Canad. Math. Bull. Vol. 58 (2), 2015 pp. 432–448
http://dx.doi.org/10.4153/CMB-2014-060-x
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Second-order Riesz Transforms and Maximal Inequalities Associated with Magnetic Schrödinger Operators

Dachun Yang and Sibei Yang

Abstract. Let $A := -(\nabla - i\vec{a}) \cdot (\nabla - i\vec{a}) + V$ be a magnetic Schrödinger operator on $\mathbb{R}^n$, where

$$\vec{a} := (a_1, \ldots, a_n) \in L^2_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n) \quad \text{and} \quad 0 \le V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$$

satisfy some reverse Hölder conditions. Let $\phi\colon \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ be such that $\phi(x,\cdot)$ for any given $x \in \mathbb{R}^n$ is an Orlicz function, $\phi(\cdot,t) \in \mathbb{A}_\infty(\mathbb{R}^n)$ for all $t \in (0,\infty)$ (the class of uniformly Muckenhoupt weights) and its uniformly critical upper type index $I(\phi) \in (0,1]$. In this article, the authors prove that second-order Riesz transforms VA^{-1} and $(\nabla - i\vec{a})^2 A^{-1}$ are bounded from the Musielak–Orlicz–Hardy space $H_{\phi,A}(\mathbb{R}^n)$, associated with A, to the Musielak–Orlicz space $L^\phi(\mathbb{R}^n)$. Moreover, we establish the boundedness of VA^{-1} on $H_{\phi,A}(\mathbb{R}^n)$. As applications, some maximal inequalities associated with A in the scale of $H_{\phi,A}(\mathbb{R}^n)$ are obtained.

1 Introduction

In this article, we study second-order Riesz transforms and maximal inequalities associated with the magnetic Schrödinger operator

$$A := -(\nabla - i\vec{a}) \cdot (\nabla - i\vec{a}) + V$$

on the Euclidean space $\mathbb{R}^n$ for $n \ge 2$ with the magnetic potential $\vec{a} := (a_1, \ldots, a_n) \in L^2_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ and the electric potential $0 \le V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

For all $k \in \{1, 2, \ldots, n\}$ and $a_k \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$, let

$$L_k := \frac{\partial}{\partial x_k} - i a_k. \tag{1.1}$$

The operator A can be defined via the following sesquilinear form $\mathcal{Q}$ by setting, for all $f, g \in \mathcal{V}$,

$$\mathcal{Q}(f,g) := \sum_{k=1}^n \int_{\mathbb{R}^n} L_k f(x) \overline{L_k g(x)}\, dx + \int_{\mathbb{R}^n} f(x) \overline{g(x)} V(x)\, dx, \tag{1.2}$$

Received by the editors July 28, 2014.
Published electronically November 24, 2014.
Dachun Yang is supported by the National Natural Science Foundation of China (Grant Nos. 11171027 and 11361020) and the Specialized Research Fund for the Doctoral Program of Higher Education of China (Grant No. 20120003110003). Sibei Yang, the corresponding author, is supported by the National Natural Science Foundation of China (Grant No. 11401276) and the Fundamental Research Funds for the Central Universities (Grant No. lzujbky-2014-18).
AMS subject classification: **42B30**, 42B35,42B25,35J10,42B37,46E30.
Keywords: Musielak–Orlicz–Hardy space, magnetic Schrödinger operator, atom, second-order Riesz transform, maximal inequality.

https://doi.org/10.4153/CMB-2014-060-x Published online by Cambridge University Press

where

$$\mathcal{V} := \left\{ f \in L^2(\mathbb{R}^n) : L_k f \in L^2(\mathbb{R}^n) \text{ for all } k \in \{1,\dots,n\} \text{ and } \sqrt{V} f \in L^2(\mathbb{R}^n) \right\}.$$

Then, by (1.2), A can be formally written as

$$Af = -\sum_{k=1}^{n} L_k^2 f + Vf, \tag{1.3}$$

for $f \in \mathcal{D}(A)$, where the *domain* of A, $\mathcal{D}(A)$, is given by

$$\mathcal{D}(A) := \Big\{ f \in \mathcal{V} : \text{ there exists } g \in L^2(\mathbb{R}^n) \text{ such that for all } \phi \in \mathcal{V}, \\ \mathcal{Q}(f,\phi) = \int_{\mathbb{R}^n} g(x)\overline{\phi(x)}\,dx \Big\}.$$

It is worth pointing out that the study of the magnetic Schrödinger operators inspires great interest because of their important applications in mathematics and physics (see, for example, [1, 2, 10, 18, 20, 26, 27]).

Recall that for $q \in (1,\infty]$, a nonnegative function w on $\mathbb{R}^n$ is said to belong to the *reverse Hölder class* $RH_q(\mathbb{R}^n)$, if when $q \in (1,\infty)$, we have $w \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ and

$$[w]_{RH_q(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} \left\{ \frac{1}{|B|}\int_B [w(x)]^q\,dx \right\}^{1/q} \left\{ \frac{1}{|B|}\int_B w(x)\,dx \right\}^{-1} < \infty, \tag{1.4}$$

and when $q = \infty$, we have $w \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$ and

$$[w]_{RH_\infty(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} \left\{ \operatorname*{ess\,sup}_{x\in B} w(x) \right\} \left\{ \frac{1}{|B|}\int_B w(x)\,dx \right\}^{-1} < \infty, \tag{1.5}$$

where the suprema are taken over all balls $B \subset \mathbb{R}^n$. A typical example of an element of the reverse Hölder class is a nonnegative polynomial on $\mathbb{R}^n$, which turns out to be in $RH_\infty(\mathbb{R}^n)$ (see, for example, [28]). Furthermore, for $0 \le U \in RH_q(\mathbb{R}^n)$ with $q \in [n/2,\infty]$, the *auxiliary function* $m(\cdot,U)$ associated with U is defined by setting

$$[m(x,U)]^{-1} := \sup\left\{ r \in (0,\infty) : \frac{r^2}{|B(x,r)|}\int_{B(x,r)} U(y)\,dy \le 1 \right\} \tag{1.6}$$

for all $x \in \mathbb{R}^n$ (see Shen [28]). It is known from [12] that $RH_q(\mathbb{R}^n)$ has the property of self-improvement. Namely, if $V \in RH_q(\mathbb{R}^n)$ for some $q \in (1,\infty)$, then there exists $\epsilon \in (0,\infty)$ such that $V \in RH_{q+\epsilon}(\mathbb{R}^n)$. Thus, for any $V \in RH_q(\mathbb{R}^n)$ with $q \in (1,\infty]$, the *critical index* q_+ for V is defined as follows:

$$q_+ := \sup\left\{ q \in (1,\infty] : V \in RH_q(\mathbb{R}^n) \right\}. \tag{1.7}$$

For all $x \in \mathbb{R}^n$, let

$$B(x) := \operatorname{curl} \vec{a}(x) := (b_{jk}(x))_{1\le j,k\le n} \tag{1.8}$$

be the *magnetic field* generated by $\vec{a}$, where for any $j,k \in \{1,\dots,n\}$, $b_{jk} := \frac{\partial a_k}{\partial x_j} - \frac{\partial a_j}{\partial x_k}$.

Assume that there exist positive constants C and c such that

$$\begin{aligned} &|B| + V \in RH_{n/2}(\mathbb{R}^n), \\ &0 \le V \le C[m(\cdot,|B|+V)]^2, \\ &|\nabla B| \le c[m(\cdot,|B|+V)]^3, \end{aligned} \tag{1.9}$$

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where B is as in (1.8),

$$|B| := \left\{ \sum_{i,j=1}^{n} |b_{ij}|^2 \right\}^{1/2}, \quad |\nabla B| := \left\{ \sum_{i,j=1}^{n} |\nabla b_{ij}|^2 \right\}^{1/2}, \tag{1.10}$$

and $m(\cdot, |B| + V)$ is as in (1.6) with U replaced by $|B| + V$. Now let

$$L := (L_1, \ldots, L_n), \tag{1.11}$$

where, for any $k \in \{1, \ldots, n\}$, L_k is as in (1.1). Shen [26, Theorem 4.7] proved that second-order Riesz transforms $L^2 A^{-1}$ are bounded on $L^p(\mathbb{R}^n)$ for $p \in (1, \infty)$ and are also of weak type $(1,1)$. Kurata and Sugano [21] studied the boundedness of the operators, VA^{-1}, $V^{1/2} LA^{-1}$, and $L^2 A^{-1}$, on $L^p(\mathbb{R}^n)$, with $p \in (1, \infty)$, under more additional assumptions for V and B. Furthermore, under assumption (1.9), Ben Ali [1, Theorem 1.10, Remark 3.15] established the boundedness of $L^2 A^{-1}$ and VA^{-1} on $L^p(\mathbb{R}^n)$ with $p \in (1, \infty)$ by using different methods from those used in [26]. Recently, also under assumption (1.9), the boundedness of VA^{-1}, $V^{1/2} LA^{-1}$, and $L^2 A^{-1}$, from the Musielak–Orlicz–Hardy space associated with the operator A, $H_{\phi,A}(\mathbb{R}^n)$, to the Musielak–Orlicz space $L^{\phi}(\mathbb{R}^n)$, was obtained in [6, Theorem 1.1] via establishing several Sobolev-type estimates for the heat kernels of A. In particular, when $\vec{a} = \vec{0}$, A and L are, respectively, the Schrödinger operator and the gradient operator on $\mathbb{R}^n$; in this case, denote the operator A by A_0. The boundedness of VA_0^{-1} and $\nabla^2 A_0^{-1}$ on the Musielak–Orlicz–Hardy space $H_{\phi,A_0}(\mathbb{R}^n)$ was studied in [5].

Recall that the Musielak–Orlicz–Hardy space is a function space of Hardy-type which unifies the classical (Orlicz–)Hardy space and the weighted (Orlicz–)Hardy space, in which the spatial and the time variables may not be separable (see, for example, [11, 14, 23, 29]). We also point out that some special Musielak–Orlicz–Hardy spaces naturally appear in many applications (see, for example, [22, 23]). Furthermore, the Musielak–Orlicz–Hardy spaces associated with operators generalize the (Orlicz–)Hardy space and the weighted (Orlicz–)Hardy space associated with operators, which have attracted great interest in recent years. Such function spaces associated with operators play important roles in the study of the boundedness of singular integrals which may have nonsmooth kernels and be beyond the scope of the classical Calderón–Zygmund theory (see, for example, [3, 4, 8, 9, 15–18]).

Moreover, assume that there exist positive constants $q \in (1, \infty]$ and $C \in (0, \infty)$ such that

$$\begin{gathered} 0 \le V \in RH_q(\mathbb{R}^n), \\ |B| \in RH_{n/2}(\mathbb{R}^n), \\ |\nabla B| \le C\big[m(\cdot, |B|) \big]^3 \end{gathered} \tag{1.12}$$

or

$$\begin{gathered} 0 \le V \in RH_q(\mathbb{R}^n), \\ \sup_{x \in Q} |B(x)| \le C \frac{1}{|Q|} \int_Q V(y)\, dy, \\ \sup_{x \in Q} |\nabla B(x)| \le C \left\{ \frac{1}{|Q|} \int_Q V(y)\, dy \right\}^{3/2}, \end{gathered} \tag{1.13}$$

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where the suprema are taken over all $Q \subset \mathbb{R}^n$, and $|B|$ and $|\nabla B|$ are as in (1.10). The boundedness of VA^{-1} and L^2A^{-1} on $L^p(\mathbb{R}^n)$ with some $p \in (1,\infty)$ was studied in [1, Theorems 1.8 and 1.11] and [2, Theorems 5.1 and 1.5], respectively, under assumptions (1.12) and (1.13). As applications, the *maximal inequality*, associated with A, stating that for all $f \in C_c^\infty(\mathbb{R}^n)$,

$$\text{(1.14)} \qquad \left\| L \cdot L(f) \right\|_{L^p(\mathbb{R}^n)} + \|Vf\|_{L^p(\mathbb{R}^n)} \leq C\|Af\|_{L^p(\mathbb{R}^n)},$$

where C is a positive constant independent of f, was obtained therein for some $p \in (1,\infty)$, under either assumptions (1.12) or (1.13).

Let $\alpha \in (0,\infty)$. For all $x \in \mathbb{R}^n$, let $a_k(x) := Q_k(x)$ with $k \in \{1,\dots,n\}$ and $V(x) := |P(x)|^\alpha$, where Q_k and P are polynomials on $\mathbb{R}^n$. Then (1.9), (1.12), and (1.13) hold true for such B and V (see [20, Remark 1(4)] or [26, p. 820] for details and more examples).

Let A and L be as in (1.3) and (1.11), respectively. The main purpose of this article is to study the boundedness of second-order Riesz transforms VA^{-1} and L^2A^{-1} on the Musielak–Orlicz–Hardy space $H_{\phi,A}(\mathbb{R}^n)$ and some maximal inequalities associated with A in the scale of $H_{\phi,A}(\mathbb{R}^n)$, under assumption (1.12) or (1.13).

To state the main results of this article, we first recall some necessary definitions and notation. Recall that a function $\Phi\colon[0,\infty)\to[0,\infty)$ is called an *Orlicz function* if it is nondecreasing, $\Phi(0)=0$, $\Phi(t)>0$ for any $t\in(0,\infty)$, and $\lim_{t\to\infty}\Phi(t)=\infty$ (see, for example, [24, 25]). Different from the classical case, the Orlicz functions in this article may not be convex. Moreover, Φ is said to be of *upper* (resp. *lower*) *type* p for some $p\in(0,\infty)$, if there exists a positive constant C such that for all $s\in[1,\infty)$ (resp. $s\in[0,1]$) and $t\in[0,\infty)$, $\Phi(st)\leq Cs^p\Phi(t)$. For a given function $\phi\colon\mathbb{R}^n\times[0,\infty)\to[0,\infty)$ such that for any given $x\in\mathbb{R}^n$, $\phi(x,\cdot)$ is an Orlicz function, ϕ is said to be of *uniformly upper* (resp. *lower*) *type* p for some $p\in(0,\infty)$ if there exists a positive constant C such that for all $x\in\mathbb{R}^n$, $s\in[1,\infty)$ (resp. $s\in[0,1]$) and $t\in[0,\infty)$, $\phi(x,st)\leq Cs^p\phi(x,t)$. Let

$$\text{(1.15)} \qquad I(\phi) := \inf\left\{ p\in(0,\infty) : \phi \text{ is of uniformly upper type } p \right\}$$

and

$$\text{(1.16)} \qquad i(\phi) := \sup\left\{ p\in(0,\infty) : \phi \text{ is of uniformly lower type } p \right\}.$$

In what follows, $I(\phi)$ (resp. $i(\phi)$) is called the *uniformly critical upper* (resp. *lower*) *type index* of ϕ. Observe that $I(\phi)$ and $i(\phi)$ may not be attainable, namely, ϕ may not be of uniformly upper (resp. lower) type $I(\phi)$ (resp. $i(\phi)$) (see [3, 14, 23, 31] for some examples). Moreover, it is easy to see that if ϕ is of uniformly upper type $p_1\in(0,\infty)$ and of uniformly lower type $p_0\in(0,\infty)$, then $p_1\geq p_0$ and hence $I(\phi)\geq i(\phi)$.

Definition 1.1 Let $\phi\colon\mathbb{R}^n\times[0,\infty)\to[0,\infty)$ satisfy that $x\mapsto\phi(x,t)$ is measurable for all $t\in[0,\infty)$. Then ϕ is said to satisfy the *uniform Muckenhoupt condition for some* $q\in[1,\infty)$, denoted by $\phi\in\mathbb{A}_q(\mathbb{R}^n)$, if for $q\in(1,\infty)$ we have

$$\mathbb{A}_q(\phi) := \sup_{t\in(0,\infty)} \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|^q} \int_B \phi(x,t)\,dx \left\{ \int_B [\phi(y,t)]^{1-q}\,dy \right\}^{q-1} < \infty,$$

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and for $q = 1$ we have

$$\mathbb{A}_1(\phi) := \sup_{t\in(0,\infty)} \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|}\int_B \phi(x,t)\,dx\Big\{\operatorname*{ess\,sup}_{y\in B}[\phi(y,t)]^{-1}\Big\} < \infty.$$

Here the first suprema are taken over all $t \in (0,\infty)$ and the second ones over all balls $B \subset \mathbb{R}^n$.

The function ϕ is said to satisfy the *uniform reverse Hölder condition for some* $q \in (1,\infty]$, denoted by $\phi \in \mathbb{RH}_q(\mathbb{R}^n)$, if $\sup_{t\in(0,\infty)}[\phi(\cdot,t)]_{RH_q(\mathbb{R}^n)} < \infty$, where $[\phi(\cdot,t)]_{RH_q(\mathbb{R}^n)}$ for any given $t \in (0,\infty)$ is defined as in (1.4) and (1.5) with w replaced by $\phi(\cdot,t)$.

The sets $\mathbb{A}_p(\mathbb{R}^n)$ for $p \in [1,\infty)$ and $\mathbb{RH}_q(\mathbb{R}^n)$ for $q \in (1,\infty]$ were introduced in [23] and [31], respectively. Let

$$\mathbb{A}_\infty(\mathbb{R}^n) := \bigcup_{q\in[1,\infty)} \mathbb{A}_q(\mathbb{R}^n).$$

The *critical indices* $q(\phi)$ and $r(\phi)$ are defined for $\phi \in \mathbb{A}_\infty(\mathbb{R}^n)$ by

$$q(\phi) := \inf\big\{q \in [1,\infty) : \phi \in \mathbb{A}_q(\mathbb{R}^n)\big\}, \tag{1.17}$$

$$r(\phi) := \sup\big\{q \in (1,\infty] : \phi \in \mathbb{RH}_q(\mathbb{R}^n)\big\}. \tag{1.18}$$

It is worth pointing that if $q(\phi) \in (1,\infty)$, then $\phi \notin \mathbb{A}_{q(\phi)}(\mathbb{R}^n)$ and there exists $\phi \notin \mathbb{A}_1(\mathbb{R}^n)$ such that $q(\phi) = 1$ (see, for example, [19]). Similarly, if $r(\phi) \in (1,\infty)$, then $\phi \notin \mathbb{RH}_{r(\phi)}(\mathbb{R}^n)$ and there exists $\phi \notin \mathbb{RH}_\infty(\mathbb{R}^n)$ such that $r(\phi) = \infty$ (see, for example, [7]).

Now we recall the notion of growth functions from Ky [23].

Definition 1.2 A function $\phi\colon \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ is called a *growth function* if the following hold true:

(i) ϕ is a *Musielak–Orlicz function*, namely,
   (a) $\phi(x,\cdot)\colon [0,\infty) \to [0,\infty)$ is an Orlicz function for all $x \in \mathbb{R}^n$;
   (b) $\phi(\cdot,t)$ is a measurable function for all $t \in [0,\infty)$.

(ii) $\phi \in \mathbb{A}_\infty(\mathbb{R}^n)$.

(iii) The function ϕ is of uniformly lower type p for some $p \in (0,1]$ and of uniformly upper type 1.

For a Musielak–Orlicz function ϕ as in Definition 1.2, a measurable function f on $\mathbb{R}^n$ is said to be in the *Musielak–Orlicz space* $L^\phi(\mathbb{R}^n)$ if $\int_{\mathbb{R}^n}\phi(x,|f(x)|)\,dx < \infty$. Moreover, for any $f \in L^\phi(\mathbb{R}^n)$, the *quasi-norm* of f is defined by

$$\|f\|_{L^\phi(\mathbb{R}^n)} := \inf\Big\{\lambda \in (0,\infty) : \int_{\mathbb{R}^n}\phi\Big(x, \frac{|f(x)|}{\lambda}\Big)\,dx \le 1\Big\}.$$

Clearly, $\phi(x,t) := \omega(x)\Phi(t)$ is a growth function if $\omega \in A_\infty(\mathbb{R}^n)$ and Φ is an Orlicz function of lower type p for some $p \in (0,1]$ and of upper type 1. Here and in what follows, $A_q(\mathbb{R}^n)$ with $q \in [1,\infty]$ denotes the *class of Muckenhoupt weights* (see, for example, [13]). A typical example of such Orlicz functions Φ is $\Phi(t) := t^p$, with $p \in (0,1]$, for all $t \in [0,\infty)$ (see [14,31] for more examples of such Φ). Another typical

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example of a growth function is $\phi(x,t) := t/\big(\ln(e+|x|)+\ln(e+t)\big)$ for $x \in \mathbb{R}^n$ and $t \in [0,\infty)$ (see [23] for the details).

Let A and ϕ be as in (1.3) and Definition 1.2, respectively. We point out that A is a nonnegative self-adjoint operator in $L^2(\mathbb{R}^n)$. Moreover, the Gaussian upper bound estimate for the kernels of the semigroup $\{e^{-tA}\}_{t>0}$ (see Lemma 2.3) further implies that the semigroup $\{e^{-tA}\}_{t>0}$ satisfies the reinforced $(1,\infty,1)$ off-diagonal estimates (see [3, Assumption (B)]). Thus, A is a nonnegative self-adjoint operator on $L^2(\mathbb{R}^n)$ satisfying the reinforced $(1,\infty,1)$ off-diagonal estimates. Now we recall the Musielak–Orlicz–Hardy space $H_{\phi,A}(\mathbb{R}^n)$ associated with A, introduced in [3].

For $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the *Lusin area function* $S_A(f)(x)$ associated with A is defined by

$$S_A(f)(x) := \left\{ \int_{\Gamma(x)} \left| t^2 A e^{-t^2 A}(f)(y) \right|^2 \frac{dy\,dt}{t^{n+1}} \right\}^{1/2},$$

where $\Gamma(x) := \{(y,t) \in \mathbb{R}^n \times (0,\infty) : |y-x| < t\}$. A function $f \in L^2(\mathbb{R}^n)$ is said to be in the set $\widetilde{H}_{\phi,A}(\mathbb{R}^n)$ if $S_A(f) \in L^\phi(\mathbb{R}^n)$; moreover, define $\|f\|_{H_{\phi,A}(\mathbb{R}^n)} := \|S_A(f)\|_{L^\phi(\mathbb{R}^n)}$. The *Musielak–Orlicz–Hardy space* $H_{\phi,A}(\mathbb{R}^n)$ is then defined as the completion of $\widetilde{H}_{\phi,A}(\mathbb{R}^n)$ with respect to the quasi-norm $\|\cdot\|_{H_{\phi,A}(\mathbb{R}^n)}$.

The first main result of this article is as follows.

Theorem 1.3 *Let A and ϕ be as in* (1.3) *and Definition* 1.2*, respectively. Assume that V and B satisfy* (1.12) *or* (1.13)*. Let*

$$q_+ \in (n/2,\infty] \cap (I(\phi)[r(\phi)]',\infty], \tag{1.19}$$

where q_+, $I(\phi)$, and $r(\phi)$ are as in (1.7)*,* (1.15)*, and* (1.18)*, respectively, and $[r(\phi)]'$ denotes the conjugate exponent of $r(\phi)$. Then the operators VA^{-1} and L^2A^{-1} are bounded from $H_{\phi,A}(\mathbb{R}^n)$ to $L^\phi(\mathbb{R}^n)$, where L is as in* (1.11)*.*

Remark 1.4 (i) Assume that A and ϕ are as in Theorem 1.3. Let $\phi(x,t) := t^p$ for all $x \in \mathbb{R}^n$ and $t \in [0,\infty)$, where $p \in (0,1]$. In this case, $H_{\phi,A}(\mathbb{R}^n)$ is just $H_A^p(\mathbb{R}^n)$ in [10,18], and Theorem 1.3 says that if $q_+ \in (n/2,\infty]$, then VA^{-1} and L^2A^{-1} are bounded from $H_A^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, which are also new.

(ii) To show Theorem 1.3, we need to apply the atomic characterization of $H_{\phi,A}(\mathbb{R}^n)$ (see [3, Theorems 5.4 and 5.9]), the functional calculus $A^{-1} = \int_0^\infty e^{-tA}\,dt$, some Sobolev type estimates for the heat kernel of A (see Proposition 2.2), and the $L^p(\mathbb{R}^n)$-boundedness of VA^{-1} and L^2A^{-1} with $p \in (1,q_+)$ (see [1, Theorems 1.8 and 1.11] and [2, Theorems 1.5 and 5.1]). To apply the atomic characterization of $H_{\phi,A}(\mathbb{R}^n)$ in terms of $(\phi,q,M)_A$-atoms (see Definition 3.1), we need to restrict $q \in ([r(\phi)]'I(\phi),\infty)$, and in order to use the $L^p(\mathbb{R}^n)$-boundedness of VA^{-1} and L^2A^{-1} with $p \in (1,q_+)$, we need to require $q < q_+$. Moreover, to apply Sobolev type estimates for the heat kernel of A in Proposition 2.2, we need $q_+ \in (n/2,\infty]$. All these lead to the restriction (1.19) in Theorem 1.3.

(iii) It is still unknown whether or not L^2A^{-1} is bounded from $H_{\phi,A}(\mathbb{R}^n)$ to $H_{\phi,A}(\mathbb{R}^n)$ or to the Musielak–Orlicz–Hardy space $H_\phi(\mathbb{R}^n)$ introduced by Ky [23] (see [6, Remark 1.2(iii)] for some reasons).

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As a corollary of Theorem 1.3, we immediately have the following maximal inequality, which further develops (1.14) in the scale of $H_{\phi,A}(\mathbb{R}^n)$.

Corollary 1.5 *Let ϕ, A, L, and V be as in Theorem 1.3. Then there exists a positive constant C such that for all $f \in H_{\phi,A}(\mathbb{R}^n)$,*

$$\|L^2 f\|_{L^\phi(\mathbb{R}^n)} + \|Vf\|_{L^\phi(\mathbb{R}^n)} \leq C\|Af\|_{H_{\phi,A}(\mathbb{R}^n)}.$$

The next result gives another maximal inequality associated to A on $H_{\phi,A}(\mathbb{R}^n)$, obtained via first establishing the boundedness of VA^{-1} on $H_{\phi,A}(\mathbb{R}^n)$.

Theorem 1.6 *Let ϕ, A, and L be as in Definition 1.2, (1.3), and (1.11), respectively. Assume further that V and B are as in Theorem 1.3 with q_+ satisfying (1.19) and*

$$n + 2 - n/q_+ > nq(\phi)/i(\phi), \tag{1.20}$$

or that V and B satisfy (1.9) and

$$n + 2 > nq(\phi)/i(\phi), \tag{1.21}$$

where q_+, $q(\phi)$, and $i(\phi)$ are as in (1.7), (1.17), and (1.16), respectively. Then

(i) *the operator VA^{-1} is bounded on $H_{\phi,A}(\mathbb{R}^n)$;*

(ii) *there exists a positive constant C such that for all $f \in H_{\phi,A}(\mathbb{R}^n)$,*

$$\left\| L \cdot L(f) \right\|_{H_{\phi,A}(\mathbb{R}^n)} + \|Vf\|_{H_{\phi,A}(\mathbb{R}^n)} \leq C\|Af\|_{H_{\phi,A}(\mathbb{R}^n)}.$$

We prove Theorem 1.6(i) by applying the atomic characterization of $H_{\phi,A}(\mathbb{R}^n)$ and the radial maximal function characterization associated with the heat semigroup $\{e^{-tA}\}_{t>0}$ of $H_{\phi,A}(\mathbb{R}^n)$ from [30, Theorem 1.6]. Theorem 1.6(ii) then follows from (i) and the differential structure of A.

Remark 1.7 (i) Let $\phi(x,t) := t^p$, with $p \in (\frac{n}{n+2-n/q_+}, 1]$, for all $x \in \mathbb{R}^n$ and $t \in [0,\infty)$. In this case, Theorem 1.6 says that if $q_+ \in (n/2, \infty]$, then VA^{-1} is bounded on $H_A^p(\mathbb{R}^n)$ and there exists a positive constant C such that for all $f \in H_A^p(\mathbb{R}^n)$,

$$\left\| L \cdot L(f) \right\|_{H_A^p(\mathbb{R}^n)} + \|Vf\|_{H_A^p(\mathbb{R}^n)} \leq C\|Af\|_{H_A^p(\mathbb{R}^n)},$$

which are also new and the last inequality generalizes (1.14) to the scale of $H_A^p(\mathbb{R}^n)$.

(ii) To prove Theorem 1.6 via the radial maximal function characterization of $H_{\phi,A}(\mathbb{R}^n)$, it suffices to prove that for any $(\phi, q, M)_A$-atom α, the radial maximal function of $VA^{-1}(\alpha)$ is uniformly bounded in $L^\phi(\mathbb{R}^n)$ (see (3.11)). To this end, we need the assumptions (1.20) when $|B|$ and V are as in Theorem 1.3 (in this case, we need to use the integral growth property of the potential V in Lemma 3.5) or (1.21) when $|B|$ and V satisfy (1.9).

The layout of this article is as follows. In Section 2, we establish some useful estimates for the heat kernels of A. Then, in Section 3, we give the proofs of Theorems 1.3 and 1.6.

Finally, we make some conventions on notation. Throughout the article, let $\mathbb{N} := \{1, 2, \dots\}$. We denote by C or C_k with $k \in \mathbb{N}$ a *positive constant* that is independent of the main parameters but may vary from line to line. We also use $C_{(\gamma,\beta,\dots)}$ to denote a *positive constant* depending on the indicated parameters γ, β, ... The *symbol* $A \lesssim B$ means that $A \le CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$. Moreover, for each ball $B := B(x_B, r_B) \subset \mathbb{R}^n$, with some $x_B \in \mathbb{R}^n$ and $r_B \in (0,\infty)$, and $\alpha \in (0,\infty)$, let $\alpha B := B(x_B, \alpha r_B)$. Finally, for $q \in [1,\infty]$, we denote by q' its *conjugate exponent*, namely, $1/q + 1/q' = 1$.

2 Estimates for Heat Kernels of A

In this section we state some Sobolev-type estimates for the heat kernels of A, which are needed in the proofs of Theorems 1.3 and 1.6. To this end, we first recall the following useful conclusions for the auxiliary function in (1.6), which is just [28, Lemma 1.4].

Lemma 2.1 *Let $0 \le U \in RH_q(\mathbb{R}^n)$ with $q \in [n/2,\infty]$ and let $m(\cdot, U)$ be as in (1.6). Then there exist positive constants C_1, C_2, and k_0 such that for all $x, y \in \mathbb{R}^n$,*

$$\frac{C_1 m(x,U)}{[1+|x-y|m(x,U)]^{k_0/(k_0+1)}} \le m(y,U) \le C_2[1+|x-y|m(x,U)]^{k_0} m(x,U).$$

Now we state the main results of this section, which play a key role in the proof of Theorem 1.3.

Proposition 2.2 *Let A be as in (1.3) with V and B satisfying (1.12) or (1.13), let L be as in (1.11), and let $\{K_t\}_{t>0}$ be the heat kernels of A. Assume that $q_+ \in (n/2,\infty)$ with q_+ as in (1.7). Then, for all $p \in [1,q_+)$ and $k \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$, there exist positive constants $C_{(k,p)}$, $\xi_{(k,p)}$ and $c_{(k,p)}$, depending on k and p, such that for all $t, s \in (0,\infty)$ and $x \in \mathbb{R}^n$, we have*

$$\left\{\int_{\{y\in\mathbb{R}^n:|y-x|\ge\sqrt{s}\}} \left|L^2 \frac{\partial^k K_t(y,x)}{\partial t^k}\right|^p dy\right\}^{1/p} \le \frac{C_{(k,p)}}{t^{1+n/(2p')+k}} \exp\Big\{-\xi_{(k,p)}\frac{s}{t}\Big\} \exp\Big\{-c_{(k,p)}(1+t[m(x,V)]^2)^{\delta}\Big\}$$

and

$$\left\{\int_{\{y\in\mathbb{R}^n:|y-x|\ge\sqrt{s}\}} \left|V(y) \frac{\partial^k K_t(y,x)}{\partial t^k}\right|^p dy\right\}^{1/p} \le \frac{C_{(k,p)}}{t^{1+n/(2p')+k}} \exp\Big\{-\xi_{(k,p)}\frac{s}{t}\Big\} \exp\Big\{-c_{(k,p)}(1+t[m(x,V)]^2)^{\delta}\Big\}; \tag{2.1}$$

here and hereafter, $m(\cdot, V)$ is as in (1.6) with U replaced by V, and $\delta := 1/[2(k_0+1)]$ with k_0 as in Lemma 2.1.

The proof of Proposition 2.2 is similar to that of [6, Proposition 2.2]. More precisely, replacing [6, Lemma 2.7] by Lemma 2.5 below and repeating the proof of [6, Proposition 2.2], we can prove Proposition 2.2, the details being omitted.

Let A be as in (1.3) with $0 \le V \in RH_q(\mathbb{R}^n)$ for some $q \in [n/2,\infty)$. Then by an argument similar to that used in the proof of [27, Theorem 1.9], we can prove the

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following Fefferman–Phong type inequality: there exists a positive constant C such that for all $u \in C_c^1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |m(x,V)u(x)|^2\,dx \le C\bigg[\int_{\mathbb{R}^n} |Lu(x)|^2\,dx + \int_{\mathbb{R}^n} V(x)|u(x)|^2\,dx\bigg].$$

Then replacing [20, Lemma 2] by this inequality and repeating the proof of [20, Theorem 1.1(b)], we obtain the following Gaussian upper bound estimates for the heat kernels of A.

Lemma 2.3 *Let A be as in (1.3) with $V \in RH_q(\mathbb{R}^n)$ for some $q \in [n/2,\infty)$. Assume that $\{K_t\}_{t>0}$ are the heat kernels of A. Then there exist positive constants C_3, C_4, and C_5 such that for all $t \in (0,\infty)$ and $x, y \in \mathbb{R}^n$,*

$$|K_t(x,y)| \le \frac{C_3}{t^{n/2}} \exp\big\{-C_4(1+t[m(x,V)]^2)^{\delta}\big\} \exp\Big\{-C_5\frac{|x-y|^2}{t}\Big\},$$

where $m(\cdot,V)$ and δ are as in Proposition 2.2.

Furthermore, replacing $q := 2p$, [6, Lemma 2.4] and $m(\cdot,|B|+V)$ in the proof of [6, Lemma 2.5] by $q := p/2+q_+$, Lemma 2.3, and $m(\cdot,V)$, respectively, and repeating the proof of [6, Lemma 2.5], we have the following Lemma 2.4, the details being omitted.

Lemma 2.4 *Let A, V, B, L, q_+, and $\{K_t\}_{t>0}$ be as in Proposition 2.2. Then for all $p \in [1, 2q+)$, there exist positive constants $\alpha_{(p)}$, $C_{(p)}$, and $c_{(p)}$, depending on p, such that for all $t \in (0,\infty)$ and $x \in \mathbb{R}^n$,*

$$\Big\{\int_{\mathbb{R}^n} \big|LK_t(y,x)\big|^p e^{\alpha_{(p)}\frac{|y-x|^2}{t}}\,dy\Big\}^{1/p} \le \frac{C_{(p)}}{t^{1/2+n/(2p')}} \exp\Big\{-c_{(p)}\big(1+t[m(x,V)]^2\big)^{\delta}\Big\},$$

where $m(\cdot,V)$ and δ are as in Proposition 2.2.

Lemma 2.5 *Let A, V, L, K_t, q_+, $m(\cdot,V)$, and δ be as in Proposition 2.2. Then, for all $p \in [1,q_+)$, there exist positive constants $\beta_{(p)}$, $C_{(p)}$ and $c_{(p)}$, depending on p, such that for all $t \in (0,\infty)$ and $x \in \mathbb{R}^n$,*

$$\Big\{\int_{\mathbb{R}^n} \big|L^2K_t(y,x)\big|^p e^{\beta_{(p)}\frac{|y-x|^2}{t}}\,dy\Big\}^{1/p} \le \frac{C_{(p)}}{t^{1+n/(2p')}} \exp\big\{-c_{(p)}(1+t[m(x,V)]^2)^{\delta}\big\}$$

and

$$\Big\{\int_{\mathbb{R}^n} \big|V(y)K_t(y,x)\big|^p e^{\beta_{(p)}\frac{|y-x|^2}{t}}\,dy\Big\}^{1/p} \le \frac{C_{(p)}}{t^{1+n/(2p')}} \exp\big\{-c_{(p)}(1+t[m(x,V)]^2)^{\delta}\big\}.$$

Replacing [6, Lemma 2.5], $0 \le V \lesssim [m(\cdot,|B|+V)]^2$ and [6, Lemma 2.8] in the proof of [6, Lemma 2.7] by Lemma 2.4, $V \in RH_q(\mathbb{R}^n)$ for some $q \in (1,q_+)$, and the following Lemma 2.6, which was established in [1, Theorems 1.8 and 1.11] and [2, Theorems 1.5 and 5.1], respectively, and repeating the proof of [6, Lemma 2.7], we can prove Lemma 2.5, the details being omitted.

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Lemma 2.6 *Let A be as in (1.3), with V and B satisfying (1.12) or (1.13), and let L be as in (1.11). Assume that q_+ is as in (1.7). Then, for all $p \in (1, q_+)$, L^2A^{-1} and VA^{-1} are bounded on $L^p(\mathbb{R}^n)$.*

3 Proofs of Theorems 1.3 and 1.6

In this section, we show Theorems 1.3 and 1.6. We begin with some necessary notions and auxiliary conclusions. We first recall the definition of $(\phi, q, M)_A$-atoms and the atomic Musielak–Orlicz–Hardy space $H^{M,q,s}_{\phi,A,\mathrm{at}}(\mathbb{R}^n)$ introduced in [3, Definitions 5.2 and 5.8].

Definition 3.1 Let A and ϕ be as in (1.3) and Definition 1.2, respectively. Assume that $q, s \in (1, \infty)$, $M \in \mathbb{N}$ and $B := B(x_B, r_B) \subset \mathbb{R}^n$ for some $x_B \in \mathbb{R}^n$ and $r_B \in (0, \infty)$ is a ball.

(i) A function $\alpha \in L^q(\mathbb{R}^n)$ is called a $(\phi, q, M)_A$-*atom* associated with the ball B, if there exists a function $b \in \mathcal{D}(A^M)$ such that

(a) $\alpha = A^M b$;

(b) for all $j \in \{0, 1, \dots, M\}$, $\mathrm{supp}(A^j b) \subset B$;

(c) $\|(r_B^2 A)^j b\|_{L^q(\mathbb{R}^n)} \le (r_B)^{2M} |B|^{1/q} \|\chi_B\|^{-1}_{L^\phi(\mathbb{R}^n)}$, where $j \in \{0, 1, \dots, M\}$.

(ii) For $f \in L^2(\mathbb{R}^n)$,

$$f = \sum_j \lambda_j \alpha_j \tag{3.1}$$

is called an *atomic* $(\phi, q, s, M)_A$-*representation* of f if every α_j is a $(\phi, q, M)_A$-atom associated with some ball $B_j \subset \mathbb{R}^n$, the summation (3.1) converges in $L^s(\mathbb{R}^n)$, and $\{\lambda_j\}_j \subset \mathbb{C}$ satisfies $\sum_j \phi(B_j, |\lambda_j| \|\chi_{B_j}\|^{-1}_{L^\phi(\mathbb{R}^n)}) < \infty$. Let

$$\widetilde{H}^{M,q,s}_{\phi,A,\mathrm{at}}(\mathbb{R}^n) := \big\{ f \in L^2(\mathbb{R}^n) : f \text{ has an atomic } (\phi, q, s, M)_A\text{-representation} \big\}$$

with the *quasi-norm*

$$\|f\|_{H^{M,q,s}_{\phi,A,\mathrm{at}}(\mathbb{R}^n)} := \inf\Big\{ \Lambda(\{\lambda_j \alpha_j\}_j) : \sum_j \lambda_j \alpha_j \text{ is a } (\phi, q, s, M)_A\text{-representation of } f \Big\},$$

where the infimum is taken over all the atomic $(\phi, q, s, M)_A$-representations of f as above and

$$\Lambda\big(\{\lambda_j \alpha_j\}_j\big) := \inf\left\{ \lambda \in (0, \infty) : \sum_j \phi\Big(B_j, \frac{|\lambda_j|}{\lambda \|\chi_{B_j}\|_{L^\phi(\mathbb{R}^n)}} \Big) \le 1 \right\}.$$

The *atomic Musielak–Orlicz–Hardy space* $H^{M,q,s}_{\phi,A,\mathrm{at}}(\mathbb{R}^n)$ is then defined as the completion of the set $\widetilde{H}^{M,q,s}_{\phi,A,\mathrm{at}}(\mathbb{R}^n)$ with respect to the quasi-norm $\|\cdot\|_{H^{M,q,s}_{\phi,A,\mathrm{at}}(\mathbb{R}^n)}$.

Then we have the following atomic characterization of $H_{\phi,A}(\mathbb{R}^n)$, which is just a corollary of [3, Theorems 5.4 and 5.9].

Lemma 3.2 *Let A and ϕ be as in (1.3) and Definition 1.2, respectively. Assume that $M \in \mathbb{N} \cap (nq(\phi)/2i(\phi), \infty)$, $s \in (1, \infty)$ and $q \in ([r(\phi)]' I(\phi), \infty)$, where $q(\phi)$, $i(\phi)$,*

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$r(\phi)$, and $I(\phi)$ are as in (1.17), (1.16), (1.18), and (1.15), respectively. Then the spaces $H_{\phi,A}(\mathbb{R}^n)$ and $H^{M,q,s}_{\phi,A,\mathrm{at}}(\mathbb{R}^n)$ coincide with equivalent quasi-norms.

Moreover, we also need some properties of ϕ in Definition 1.2. In what follows, for any measurable subset E of $\mathbb{R}^n$ and $t \in [0,\infty)$, let $\phi(E,t) := \int_E \phi(x,t)\,dx$. Then we have the following properties for ϕ from [3, Lemma 2.5], based on the corresponding results of [13, 23].

Lemma 3.3 *Let ϕ be as in Definition 1.2.*

(i) *There exists a positive constant C such that for all $(x,t_j) \in \mathbb{R}^n \times [0,\infty)$ with $j \in \mathbb{N}$, $\phi(x, \sum_{j=1}^\infty t_j) \le C \sum_{j=1}^\infty \phi(x,t_j)$.*

(ii) $\mathbb{A}_\infty(\mathbb{R}^n) = \bigcup_{p\in[1,\infty)} \mathbb{A}_p(\mathbb{R}^n) = \bigcup_{q\in(1,\infty]} \mathbb{RH}_q(\mathbb{R}^n)$.

(iii) *If $\phi \in \mathbb{A}_p(\mathbb{R}^n)$ with $p \in [1,\infty)$, then there exists a positive constant C such that for all balls $B_1, B_2 \subset \mathbb{R}^n$ with $B_1 \subset B_2$ and $t \in (0,\infty)$,*

$$\frac{\phi(B_2,t)}{\phi(B_1,t)} \le C\Big[\frac{|B_2|}{|B_1|}\Big]^p.$$

Now we prove Theorem 1.3 using Proposition 2.2 and Lemmas 3.2 and 3.3.

Proof of Theorem 1.3 Since the proof of Theorem 1.3 under assumption (1.13) is similar to that under assumption (1.12), we only give the proof of Theorem 1.3 under assumption (1.12).

We first prove the boundedness of VA^{-1} from $H_{\phi,A}(\mathbb{R}^n)$ to $L^\phi(\mathbb{R}^n)$. From the assumption $q_+ > I(\phi)[r(\phi)]'$, we deduce that there exists $q \in (I(\phi)[r(\phi)]', q_+)$. Let

$$s \in (1,q_+), \quad M \in \mathbb{N} \cap \Big(\frac{nq(\phi)}{2i(\phi)}, \infty\Big), \quad \text{and} \quad f \in \widetilde{H}^{M,q,s}_{\phi,A,\mathrm{at}}(\mathbb{R}^n).$$

By this, we know that there exist $\{\lambda_j\}_j \subset \mathbb{C}$ and a sequence $\{\alpha_j\}_j$ of $(\phi,q,M)_A$-atoms, associated with the balls $\{B_j\}_j$, such that

$$(3.2) \qquad f = \sum_j \lambda_j \alpha_j \text{ in } L^s(\mathbb{R}^n) \quad \text{and} \quad \|f\|_{H^{M,q,s}_{\phi,A,\mathrm{at}}(\mathbb{R}^n)} \sim \Lambda\big(\{\lambda_j\alpha_j\}_j\big).$$

By using (2.1), Lemma 2.6, $A^{-1} = \int_0^\infty e^{-tA}\,dt$, Lemma 3.3, and the uniformly upper and the uniformly lower properties of ϕ, as in the proof of [6, (3.3)], we conclude that for all $\lambda \in \mathbb{C}$ and $(\phi,q,M)_A$-atoms α associated with the ball B,

$$(3.3) \qquad \int_{\mathbb{R}^n} \phi\big(x, \big|VA^{-1}(\lambda\alpha)(x)\big|\big)\,dx \lesssim \phi\big(B, |\lambda|\|\chi_B\|^{-1}_{L^\phi(\mathbb{R}^n)}\big).$$

Then from (3.3), (3.2), and Lemmas 3.3(i) and 2.6, it follows that for all $\lambda \in (0,\infty)$,

$$\begin{aligned}\int_{\mathbb{R}^n} \phi\Big(x, \frac{|VA^{-1}(f)(x)|}{\lambda}\Big)\,dx &\lesssim \sum_j \int_{\mathbb{R}^n} \phi\Big(x, \frac{|VA^{-1}(\lambda_j\alpha_j)(x)|}{\lambda}\Big)\,dx \\ &\lesssim \sum_j \phi\Big(B_j, \frac{|\lambda_j|}{\lambda\|\chi_{B_j}\|_{L^\phi(\mathbb{R}^n)}}\Big),\end{aligned}$$

which, together with (3.2) again, implies that

$$\|VA^{-1}(f)\|_{L^\phi(\mathbb{R}^n)} \lesssim \|f\|_{H^{M,q,s}_{\phi,A,\mathrm{at}}(\mathbb{R}^n)}.$$

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This combined with the fact that $\widetilde{H}^{M,q,s}_{\phi,A,\mathrm{at}}(\mathbb{R}^n)$ is dense in $H^{M,q,s}_{\phi,A,\mathrm{at}}(\mathbb{R}^n)$ and Lemma 3.2 further yields that VA^{-1} is bounded from $H_{\phi,A}(\mathbb{R}^n)$ to $L^{\phi}(\mathbb{R}^n)$.

The proof of the boundedness of L^2A^{-1} from $H_{\phi,A}(\mathbb{R}^n)$ to $L^{\phi}(\mathbb{R}^n)$ is similar to that of the boundedness of VA^{-1} from $H_{\phi,A}(\mathbb{R}^n)$ to $L^{\phi}(\mathbb{R}^n)$, the details being omitted here. This concludes the proof of Theorem 1.3. ∎

In order to prove Theorem 1.6, we need the radial maximal function characterization of the space $H_{\phi,A}(\mathbb{R}^n)$, which was established in [30, Theorem 1.6]. We first recall the definition of the radial maximal function associated with A. For $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the *radial maximal function* $\mathcal{R}_h(f)$ of f, associated with $\{e^{-tA}\}_{t>0}$, is defined by setting $\mathcal{R}_h(f)(x) := \sup_{t\in(0,\infty)} |e^{-tA}(f)(x)|$. Let

$$\widetilde{H}_{\phi,\mathcal{R}_h}(\mathbb{R}^n) := \left\{ f \in L^2(\mathbb{R}^n) : \mathcal{R}_h(f) \in L^{\phi}(\mathbb{R}^n) \right\}$$

equipped with the quasi-norm

$$\|f\|_{H_{\phi,\mathcal{R}_h}(\mathbb{R}^n)} := \|\mathcal{R}_h(f)\|_{L^{\phi}(\mathbb{R}^n)}.$$

Then the space $H_{\phi,\mathcal{R}_h}(\mathbb{R}^n)$ is defined as the *completion* of $\widetilde{H}_{\phi,\mathcal{R}_h}(\mathbb{R}^n)$ with respect to the quasi-norm $\|\cdot\|_{H_{\phi,\mathcal{R}_h}(\mathbb{R}^n)}$.

Lemma 3.4 ([30]) *Let A and ϕ be as in (1.3) and Definition 1.2, respectively. Then the spaces $H_{\phi,A}(\mathbb{R}^n)$ and $H_{\phi,\mathcal{R}_h}(\mathbb{R}^n)$ coincide with equivalent quasi-norms.*

Furthermore, we also need the following estimate for the potential V, which was established in [28, Lemma 1.2].

Lemma 3.5 *Let $V \in RH_{q_0}(\mathbb{R}^n)$ with $q_0 \in [n/2,\infty)$. Then there exists a positive constant C such that for all $0 < r < R < \infty$ and $x \in \mathbb{R}^n$,*

$$\frac{1}{r^{n-2}} \int_{B(x,r)} V(y)\,dy \le C\Big(\frac{R}{r}\Big)^{\frac{n}{q_0}-2} \frac{1}{R^{n-2}} \int_{B(x,R)} V(y)\,dy.$$

Moreover, if $r := [m(x,V)]^{-1}$ with $x \in \mathbb{R}^n$, then $\frac{1}{r^{n-2}} \int_{B(x,r)} V(y)\,dy = 1$.

Now we prove Theorem 1.6 using Lemmas 3.2–3.5.

Proof of Theorem 1.6

Step 1 We prove Theorem 1.6 under the assumption that B and V are as in Theorem 1.3 with q_+ satisfying (1.19) and (1.20).

We first prove (i). From the assumption $n + 2 - n/q_+ > nq(\phi)/i(\phi)$, it follows that there exist $p_0 \in (0, i(\phi))$, $\widetilde{q} \in (q(\phi),\infty)$ and $q_1 \in (1, q_+)$ such that $n + 2 - n/q_1 > n\widetilde{q}/p_0$, $V \in RH_{q_1}(\mathbb{R}^n)$, $\phi \in \mathbb{A}_{\widetilde{q}}(\mathbb{R}^n)$ and ϕ is of uniformly lower type p_0. Let $\varepsilon_0 := 2 - n/q_1$. Then

$$n + \varepsilon_0 > n\widetilde{q}/p_0. \tag{3.4}$$

Let $f \in \widetilde{H}^{M,q,s}_{\phi,A,\mathrm{at}}(\mathbb{R}^n)$ with

$$s \in (1, q_+), \quad M \in \mathbb{N} \cap \Big(\frac{nq(\phi)}{2i(\phi)}, \infty\Big), \quad \text{and} \quad q \in \Big(\max\{I(\phi)[r(\phi)]', q_1'\}, \infty\Big).$$

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Then there exist $\{\lambda_j\}_j \subset \mathbb{C}$ and a sequence $\{\alpha_j\}_j$ of $(\phi, q, M)_A$-atoms, associated with the balls $\{B_j\}_j$, such that (3.2) holds true.

Let $\lambda \in \mathbb{C}$ and α be a $(\phi, q, M)_A$-atom associated with the ball $B := B(x_B, r_B)$ for some $x_B \in \mathbb{R}^n$ and $r_B \in (0, \infty)$. Then it follows from Definition 3.1 that there exists $b \in L^q(\mathbb{R}^n)$ such that $\alpha = Ab$, $\operatorname{supp}(b) \subset B$ and

$$\|b\|_{L^q(\mathbb{R}^n)} \le (r_B)^2 |B|^{1/q} \|\chi_B\|^{-1}_{L^\phi(\mathbb{R}^n)}. \tag{3.5}$$

Since, for all $t \in (0, \infty)$ and $x \in \mathbb{R}^n$,

$$e^{-tA}\big(VA^{-1}(\lambda\alpha)\big)(x) = \lambda \int_B K_t(x, y)(Vb)(y)\,dy \tag{3.6}$$

and, by Lemma 2.3, $|K_t(x, y)| \lesssim \frac{1}{t^{n/2}} e^{-C_5 \frac{|x-y|^2}{t}}$, it follows that for all $x \in 4B$,

$$\mathcal{R}_h\big(VA^{-1}(\lambda\alpha)\big)(x) \lesssim |\lambda| \mathcal{M}(Vb)(x), \tag{3.7}$$

where $\mathcal{R}_h$ and $\mathcal{M}$ denote the radial maximal function as in Lemma 3.4 and the classical Hardy–Littlewood operator on $\mathbb{R}^n$, respectively. Recall that for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, $\mathcal{M}(f)(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|\,dy$, where the supremum is taken over all balls B containing x. From the assumption $I(\phi)[r(\phi)]' < q_+$, it follows that there exists $q_2 \in (I(\phi)[r(\phi)]', \min\{q, q_+\})$, which further implies that there exists $p_1 \in (I(\phi), 1]$ such that ϕ is of uniformly upper type p_1 and $\phi \in \mathbb{RH}_{(q_2/p_1)'}(\mathbb{R}^n)$. By this, the uniformly upper type p_1 property of ϕ, (3.7), the boundedness of $\mathcal{M}$ on $L^{q_2}(\mathbb{R}^n)$, Lemma 2.6, $\phi \in \mathbb{RH}_{(q_2/p_1)'}(\mathbb{R}^n)$, Hölder's inequality, and Lemma 3.3(iii), we further conclude that

$$\begin{aligned}
&\int_{4B} \phi\big(x, \mathcal{R}_h\big(VA^{-1}(\lambda\alpha)\big)(x)\big)\,dx \\
&\quad\lesssim \int_{4B} \phi\big(x, |\lambda| \|\chi_B\|^{-1}_{L^\phi(\mathbb{R}^n)}\big) \big[1 + \mathcal{M}(Vb)(x) \|\chi_B\|_{L^\phi(\mathbb{R}^n)}\big]^{p_1}\,dx \\
&\quad\lesssim \phi\big(4B, |\lambda| \|\chi_B\|^{-1}_{L^\phi(\mathbb{R}^n)}\big) + \|\chi_B\|^{p_1}_{L^\phi(\mathbb{R}^n)} \\
&\qquad\times \|\mathcal{M}(Vb)\|^{p_1}_{L^{q_2}(\mathbb{R}^n)} \Big\| \phi\big(\cdot, |\lambda| \|\chi_B\|^{-1}_{L^\phi(\mathbb{R}^n)}\big) \Big\|_{L^{(q_2/p_1)'}(4B)} \\
&\quad\lesssim \phi\big(4B, |\lambda| \|\chi_B\|^{-1}_{L^\phi(\mathbb{R}^n)}\big) + \|\chi_B\|^{p_1}_{L^\phi(\mathbb{R}^n)} \|\alpha\|^{p_1}_{L^{q_2}(\mathbb{R}^n)} \\
&\qquad\times |4B|^{-p_1/q_2} \phi\big(4B, |\lambda| \|\chi_B\|^{-1}_{L^\phi(\mathbb{R}^n)}\big) \lesssim \phi\big(B, |\lambda| \|\chi_B\|^{-1}_{L^\phi(\mathbb{R}^n)}\big).
\end{aligned} \tag{3.8}$$

When $x \in \mathbb{R}^n \setminus (4B)$, we estimate $\mathcal{R}_h(VA^{-1}(\lambda\alpha))(x)$ by considering the following two cases for r_B.

Case 1 $r_B \in [\{m(x_B, V)\}^{-1}, \infty)$. In this case, we see that $r_B m(x_B, V) \ge 1$. From this, (3.6), Lemmas 2.3, 2.1, and 2.6, the fact that for any $y \in B$ and $x \in S_j(B) := (2^{j+1}B) \setminus (2^j B)$ with $j \ge 2$, $|x - y| \sim 2^j r_B$, and Hölder's inequality, we further deduce

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that for any $t \in (0,\infty)$, $m \in \mathbb{N}$ with $m \geq n/2$, and $x \in S_j(B)$ with $j \geq 2$,

$$\begin{aligned}
(3.9)\qquad &\big|e^{-tA}\big(VA^{-1}(\lambda\alpha)\big)(x)\big| \\
&\lesssim \frac{|\lambda|}{t^{n/2}}\int_B e^{-C_4\{1+t[m(y,V)]^2\}^\delta} e^{-\frac{C_5|x-y|^2}{t}}|V(y)b(y)|\,dy \\
&\lesssim t^{-n/2}\Big[\frac{t}{2^{2j}(r_B)^2}\Big]^m\big(t[m(x_B,V)]^2\big)^{n/2-m} \\
&\quad\times[r_B m(x_B,V)]^{\frac{2k_0(m-n/2)}{1+k_0}}|\lambda||B|\|\chi_B\|^{-1}_{L^\phi(\mathbb{R}^n)} \\
&\lesssim 2^{-2mj}[r_B m(x_B,V)]^{\frac{n-2m}{1+k_0}}|\lambda|\|\chi_B\|^{-1}_{L^\phi(\mathbb{R}^n)} \lesssim 2^{-2mj}|\lambda|\|\chi_B\|^{-1}_{L^\phi(\mathbb{R}^n)}.
\end{aligned}$$

Case 2 $r_B \in (0,[m(x_B,V)]^{-1})$. In this case, by Hölder's inequality, $V \in RH_{q_1}(\mathbb{R}^n)$, $q_1' < q$, (3.5), and Lemma 3.5, we know that

$$\begin{aligned}
\int_B |V(y)b(y)|\,dy &\leq \|V\|_{L^{q_1}(B)}\|b\|_{L^{q_1'}(B)} \\
&\lesssim |B|^{-1/q_1'}\Big[\int_B V(y)\,dy\Big](r_B)^2|B|^{1/q_1'}\|\chi_B\|^{-1}_{L^\phi(\mathbb{R}^n)} \\
&\lesssim |B|\|\chi_B\|^{-1}_{L^\phi(\mathbb{R}^n)}[r_B m(x_B,V)]^{\varepsilon_0},
\end{aligned}$$

which, combined with (3.6) and Lemmas 2.3 and 2.1, implies that for any $t \in (0,\infty)$ and $x \in S_j(B)$ with $j \geq 2$,

$$\begin{aligned}
(3.10)\qquad &\big|e^{-tA}\big(VA^{-1}(\lambda\alpha)\big)(x)\big| \\
&\lesssim \frac{|\lambda|}{t^{n/2}}\int_B e^{-C_4\{1+t[m(y,V)]^2\}^\delta} e^{-\frac{C_5|x-y|^2}{t}}|V(y)b(y)|\,dy \\
&\lesssim t^{-n/2}\Big[\frac{t}{2^{2j}(r_B)^2}\Big]^{(n+\varepsilon_0)/2}\big(t[m(x_B,V)]^2\big)^{-\varepsilon_0/2} \\
&\quad\times|\lambda||B|\|\chi_B\|^{-1}_{L^\phi(B)}[r_B m(x_B,V)]^{\varepsilon_0} \lesssim 2^{-(n+\varepsilon_0)j}|\lambda|\|\chi_B\|^{-1}_{L^\phi(\mathbb{R}^n)}.
\end{aligned}$$

This, together with (3.9), yields that for any $x \in S_j(B)$ with $j \geq 2$,

$$\mathcal{R}_h\big(VA^{-1}(\lambda\alpha)\big)(x) \lesssim 2^{-(n+\varepsilon_0)j}|\lambda|\|\chi_B\|^{-1}_{L^\phi(\mathbb{R}^n)},$$

which, combined with the uniformly lower type p_0 property of ϕ, $\phi \in \mathbb{A}_{\widetilde{q}}(\mathbb{R}^n)$, Lemma 3.3(iii), and (3.4), further implies that

$$\begin{aligned}
&\sum_{j=2}^\infty \int_{S_j(B)} \phi\big(x,\mathcal{R}_h\big(VA^{-1}(\lambda\alpha)\big)(x)\big)\,dx \\
&\lesssim \sum_{j=2}^\infty 2^{-(n+\varepsilon_0)p_0 j}\int_{S_j(B)}\phi\big(x,|\lambda|\|\chi_B\|^{-1}_{L^\phi(\mathbb{R}^n)}\big)\,dx \\
&\lesssim \sum_{j=2}^\infty 2^{-(n+\varepsilon_0-n\widetilde{q}/p_0)p_0 j}\phi\big(B,|\lambda|\|\chi_B\|^{-1}_{L^\phi(\mathbb{R}^n)}\big) \lesssim \phi\big(B,|\lambda|\|\chi_B\|^{-1}_{L^\phi(\mathbb{R}^n)}\big).
\end{aligned}$$

From this and (3.8), we deduce that

$$(3.11)\qquad \int_{\mathbb{R}^n}\phi\big(x,\mathcal{R}_h\big(VA^{-1}(\lambda\alpha)\big)(x)\big)\,dx \lesssim \phi\big(B,|\lambda|\|\chi_B\|^{-1}_{L^\phi(\mathbb{R}^n)}\big).$$

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Thus, by Lemma 3.3(i), (3.2), and (3.11), we conclude that for all $\lambda \in (0,\infty)$,

$$\int_{\mathbb{R}^n} \phi\Big(x, \frac{\mathcal{R}_h(VA^{-1}(f))(x)}{\lambda}\Big)\,dx \lesssim \sum_j \int_{\mathbb{R}^n} \phi\Big(x, \frac{|\lambda_j| \mathcal{R}_h(VA^{-1}(\alpha_j))(x)}{\lambda}\Big)\,dx$$
$$\lesssim \sum_j \phi\Big(B_j, \frac{|\lambda_j|}{\lambda \|\chi_B\|_{L^\phi(\mathbb{R}^n)}}\Big),$$

which, together with (3.2) again and Lemma 3.4, implies that

$$\big\|VA^{-1}(f)\big\|_{H_{\phi,A}(\mathbb{R}^n)} \lesssim \|f\|_{H^{M,q,s}_{\phi,A,\mathrm{at}}(\mathbb{R}^n)}.$$

This, combined with the arbitrariness of $f \in \widetilde{H}^{M,q,s}_{\phi,A,\mathrm{at}}(\mathbb{R}^n)$, the fact that $\widetilde{H}^{M,q,s}_{\phi,A,\mathrm{at}}(\mathbb{R}^n)$ is dense in $H^{M,q,s}_{\phi,A,\mathrm{at}}(\mathbb{R}^n)$, and Lemma 3.2, further yields that VA^{-1} is bounded on $H_{\phi,A}(\mathbb{R}^n)$, which proves (i) in this case.

Now we show (ii) by using (i). Let $f \in H_{\phi,A}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then by (1.3) we see that

$$L\cdot LA^{-1}(f) = -f + VA^{-1}(f).$$

This, together with the boundedness obtained in (i), further implies that

$$\big\|L\cdot LA^{-1}(f)\big\|_{H_{\phi,A}(\mathbb{R}^n)} \lesssim \|f\|_{H_{\phi,A}(\mathbb{R}^n)} + \big\|VA^{-1}(f)\big\|_{H_{\phi,A}(\mathbb{R}^n)} \lesssim \|f\|_{H_{\phi,A}(\mathbb{R}^n)}.$$

From this and the fact that $H_{\phi,A}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is dense in $H_{\phi,A}(\mathbb{R}^n)$, we deduce that for any $f \in H_{\phi,A}(\mathbb{R}^n)$, $\|L\cdot L(f)\|_{H_{\phi,A}(\mathbb{R}^n)} + \|Vf\|_{H_{\phi,A}(\mathbb{R}^n)} \lesssim \|Af\|_{H_{\phi,A}(\mathbb{R}^n)}$, which completes the proof of (ii) in this case.

Step 2 We prove Theorem 1.6 under the assumption that B and V satisfy (1.9) and (1.21). In this case, similar to the proof in Step 1, it suffices to prove that for all $\lambda \in \mathbb{C}$ and $(\phi,q,M)_A$-atoms α associated with some ball $B := B(x_B, r_B)$ for some $x_B \in \mathbb{R}^n$ and $r_B \in (0,\infty)$, with $q \in (I(\phi)[r(\phi)]',\infty)$ and $M \in \mathbb{N} \cap (\frac{nq(\phi)}{2i(\phi)},\infty)$, (3.11) holds true in this case.

By [20, Theorem 1(b)], we see that there exist positive constants C and c such that for all $t \in (0,\infty)$ and $x, y \in \mathbb{R}^n$,

$$|K_t(x,y)| \lesssim \frac{1}{t^{n/2}} \exp\big\{-C(1+t[m(x,|B|+V)]^2)^\delta\big\} \exp\Big\{-c\frac{|x-y|^2}{t}\Big\}. \tag{3.12}$$

For any $x \in 4B$, it follows from (3.12) that (3.7) holds true in this case. Via this and repeating the proof of (3.8), we know that (3.8) also holds true in this case.

When $x \in \mathbb{R}^n \setminus (4B)$, we consider the following two cases for r_B.

Case 1 $r_B \in [\{m(x_B,|B|+V)\}^{-1},\infty)$. Similar to (3.9), we see that (3.9) holds true in this case.

Case 2 $r_B \in (0,[m(x_B,|B|+V)]^{-1})$. In this case, by the assumption $0 \le V \lesssim [m(\cdot,|B|+V)]^2$, Lemma 2.1, (3.5), and Hölder's inequality, we conclude that

$$\int_B |V(y)b(y)|\,dy \lesssim \big[m(x_B,|B|+V)\big]^2 (r_B)^2 |B| \|\chi_B\|^{-1}_{L^\phi(\mathbb{R}^n)}.$$

From this, (3.12), and the fact that for any $y \in B$ and $x \in S_j(B)$ with $j \ge 2$, $|x-y| \sim 2^j r_B$, similar to (3.10), we deduce that for any $t \in (0,\infty)$ and $x \in S_j(B)$ with $j \ge 2$,

$$\big|e^{-tA}\big(VA^{-1}(\lambda\alpha)\big)(x)\big| \lesssim 2^{-(n+2)j}|\lambda| \|\chi_B\|^{-1}_{L^\phi(\mathbb{R}^n)}.$$

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Via this estimate and (3.9), similar to the proof of Step 1, we complete the proof of this case and hence Theorem 1.6. ■

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School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, P. R. China
e-mail: dcyang@bnu.edu.cn

School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu 730000, P. R. China
e-mail: yangsb@lzu.edu.cn

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