}x) \leq 1$. But this implies that $P^{k-1}x \subseteq T$, a contradiction. It follows that $X = T$ and hence $k(X) \leq 1$. ■

We can extend Theorem 2.10 in case the ring R is a domain. Recall that if R is an integral domain then an R-module M is called *divisible* if $M = cM$ for every non-zero element c of R. Note that if X is an injective module over R, then, by [13, Lemma 2.4 and Proposition 2.6], every homomorphic image of X is divisible .

Theorem 2.11 *Let R be a Noetherian integral domain and let M be a divisible R-module having Krull dimension. Then M has Krull dimension at most* 1.

Proof Suppose that M is non-zero. By Lemma 1.1(ii), M is finite dimensional. If E denotes the injective hull of M, then $E = E_1 \oplus \cdots \oplus E_n$ for some positive integer n and indecomposable injective submodules E_i $(1 \leq i \leq n)$ of E. For each $1 \leq i \leq n$ let $\pi_i \colon E \to E_i$ denote the canonical projection and let K_i denote the kernel of π_i. Note that $K_1 \cap \cdots \cap K_n = 0$ and hence M embeds in $(M/K_1) \oplus \cdots \oplus (M/K_n)$. Moreover, for each $1 \leq i \leq n$, $M/K_i \cong \pi_i(M)$, which is a non-zero divisible submodule of E_i. Thus, by Lemma 1.1, we can suppose without loss of generality that M is a submodule of an indecomposable injective R-module X.

We now adapt the proof of Theorem 2.10. As before there exists an essential submodule Y of X such that $Y \cong R/P$, for some prime ideal P of R, and we again set $Z = \{x \in X : Px = 0\}$. Note that Z is an essential submodule of X, Z is a torsion-free (R/P)-module and $M \cap Z \neq 0$. Let $m \in M \cap Z, c \in R \backslash P$. Since M is divisible it follows that $m = cm'$ for some $m' \in M$. However, $cPm' = Pm = 0$ implies that $Pm' = 0$, because Y is essential in X. Thus $m' \in M \cap Z$. It follows that $M \cap Z$ is a non-zero torsion-free divisible module over the integral domain R/P. By [13, Proposition 2.7], $M \cap Z$ is injective. Applying Theorem 2.9, we conclude that the ring R/P is semilocal and one-dimensional.

Let N denote the sum of all R-submodules L of M such that $k(L) \leq 1$. Note that $M \cap Z \subseteq N$ by Theorem 2.9. Finally the argument at the end of the proof of Theorem 2.10 gives that $M = N$ and hence $k(M) \leq 1$. ■

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