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Power and commutator structure of groups

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The purpose of this paper is to prove a result about the power and commutator structure of groups which generalises some results of Philip Hall. The results presented here are the key to determining the class of a nilpotent wreath product.

Introduction

0.1 SCOPE

If α and β are two elements of a group, their commutator is defined to be

$$[\alpha, \beta] = \alpha^{-1}\beta^{-1}\alpha\beta \ .$$

An arbitrary group element may be regarded as a commutator

$$[\alpha] = \alpha$$

with only one entry, namely, α ; if α and β are commutators, then $[\alpha, \beta]$ is a commutator whose entries are elements of the disjoint union of the sets of entries of α and β .

In Philip Hall's well-known "contribution to the theory of groups of prime-power order" [*4*] it is established that in an arbitrary group there

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are relationships between the operations of commutation, powering, and multiplication. In particular, his Theorem 3.2, as modified in 3.3 of the present paper, states that for arbitrary elements α and β of a group, prime p, and positive integer h,

$$(\alpha\beta)^{p^h} = \alpha^{p^h}\beta^{p^h} \prod \left\{\kappa_g^{p^{h(g)}} : g \in \Gamma\right\}$$

where Γ is a finite ordered index set, and for each g in Γ the integer $h(g)$ satisfies $0 \le h(g) \le h$, and κ_g is a commutator with at least $\max\{2, p^{h-h(g)}\}$ entries from the set $\{\alpha, \beta\}$.

This and similar relationships are studied in the present paper. Essentially, the conclusion is that if a group element is expressed in terms of a particular subset of the elements of the group by finitely many applications of the operations of commutation, raising to the pth power, and multiplication, then that element may also be expressed in terms of the same subset by a finite sequence of operations in which all commutations precede all pth powerings which in turn precede all multiplications – that is, the element is expressible as a product of p-power powers of commutators. Moreover, the final expression has weight at least as great, measured by each of a family of weight functions, as has the given one.

The weight functions themselves are of some interest. They are used, in 2.4, to define a family of descending central series of a group, which includes as a special case the lower central series. These series and weight functions, as well as main results already mentioned, are used in [6] to determine the class of a nilpotent wreath product.

0.2 APPROACH

The use of the word "expression" in the preceding subsection reflects one of the major difficulties in this undertaking. Suppose $\alpha^p = [\beta, \gamma]$ where α, β, and γ are elements of a group; that is, the same group element may be expressed in terms of α, β, and γ in two essentially different ways. We wish to discuss the expressions rather than the element itself, but how can they be named? The symbols "α^p" and "$[\beta, \gamma]$" denote an element – the same element – of the group.

This difficulty is here resolved by the recognition (compare Ward,

[7], p. 346) that the expressions are themselves elements of a free universal algebra (see Cohn [2]) with operations corresponding to commutation, pth powering, and multiplication in the group. There is a homomorphism from this algebra to the group, under which each expression is mapped to the group element which it represents. Although this paper aims at results about group theory, its arguments are carried out mainly in terms of universal algebras. A reader to whom these are unfamiliar should keep in mind the idea that each element of the universal algebra is essentially a way of obtaining a group element from a generating set.

It may appear surprising that the operation of inversion has been omitted from the universal algebra. The need for it is avoided by the choice of a set of generators for the group which includes the inverses of all its elements, and the observation that the inverse of a commutator $[\beta, \alpha]$ in a group is simply the commutator $[\alpha, \beta]$.

0.3 OUTLINE

In the first two sections, $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebras are introduced, with some special types of elements (notably cpp-elements and scpp-elements) and, in 1.3, several weight functions. Weight ideals, based on these functions, correspond with important descending central series in groups.

When a group with the operations of commutation, raising to the pth power, and multiplication is considered as a $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebra, several laws, or identical relations (defined in 3.1) hold in it. Some of these are used in 3.1 to define group-like varieties. The central theme of the paper is a derivation, from these laws, of laws which express an arbitrary element as a product of scpp-elements without loss of weight under any of the functions defined in 1.3.

One step, in Section 4, is to derive laws linking an arbitrary element with a product of cpp-elements, without loss of weight.

Section 5 interrupts consideration of laws to introduce subword arrays for a cpp-word. In section 6, these are used to obtain further laws which link a cpp-element with a product of scpp-elements.

0.4 NOTATION AND TERMINOLOGY

Elements of groups and algebras will be denoted by lower case Greek letters, such as α, β, φ, and ψ; mappings (and elements of operator sets in universal algebras) by underlined Greek letters such as $\underline{\alpha}, \underline{\gamma}$, and $\underline{\pi}$; and integers and integer-valued functions by lower case Roman letters such as a, b, and f. The symbols $\mathbb{Z}, \mathbb{N}$, and $\mathbb{Z}^+$ respectively denote the sets of all integers, all non-negative integers, and all positive integers. For l in $\mathbb{Z}^+$, the underlined symbol $\underline{l}$ denotes the set

$$\underline{l} = \{i \in \mathbb{N} : 0 \leq i < l\} .$$

Upper case Greek letters Γ, Δ, and Θ are used to represent index sets; it will save repetition later if the convention is laid down now that unless otherwise stated, all sets denoted by these symbols are finite and ordered, so that, for example, such a product as $\prod \{\delta_d : d \in \Delta\}$ of group elements is well-defined. Finally, upper case German script, here represented by double underlining, for example $\underline{\underline{V}}$, is used to represent varieties.

1. A $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-word algebra

1.1 WORDS AND SUBWORDS

Let Ξ be a countably infinite set, and let B be the $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-word algebra on Ξ, where $\underline{\gamma}$ and $\underline{\mu}$ are binary operations and $\underline{\pi}$ a unary operation (see, for example, Cohn [2], III.2, p. 117). The elements of B are of precisely four possible types:

(i) elements of Ξ;

(ii) $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-rows of the form $\alpha\beta\underline{\gamma}$ where α and β are words;

(iii) $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-rows of the form $\alpha\underline{\pi}$ where α is a word; and

(iv) $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-rows of the form $\alpha\beta\underline{\mu}$ where α and β are words.

Correspondingly, given a $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-word ψ in B, define φ to be a subword of ψ (written "$\varphi \leq \psi$") if and only if either

(i) $\varphi = \psi$; or

(ii) $\psi = \alpha\beta\underline{\gamma}$ and either $\varphi \leq \alpha$ or $\varphi \leq \beta$; or

(iii) $\psi = \alpha\underline{\pi}$ and $\varphi \leq \alpha$; or

(iv) $\psi = \alpha\beta\underline{\mu}$ and either $\varphi \le \alpha$ or $\varphi \le \beta$.

If one of the conditions (ii), (iii), or (iv) holds, then φ is a proper subword of ψ (written "$\varphi < \psi$").

If φ is a proper subword of ψ , then there exists a finite sequence $\{\varphi_i : i \in \underline{n+1}\}$ of subwords of ψ such that

$$\varphi = \varphi_0 < \varphi_1 < \ldots < \varphi_n = \psi$$

which is maximal in the sense that for i in $\underline{n}$, there is no subword φ' of ψ satisfying

$$\varphi_i < \varphi' < \varphi_{i+1} .$$

Corresponding to each word ψ in B there exists a finite subset, say $\{\xi_j : j \in \underline{m}\}$, of Ξ such that every element of Ξ which is equal (as a word) to a subword of ψ is contained in $\{\xi_j : j \in \underline{m}\}$. To emphasise this, the notation $\psi = \psi(\xi_0, \ldots, \xi_{m-1})$ will sometimes be used. There is no implication that every element of $\{\xi_j : j \in \underline{m}\}$ should occur as a subword of $\psi(\xi_0, \ldots, \xi_{m-1})$.

If $\underline{\alpha}$ is a homomorphism from B to a $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebra D such that for i in $\underline{m}$, $\xi_i\underline{\alpha} = \rho_i$, then $\psi\underline{\alpha}$ is denoted

$$\psi(\rho_0, \ldots, \rho_{m-1}) .$$

A subword of a word ψ which belongs to Ξ is called an initial subword of ψ .

Words in B which contain neither symbol $\underline{\pi}$ or $\underline{\mu}$ are called c-words. Those which do not contain the symbol $\underline{\mu}$ are called cpp-words. (The reference is to "commutator" and "commutator p-power".) Words of the form $\alpha\underline{\pi}^n$ where α is a c-word and $n \ge 0$ will be called simple cpp-words, abbreviated to scpp-words.

In order to simplify later notation, mappings $\underline{\gamma}'$ and $\underline{\mu}'$ from the set of finite, non-empty ordered sets of elements in B to B are defined inductively as follows:

Let Δ be a finite, non-empty, ordered set of elements in B . If

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$|\Delta| = 1$, say $\Delta = \{\alpha\}$, then

$$\Delta\underline{\mu}' = \alpha \text{ and } \Delta\underline{\gamma}' = \alpha .$$

If $|\Delta| > 1$, then let Δ^* be the ordered set obtained by deleting the "last element", α say, of Δ ; and define

$$\Delta\underline{\mu}' = \Delta^*\underline{\mu}'\alpha\underline{\mu} \text{ and } \Delta\underline{\gamma}' = \Delta^*\underline{\gamma}'\alpha\underline{\gamma} .$$

That is, the effect of $\underline{\mu}'$ or $\underline{\gamma}'$ on an ordered set is that of successive application of the operation $\underline{\mu}$ or $\underline{\gamma}$ respectively to the elements of the set arranged with left-normed bracketing.

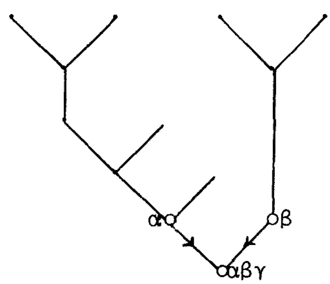
1.2 THE TREE OF A cpp-WORD

In the following pages, much attention will be given to cpp-words; that is, words not containing the symbol $\underline{\mu}$. The structure of a cpp-word may conveniently be visualised in terms of a graph which is in fact a rooted tree, or arborescence, except that the directions of the arrows constituting its arcs are here reversed (for example, see Berge [1], Chapter 3, §3, p. 33). There is a one-one correspondence between vertices of the tree and subwords of the word, the root corresponding to the word itself. Every vertex is considered labelled with the corresponding subword. To each symbol $\underline{\gamma}$ in the word corresponds a pair of arcs directed toward the same vertex, and to each symbol $\underline{\pi}$ a single arc.

A formal definition may be made inductively, as follows:

(i) The tree representing an element ξ of Ξ consists of a single vertex labelled ξ .

(ii) The tree representing a word $\alpha\beta\underline{\gamma}$ is obtained from the disjoint union of the trees representing α and β , drawn with that representing α on the left, by adjoining a new vertex labelled $\alpha\beta\underline{\gamma}$ which becomes its root, and an arc directed toward this new vertex from each of the previous roots labelled α and β . The new arcs are referred to as $\underline{\gamma}$-arcs, sometimes as left- and right-$\underline{\gamma}$-arcs, respectively.



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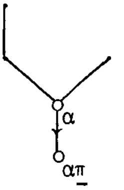
(iii) The tree representing a word $\alpha\underline{\pi}$ is obtained from the tree representing α by adjoining a new vertex labelled $\alpha\underline{\pi}$ which becomes its root, and an arc directed toward it from the previous root labelled α . The new arc is called a $\underline{\pi}$-arc.

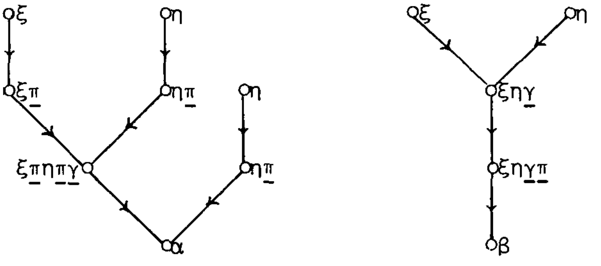


The introduction of a distinction between left- and right-$\underline{\gamma}$-arcs directed toward a vertex makes it possible to establish a one-one correspondence between such trees with labelled vertices and elements of B .

As examples, if ξ and η are elements of Ξ , and α and β are the words

$$\alpha = \xi\underline{\pi}\eta\underline{\pi}\underline{\gamma}\eta\underline{\pi}\underline{\gamma} \quad \text{and} \quad \beta = \xi\eta\underline{\gamma}\underline{\pi}\underline{\pi} ,$$

then the trees representing α and β are:



When distinct subwords of a given word are equal as words, then different vertices of the tree of that given word have the same label.

In the tree corresponding to a word ψ , a vertex which is not the terminal point of an arc must correspond to an initial subword of ψ . Such a vertex is called an initial vertex of the tree.

Through the remainder of this paper, tree diagrams will be drawn only occasionally. However, large numbers of them were used in the process of formulating the various definitions, statements, and proofs; and the reader will probably find it an advantage to draw his or her own diagrams quite frequently, and to formulate definitions in terms of the diagrams.

1.3 INTEGERS ASSOCIATED WITH WORDS

The first two functions to be defined are from the set of ordered

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pairs $\{(\varphi, \psi) : \psi \in B$ and $\varphi \le \psi\}$ to the set $\mathbb{N}$ of non-negative integers. Let ψ be a word in B and φ a subword of ψ, and let

$$\varphi = \varphi_0 < \varphi_1 < \ldots < \varphi_n = \psi$$

be the maximal sequence of subwords linking φ with ψ described earlier. Define $k(\varphi, \psi)$ and $l(\varphi, \psi)$ to be the number of subwords in the set $\{\varphi_i : 1 \le i \le n\}$ which terminate in $\underline{\gamma}$ and in $\underline{\pi}$, respectively. (Note that $\varphi_0 = \varphi$ is *not* contained in the set. Subwords ending in $\underline{\mu}$ are omitted from the counts.) When ψ is a cpp-word, the directed path in the tree representing ψ from the vertex labelled φ to the root passes through precisely $k(\varphi, \psi)$ $\underline{\gamma}$-arcs and $l(\varphi, \psi)$ $\underline{\pi}$-arcs.

Several weight functions from B to $\mathbb{Z}^+$ will now be introduced. The most important of these, to be called c-weight and cpp-weight (again referring to "commutator" and "commutator p-power" respectively) are in fact special cases of more general functions which will be defined first.

Let p be a fixed prime, e an integer which may be either 1 or p, and a and b integers such that $a \ge b \ge 0$ and $a \ge 1$ (this use of the symbols "e", "a", and "b" will recur throughout this paper). The function $w^e_{a,b} : B \to \mathbb{Z}^+$ is defined inductively as follows:

(i) if $\psi \in \Xi$, then $w^e_{a,b}(\psi) = a$;

(ii) if $\psi = \alpha\beta\underline{\gamma}$, then $w^e_{a,b}(\psi) = w^e_{a,b}(\alpha) + w^e_{a,b}(\beta)$;

(iii) if $\psi = \alpha\underline{\pi}$, then

(a) $w^1_{a,b}(\psi) = w^1_{a,b}(\alpha) + b$ and

(b) $w^p_{a,b}(\psi) = pw^p_{a,b}(\alpha)$; and

(iv) if $\psi = \alpha\beta\underline{\mu}$, then $w^e_{a,b}(\psi) = \min\left\{w^e_{a,b}(\alpha), w^e_{a,b}(\beta)\right\}$.

For example, if ξ and η are elements of Ξ, then

$$w^1_{a,b}(\xi\eta\underline{\gamma}\eta\underline{\gamma}) = 3a ,$$

$$w^p_{a,b}(\xi\eta\underline{\gamma}\eta\underline{\gamma}) = 3a \ ,$$

$$w^1_{a,b}(\xi\underline{\pi}\eta\underline{\gamma}\eta\underline{\gamma}\underline{\pi}\underline{\pi}) = 3a + 3b \ ,$$

$$w^p_{a,b}(\xi\underline{\pi}\eta\underline{\gamma}\eta\underline{\gamma}\underline{\pi}\underline{\pi}) = a\left(p^3+2p^2\right) \ ,$$

and

$$w^p_{a,b}(\xi\underline{\pi}\eta\underline{\gamma}\eta\underline{\gamma}\underline{\pi}\underline{\pi}\xi\underline{\mu}) = a \ .$$

The function $w^p_{a,b}$ is independent of b ; in fact it is easy to see that for all ψ in B ,

$$w^p_{a,b}(\psi) = aw^p_{1,0}(\psi) \ .$$

The function $w^1_{a,b}$ does depend on b . An easy way to compute its value is to note that if a cpp-word ψ contains l symbols $\underline{\gamma}$ and m symbols $\underline{\pi}$, then $w^1_{a,b}(\psi) = a(l+1) + bm$.

In the application to wreath products in [6] these weight functions are used in a normal subgroup of finite p-power index in a group, and the integers a and b depend on the quotient group. When the quotient group is trivial, $a = 1$ and $b = 0$. This special case is the important one already referred to: $w^1_{1,0}$ is the c-weight function and $w^p_{1,0}$ the cpp-weight function.

These simpler functions may be defined for a word, not only as a whole, but also with respect to each element of the generating set Ξ . That is, functions from $B \times \Xi$ to $\mathbb{N}$ are defined as follows: for all ξ in Ξ ,

(i) if $\psi \in \Xi$, then

$$\text{c-wt}(\psi, \xi) = \text{cpp-wt}(\psi, \xi) = \begin{cases} 1 & \text{if } \psi = \xi \ , \\ 0 & \text{if } \psi \neq \xi \ ; \end{cases}$$

(ii) if $\psi = \alpha\beta\underline{\gamma}$, then

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$$\text{c-wt}(\psi, \xi) = \text{c-wt}(\alpha, \xi) + \text{c-wt}(\beta, \xi)$$

and

$$\text{cpp-wt}(\psi, \xi) = \text{cpp-wt}(\alpha, \xi) + \text{cpp-wt}(\beta, \xi) \;;$$

(iii) if $\psi = \alpha\underline{\pi}$, then

$$\text{c-wt}(\psi, \xi) = \text{c-wt}(\alpha, \xi)$$

and

$$\text{cpp-wt}(\psi, \xi) = p.\text{cpp-wt}(\alpha, \xi) \;;$$

and

(iv) if $\psi = \alpha\beta\underline{\mu}$, then

$$\text{c-wt}(\psi, \xi) = \min\{\text{c-wt}(\alpha, \xi), \text{c-wt}(\beta, \xi)\} ,$$

and

$$\text{cpp-wt}(\psi, \xi) = \min\{\text{cpp-wt}(\alpha, \xi), \text{cpp-wt}(\beta, \xi)\} .$$

Note that for all words ψ in B ,

$$\text{c-wt}(\psi) = w^{1}_{1,0}(\psi) = \sum \{\text{c-wt}(\psi, \xi) : \xi \in \Xi\}$$

and

$$\text{cpp-wt}(\psi) = w^{p}_{1,0}(\psi) = \sum \{\text{cpp-wt}(\psi, \xi) : \xi \in \Xi\} .$$

In the special case that ψ is a cpp-word, simple interpretations of these weights are available. For example, it is easy to see that c-wt(ψ, ξ) is equal to the number of occurrences of the symbol ξ in the word ψ . This in turn is equal to the number of vertices labelled ξ in the tree representing ψ . Correspondingly,

$$\text{cpp-wt}(\psi, \xi) = \sum \{p^{l(\rho,\psi)} : \rho \leq \psi \text{ and } \rho = \xi\} .$$

For words not containing the symbol $\underline{\pi}$, the c-wt and cpp-wt take equal values. In particular on a c-word, which corresponds to a commutator in a group, both take values equal to the usual commutator weight.

The following lemma deals with the behaviour of the weight functions on a cpp-word under operations that may be regarded either as endomorphisms of B or as "substitutions" into words.

1.4 LEMMA

Let $\varphi = \varphi(\xi_0, \ldots, \xi_m)$ *be a cpp-word, and for* i *in* $\underline{m}$ *let* $\alpha_i = \alpha_i(\xi_0, \ldots, \xi_{l(i)-1})$ *also be cpp-words. (Without loss of generality, it may be assumed that* $m \geq l(i)$ *for all* i *in* $\underline{m}$ *; then set* $\alpha_i = \alpha_i(\xi_0, \ldots, \xi_m)$ *for all* i *in* $\underline{m}$ *.) Let* $\underline{\alpha}$ *be an endomorphism of* B *such that* $\xi_i\underline{\alpha} = \alpha_i$ *for all* i *in* $\underline{m}$ *. Then, for all* h *in* $\underline{m}$ *and all integers* a *and* b *satisfying* $a \geq b \geq 0$ *and* $a \geq 1$,

(a) $w^1_{a,b}(\varphi\underline{\alpha}) = \sum\left\{\text{c-wt}(\varphi, \xi_i)w^1_{a,b}(\alpha_i) : i \in \underline{m}\right\} + w^1_{a,b}(\varphi) - w^1_{a,0}(\varphi)$,

(b) $w^p_{a,b}(\varphi\underline{\alpha}) = \sum\left\{\text{cpp-wt}(\varphi, \xi_i)w^p_{a,b}(\alpha_i) : i \in \underline{m}\right\}$,

(c) $\text{c-wt}(\varphi\underline{\alpha}, \xi_h) = \sum\left\{\text{c-wt}(\varphi, \xi_i)\text{c-wt}(\alpha_i, \xi_h) : i \in \underline{m}\right\}$, *and*

(d) $\text{cpp-wt}(\varphi\underline{\alpha}, \xi_h) = \sum\left\{\text{cpp-wt}(\varphi, \xi_i)\text{cpp-wt}(\alpha_i, \xi_h) : i \in \underline{m}\right\}$.

Proof. The proofs of *(a)*, *(b)*, *(c)*, and *(d)* are very similar in outline, proceeding by induction on the number of symbols $\underline{\gamma}$ or $\underline{\pi}$ in the word φ , and treating separately

(i) the initial case, where without loss of generality $\varphi = \xi_0$ and $\varphi\underline{\alpha} = \alpha_0$,

(ii) the case in which φ ends in the symbol $\underline{\gamma}$, and

(iii) the case in which φ ends in the sumbol $\underline{\pi}$.

The details are routine, and are omitted.

1.5 PRE-ORDERS ON B

A relation denoted $\leq'$ is defined on B by the condition that $\alpha \leq' \beta$ if and only if every weight function defined in 1.3 takes at α a value less than or equal to its value at β . More formally, $\alpha \leq' \beta$ if and only if for all integers a and b such that $a \geq b \geq 0$ and $a \geq 1$, all e in $\{1, p\}$, and all ξ in Ξ ,

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$$w_{a,b}(\alpha) \leq w^{e}_{a,b}(\beta) \ ,$$

$$\text{c-wt}(\alpha,\ \xi) \leq \text{c-wt}(\beta,\ \xi) \ ,$$

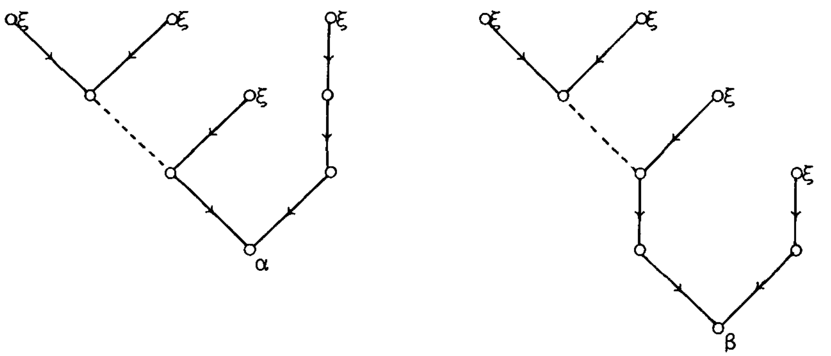
and

$$\text{cpp-wt}(\alpha,\ \xi) \leq \text{cpp-wt}(\beta,\ \xi) \ .$$

The relation $\leq'$ is clearly transitive, and is therefore a pre-order. It is reflexive, but is not antisymmetric, even on cpp-words.

For example, if

$$\alpha = \xi(\xi\underline{\gamma})^{p-1}\xi\underline{\pi}\underline{\pi}\underline{\gamma} \quad \text{and} \quad \beta = \xi(\xi\underline{\gamma})^{p-1}\underline{\pi}\xi\underline{\pi}\underline{\gamma} \ ,$$



then

$$w^{1}_{a,b}(\alpha) = w^{1}_{a,b}(\beta) = a(p+1) + 2b \ ,$$

$$w^{p}_{a,b}(\alpha) = w^{p}_{a,b}(\beta) = a\left(p^2+p\right) \ ,$$

$$\text{c-wt}(\alpha,\ \xi) = \text{c-wt}(\beta,\ \xi) = p + 1 \ ,$$

and

$$\text{cpp-wt}(\alpha,\ \xi) = \text{cpp-wt}(\beta,\ \xi) = p^2 + p \ .$$

Nevertheless the structures of the words α and β are quite different, as can be seen by considering values of $l(\rho,\ \alpha)$ and $l(\rho,\ \beta)$ for various initial subwords ρ .

A non-reflexive relation $<''$ is now defined as follows: $\alpha <'' \beta$ if and only if $\alpha \leq' \beta$, and for all suitable integer triples $(a,\ b,\ e)$,

$$w^e_{a,b}(\alpha) < w^p_{a,b}(\beta) \;.$$

Note that $<''$ is also a pre-order. The preceding example makes clear that " $\alpha <'' \beta$ " is a stronger statement than " $\alpha \le' \beta$ and $\alpha \ne \beta$ ".

The pre-orders are affected in the obvious ways by operations $\underline{\gamma}$, $\underline{\pi}$, and $\underline{\mu}$; namely:

(a) $\alpha <'' \alpha\beta\underline{\gamma}$;

(b) $\alpha \le' \alpha\underline{\pi}$;

(c) $\alpha\beta\underline{\mu} \le' \alpha$;

and "respect" the operations, in the sense that if $\alpha_1 \le' \beta_1$ and $\alpha_2 \le' \beta_2$, then:

(d) $\alpha_1\alpha_2\underline{\gamma} \le' \beta_1\beta_2\underline{\gamma}$;

(e) $\alpha_1\underline{\pi} \le' \beta_1\underline{\pi}$;

(f) $\alpha_1\alpha_2\underline{\mu} \le' \beta_1\beta_2\underline{\mu}$.

Implications similar to (d), (e), and (f) with $<''$ replacing $\le'$ also hold. Note also that

(g) $\alpha \le' \beta_1$ and $\alpha \le' \beta_2 \Leftrightarrow \alpha \le' \beta_1\beta_2\underline{\mu}$.

One special situation in which these relations will frequently be used deserves comment. If Γ and Δ are finite (non-empty) ordered sets of cpp-words, then the relation $\Gamma\underline{\mu}' \le' \Delta\underline{\mu}'$ holds if and only if for each weight function w defined earlier and each element β in Δ , there exists α in Γ such that $w(\alpha) \le w(\beta)$. Since in general the choice of α depends on w as well as on β , it does not necessarily follow that for each β in Δ there exists α in Γ such that $\alpha \le' \beta$. A useful exception holds when $|\Gamma| = 1$; the statements " $\alpha \le' \Delta\underline{\mu}'$ " and " $\alpha \le' \beta$ for all β in Δ " are equivalent.

1.6 LEMMA.

Let $\varphi = \varphi(\xi_0, \ldots, \xi_{m-1})$ *be a word (in* B *) and for* i *in* $\underline{m}$ *let* α_i *and* β_i *be words such that* $\alpha_i \le' \beta_i$. *Then*

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$$\varphi(\alpha_0, \ldots, \alpha_{m-1}) \leq' \varphi(\beta_0, \ldots, \beta_{m-1}) .$$

If, further, for some i *in* $\underline{m}$,

$$\text{c-wt}(\varphi, \xi_i) \geq 1 \quad \textit{and} \quad \alpha_i <'' \beta_i ,$$

then

$$\varphi(\alpha_0, \ldots, \alpha_{m-1}) <'' \varphi(\beta_0, \ldots, \beta_{m-1}) .$$

Proof. (i) If φ contains no symbols $\underline{\gamma}$, $\underline{\pi}$, or $\underline{\mu}$, then without loss of generality $\alpha = \xi_0$, and all conclusions of the lemma clearly hold.

For the remaining cases in which φ does contain at least one symbol $\underline{\gamma}$, $\underline{\pi}$, or $\underline{\mu}$, suppose inductively that the result is already established for all words with fewer such symbols. Let $\underline{\alpha}$ and $\underline{\beta}$ be endomorphisms of B defined by

$$\xi_i\underline{\alpha} = \alpha_i \quad \text{and} \quad \xi_i\underline{\beta} = \beta_i \quad \text{for all } i \text{ in } \underline{m} ,$$

and

$$\xi\underline{\alpha} = \xi_0 \quad \text{and} \quad \xi\underline{\beta} = \xi_0 \quad \text{for all } \xi \text{ in } \Xi\backslash\{\xi_i : i \in \underline{m}\} .$$

Hence $\varphi(\alpha_0, \ldots, \alpha_{m-1}) = \varphi\underline{\alpha}$ and $\varphi(\beta_0, \ldots, \beta_{m-1}) = \varphi\underline{\beta}$.

(ii) If $\varphi = \psi_1\psi_2\underline{\gamma}$, then by the inductive hypothesis $\psi_1\underline{\alpha} \leq' \psi_1\underline{\beta}$ and $\psi_2\underline{\alpha} \leq' \psi_2\underline{\beta}$. Hence, by property (d) in 1.5.

$$\varphi\underline{\alpha} = (\psi_1\psi_2\underline{\gamma})\underline{\alpha} \leq' (\psi_1\psi_2\underline{\gamma})\underline{\beta} = \varphi\underline{\beta} ,$$

as required. If for some i in $\underline{m}$, $\text{c-wt}(\varphi, \xi_i) \geq 1$ and $\alpha_i <'' \beta_i$, then there is an element d in $\{1, 2\}$ such that $\text{c-wt}(\psi_d, \xi_i) \geq 1$, whence by induction $\psi_d\underline{\alpha} <'' \psi_d\underline{\beta}$; and again the required conclusion follows.

(iii) If $\varphi = \psi\underline{\pi}$, then from either of the relations $\psi\underline{\alpha} \leq' \psi\underline{\beta}$ or $\psi\underline{\alpha} <'' \psi\underline{\beta}$, the corresponding relation between $\varphi\underline{\alpha}$ and $\varphi\underline{\beta}$ follows immediately, by property (e) in 1.5.

(iv) If $\varphi = \psi_1\psi_2\underline{\mu}$, then the inductive hypothesis states that $\psi_1\underline{\alpha} \leq' \psi_1\underline{\beta}$ and $\psi_2\underline{\alpha} \leq' \psi_2\underline{\beta}$, whence by property (f) in 1.5, the required

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result follows. If for some i in $\underline{m}$, $\text{c-wt}(\varphi, \xi_i) \geq 1$, then both $\text{c-wt}(\psi_1, \xi_i) \geq 1$ and $\text{c-wt}(\psi_2, \xi_i) \geq 1$. Thus, inductively, if $\alpha_i <'' \beta_i$ then both $\psi_1\underline{\alpha} <'' \psi_1\underline{\beta}$ and $\psi_2\underline{\alpha} <'' \psi_2\underline{\beta}$, and the required result follows as before. □

2. Other $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebras

2.1 DEFINITIONS

Every countable $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebra is a homomorphic image of the word algebra B described in 1.1. The homomorphic images of c-words, cpp-words, and scpp-words are referred to as c-elements, cpp-elements, and scpp-elements respectively. In a group, a c-element is more usually called a commutator, and the homomorphic images of the initial subwords of the corresponding c-word are called its entries.

There are difficulties in extending definitions of weight functions, or the pre-order relations $\leq'$ and $<''$ to general $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebras. For example, there may be many different words in B , with different weights, mapped by a homomorphism to the same image.

For a given weight function w on B , a countable $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebra D , and a surjective homomorphism $\underline{\alpha} : B \to D$, it is possible to define a function from D to $Z^+ \cup \{\infty\}$ by setting, for all δ in D ,

$$w(\delta, \underline{\alpha}) = \begin{cases} \max\{w(\varphi) : \varphi\underline{\alpha} = \delta\} & \text{if this exists} \\ \infty & \text{if no such maximum exists.} \end{cases}$$

Correspondingly, a weight function from D to $Z^+ \cup \{\infty\}$ which is independent of a particular homomorphism may be defined by setting, for all δ in D ,

$$w(\delta) = \begin{cases} \max\{w(\delta, \underline{\alpha}) : \underline{\alpha} \text{ a surjective homomorphism from } B \text{ to } D\} \\ \qquad\qquad\qquad\qquad \text{if such a maximum exists} \\ \infty \qquad\qquad\qquad\qquad \text{if no such maximum exists.} \end{cases}$$

This procedure involves the possibility that different weight functions might, for a fixed element of D , take values related to quite differently-structured elements of B , so that the relationship between values taken by different weight functions on a fixed element of D is lost. Such an approach is implicit in the use of the weight ideals defined in the next subsection.

Comparisons between elements of D may be made in terms of relations, again denoted $\leq'$ and $<''$, defined as follows:

> for arbitrary ρ and σ in D , $\rho \leq' \sigma$ if and only if for every word φ in B and surjective homomorphism $\underline{\alpha}^*$ from B to D such that $\varphi\underline{\alpha}^* = \rho$, there exist an element ψ in B and surjective homomorphism $\underline{\alpha}$ from B to D such that $\varphi \leq' \psi$, $\psi\underline{\alpha} = \sigma$, and for all ξ in Ξ which occur as subwords of φ , $\xi\underline{\alpha} = \xi\underline{\alpha}^*$. (The last condition implies that $\varphi\underline{\alpha} = \varphi\underline{\alpha}^* = \rho$.)

Relation $<''$ is defined similarly. Both relations defined in this way are transitive; the proof is routine, and is omitted.

2.2 WEIGHT IDEALS OF $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-ALGEBRAS

As before, let p be a fixed prime, and a and b integers such that $a \geq b \geq 0$ and $a \geq 1$. Suppose that the generating set Ξ for B is $\Xi = \{\xi_i : i \in \mathbb{N}\}$. Now, corresponding to sets of non-negative integers

$$\{l(e) : e = 1 \text{ or } e = p\}, \{w(i) : i \in \mathbb{N}\} \text{ , and } \{v(i) : i \in \mathbb{N}\} \text{ ,}$$

satisfying the conditions that $\sum \{w(i) : i \in \mathbb{N}\}$ and $\sum \{v(i) : i \in \mathbb{N}\}$ are finite, let I be the set of words α in B satisfying the conditions:

(a) for all $e \in \{1, p\}$, $w^e_{a,b}(\alpha) \geq l(e)$,

(b) for all $i \in \mathbb{N}$, $\text{c-wt}(\alpha, \xi_i) \geq w(i)$, and

(c) for all $i \in \mathbb{N}$, $\text{cpp-wt}(\alpha, \xi_i) \geq v(i)$.

The set I is closed under the operations $\underline{\gamma}$, $\underline{\pi}$, and $\underline{\mu}$. In fact, if $\alpha \in I$ and $\beta \in B$, then $\alpha\beta\underline{\gamma} \in I$ and $\beta\alpha\underline{\gamma} \in I$. If the operation $\underline{\mu}$ is regarded as playing the role normally played in an algebra by addition, the set I may be called an *ideal* of B .

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If D is an arbitrary $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebra and $\underline{\alpha}$ is a surjective homomorphism from B to D, then the image in D of an ideal in B is again an ideal, in the same sense.

Conditions (b) and (c) may be made trivial by setting $w_i = v_i = 0$ for all i in $\mathbb{N}$. The ideals defined in this way by condition (a) only are easily seen to be fully invariant, mapped into themselves by every endomorphism of B. Such ideals of B, and their images in other $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebras, will be called *weight ideals*. Lemma 2.3 shows that a weight ideal in an arbitrary $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebra is independent of the surjective homomorphism from B to the algebra used in its definition.

2.3 LEMMA

If I is a weight ideal of the $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-word algebra B, and if $\underline{\alpha}$ and $\underline{\beta}$ are surjective homomorphisms from B to a $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-word algebra D, then $I\underline{\alpha} = I\underline{\beta}$.

Proof. From the symmetry of the situation, it is sufficient to show that $I\underline{\alpha} \subseteq I\underline{\beta}$. Let $\rho \in I\underline{\alpha}$. Then there exists a word, say $\varphi = \varphi(\xi_0, \ldots, \xi_{z-1})$ in I such that $\varphi\underline{\alpha} = \rho$. Since $\underline{\beta}$ is a surjection, for each i in $\underline{z}$ there exists a word χ_i in B such that $\chi_i\underline{\beta} = \xi_i\underline{\alpha}$; then $\varphi(\chi_0, \ldots, \chi_{z-1})\underline{\beta} = \varphi\underline{\alpha} = \rho$. However, by Lemma 1.4 the word $\varphi(\chi_0, \ldots, \chi_{z-1})$ has at least as great a value under each weight function $w^e_{a,b}$ used in the definition of I as has $\varphi = \varphi(\xi_0, \ldots, \xi_{z-1})$; and hence $\varphi(\chi_0, \ldots, \chi_{z-1}) \in I$. Thus $\rho \in I\underline{\beta}$, as required. □

2.4 DESCENDING CENTRAL SERIES

Let $w^e_{a,b}$ be one of the weight functions defined in 1.3, and $\underline{\alpha}$ a surjective homomorphism from B to a $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebra D. Define

$$\gamma_v^{a,b,e}(B) = \left\{\varphi \in B : w^e_{a,b}(\varphi) \geq v\right\}$$

and

$$\gamma_v^{a,b,e}(D) = \left\{\varphi\underline{\alpha} : \varphi \in \gamma_v^{a,b,e}(B)\right\} .$$

The series

$$D = \gamma_1^{a,b,e}(D) \supseteq \gamma_2^{a,b,e}(D) \supseteq \dots \supseteq \gamma_i^{a,b,e}(D) \supseteq \dots$$

obtained in this way from each of the weight functions is *strongly central* in the sense that for all i, j in Z^+,

$$\varphi \in \gamma_i^{a,b,e}(D) \;,\quad \psi \in \gamma_j^{a,b,e}(D) \text{ only if } \varphi\psi\underline{\gamma} \in \gamma_{i+j}^{a,b,e}(D) \;.$$

It is not hard to see that the series based on the weight function $w_{1,0}^1$ is the most rapidly descending of these series. In fact, a routine inductive proof shows that it descends more rapidly than any other central series - which, when D is a group, identifies it as the lower central series.

For each weight function other than $w_{1,0}^1$, it is also true that

$$\varphi \in \gamma_v^{a,b,e}(D) \text{ only if } \varphi\underline{\pi} \in \gamma_{v+1}^{a,b,e}(D) \;.$$

In a group, this means that the factor $\gamma_v^{a,b,e}(D)/\gamma_{v+1}^{a,b,e}(D)$ is not only central, but elementary. A similar inductive proof shows that the series based on the function $w_{1,1}^1$ is the most rapidly descending elementary central series.

A series which appears particularly important in the study of extensions by p-groups is that defined by the function $w_{1,0}^p$. This is the most rapidly descending central series such that

$$\varphi \in \gamma_v^{a,b,e}(D) \text{ only if } \varphi\underline{\pi} \in \gamma_{pv}^{a,b,e}(D) \;,$$

and is here called the *cpp-series*.

When D is a group G, set

$$\gamma_i(G) = \gamma_i^{1,0,1}(G) \;,$$

$$\varepsilon_i(G) = \gamma_i^{1,1,1}(G) \;,$$

and

$$\pi_i(G) = \gamma_i^{1,0,p}(G) \;.$$

Note, however, that the lower elementary central series and the cpp-series are well-defined only after the prime p has been chosen.

3. Group-like varieties of $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebras

In this section, the idea of a law is introduced. When a group is considered as a $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebra, with commutation, raising to the pth power, and multiplication as the three operations, many laws hold. Some obvious ones are not considered in this section, but those that are chosen for the definition of a grouplike variety, and checked to hold in all groups, are sufficient for the calculations in later sections.

3.1 DEFINITIONS AND NOTATION

Following Cohn [2], IV.1, p. 162, define a *law* in a $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebra to be a pair of words in $B \times B$. The law (θ, φ) is said to *hold* in a $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebra D if under every homomorphism $\underline{\alpha} : B \to D$, the words θ and φ have the same image, that is, $\theta\underline{\alpha} = \varphi\underline{\alpha}$.

The *variety* defined by a set of laws is the class of algebras in which all laws of the set hold. The statement that a law (θ, φ) holds in a variety $\underline{\underline{V}}$ will be denoted $\theta \overset{\underline{\underline{V}}}{=} \varphi$.

Let ξ, η, ζ, ξ_i for i in N , and η_i for i in N be elements of Ξ . A *group-like variety* of $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebras is defined to be a variety $\underline{\underline{V}}$ with laws of the form:

(i) $\xi\eta\underline{\mu}\zeta\underline{\mu} \overset{\underline{\underline{V}}}{=} \xi\eta\zeta\underline{\mu}\underline{\mu}$ (that is, the operation $\underline{\mu}$ is associative);

(ii) $\eta\xi\underline{\mu} \overset{\underline{\underline{V}}}{=} \xi\eta\underline{\mu}\eta\xi\underline{\gamma}\underline{\mu}$ (if $\underline{\mu}$ is identified as group multiplication, then this law identifies $\underline{\gamma}$ as the operation of commutation);

(iii) $\xi(\xi\underline{\mu})^{p-1} \overset{\underline{\underline{V}}}{=} \xi\underline{\pi}$ (again, if $\underline{\mu}$ is identified as group multiplication, then this law identifies $\underline{\pi}$ as the operation of raising to the pth power);

(iv) for arbitrary finite sub-ordered-sets Γ and Δ of N ,

$$\{\xi_g : g \in \Gamma\}\underline{\mu}'\{\eta_d : d \in \Delta\}\underline{\mu}'\underline{\gamma} \overset{\underline{\underline{V}}}{=} \{\xi_g\eta_d\underline{\gamma} : (g, d) \in \Gamma \times \Delta\}\underline{\mu}'\{\zeta_t : t \in \Theta_1\}\underline{\mu}'\underline{\mu}$$

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where for t in Θ_1 , ζ_t is a c-word and there exists a triple $(g(t), d(t), h(t))$ either belonging to $\Gamma \times \Delta \times \Gamma$ such that $g(t) \neq h(t)$ and

$$\xi_{g(t)}\eta_{d(t)}\underline{\gamma}\xi_{h(t)}\underline{\gamma} \leq' \zeta_t ,$$

or belonging to $\Gamma \times \Delta \times \Delta$ such that $d(t) \neq h(t)$ and

$$\xi_{g(t)}\eta_{d(t)}\underline{\gamma}\eta_{h(t)}\underline{\gamma} \leq' \zeta_t .$$

(If the product indexed by Θ_1 were deleted from the right-hand side, this law would say that the operation $\underline{\gamma}$ distributed over the operation $\underline{\mu}$.)

(v) For arbitrary finite sub-ordered-set Δ of $\mathbb{N}$, and m in $\mathbb{Z}^+$,

$$\{\xi_d : d \in \Delta\}\underline{\mu}'\underline{\pi}^m \overset{V}{=} \left\{\zeta_t\underline{\pi}^{l(t)} : t \in \Theta_2\right\}\underline{\mu}'$$

where for all t in Θ_2 , ζ_t is a c-word and

$$\{\text{c-wt}(\zeta_t, \xi_d) : d \in \Delta\} \geq p^{m-l(t)} .$$

Further, there is a subset Θ_2^* of Θ_2 such that

$$\left\{\zeta_t\underline{\pi}^{l(t)} : t \in \Theta_2^*\right\} = \left\{\xi_d\underline{\pi}^m : d \in \Delta\right\} ,$$

and for each t in $\Theta_2 \backslash \Theta_2^*$, there are at least two distinct elements d in Δ such that $\text{c-wt}(\zeta_t, \xi_d) \geq 1$.

(If the terms indexed by elements of $\Theta_2 \backslash \Theta_2^*$ were deleted from the right-hand side, this law would say that the operation $\underline{\pi}$ distributed over the operation $\underline{\mu}$.)

(vi) For arbitrary m in $\mathbb{Z}^+$,

(a) $\xi\underline{\pi}^m\eta\underline{\gamma} \overset{V}{=} \xi\eta\underline{\gamma}\underline{\pi}^m\left\{\zeta_t\underline{\pi}^{l(t)} : t \in \Theta_3\right\}\underline{\mu}'\underline{\mu}$, and

(b) $\xi\eta\underline{\pi}^m\underline{\gamma} \overset{V}{=} \left\{\zeta_t\underline{\pi}^{l(t)} : t \in \Theta_4\right\}\underline{\mu}'\xi\eta\underline{\gamma}\underline{\pi}^m\underline{\mu}$;

where for each t in Θ_3 , ζ_t is a c-word such that

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$$\text{c-wt}(\zeta_t, \xi) \geq \max\{2, p^{m-l(t)}\} \text{ and } \text{c-wt}(\zeta_t, \eta) \geq 1 \text{ , and}$$

for each t in Θ_4 , ζ_t. is a c-word such that

$$\text{c-wt}(\zeta_t, \xi) \geq 1 \text{ and } \text{c-wt}(\zeta_t, \eta) \geq \max\{2, p^{m-l(t)}\} .$$

(If all terms indexed by Θ_3 and Θ_4 were deleted from the right-hand sides, and if the language were stretched a little to refer in this way to a binary and a unary operation, then this law would say that the operations $\underline{\gamma}$ and $\underline{\pi}$ commuted.)

It should be noted that because elements of the sets $\{\zeta_t : t \in \Theta_i\}$ for $1 \leq i \leq 4$ are not fully specified, (iv), (v), and (vi) are not, strictly speaking, laws; but are conditions which laws must satisfy.

Note that in each of the six types of law, the expression on the right-hand side is a $\underline{\mu}$-product of scpp-words. Also, if the left-hand side is α and the right-hand side is β , it can be checked that $\alpha \leq' \beta$.

Note also that when ζ is a cpp-word, the statement " $\text{c-wt}(\zeta, \xi) \geq n$" is equivalent to the statement "the word ζ has at least n distinct subwords, all equal as words to ξ ". The latter interpretation will be used in some applications of these laws, particularly in Section 6.

Simplified notation and terminology is now introduced, in the light of the application of this theory of $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebras to groups. The operation $\underline{\mu}$ will be referred to as multiplication and denoted simply by juxtaposition; that is, $\alpha\beta\underline{\mu}$ will be written as $\alpha\beta$. The familiar symbol $\prod$ before an ordered set of elements of a $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebra will replace $\underline{\mu}'$ after the set, and α and β will be called factors of the product $\alpha\beta$. Similarly, the expressions $\alpha\beta\underline{\gamma}$, $\{\alpha_i : i \in \underline{m}\}\underline{\gamma}'$, and $\alpha(\beta\underline{\gamma})^n$ will be written $[\alpha, \beta]$, $[\alpha_0, \alpha_1, \ldots, \alpha_{m-1}]$, and $[\alpha, n\beta]$, respectively. However the expression $\alpha\underline{\pi}$ is retained, because the expression α^p now means $\alpha(\alpha\underline{\mu})^{p-1}$.

The laws, in this notation, are:

(i) $$(\xi\eta)\zeta \overset{\text{V}}{=} \xi(\eta\zeta) \text{ ;}$$

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(ii) $$\eta\xi \overset{\underline{V}}{=} \xi\eta[\eta,\ \xi]\ ;$$

(iii) $$\xi\underline{\pi} \overset{\underline{V}}{=} \xi^p\ ;$$

(iv) $$\left[\prod\{\xi_d : d \in \Delta\},\ \prod\{\eta_g : g \in \Gamma\}\right] \overset{\underline{V}}{=} \prod\{[\xi_d,\ \eta_g] : (d,\ g) \in \Delta \times \Gamma\} \prod\{\zeta_t : t \in \Theta_1\}\ ;$$

(v) $$\left(\prod\{\xi_d : d \in \Delta\}\right)\underline{\pi}^m \overset{\underline{V}}{=} \prod\{\zeta_t : t \in \Theta_2\}\ ;$$

(vi) (a) $$[\xi\underline{\pi}^m,\ \eta] \overset{\underline{V}}{=} [\xi,\ \eta]\underline{\pi}^m \prod\{\zeta_t : t \in \Theta_3\}\ ,$$

(b) $$[\xi,\ \eta\underline{\pi}^m] \overset{\underline{V}}{=} \prod\{\zeta_t : t \in \Theta_4\}[\xi,\ \eta]\underline{\pi}^m$$

where the symbols and sets are as described earlier.

The first consequence of these laws to be worked out deals with the type of substitution considered in Lemmas 1.4 and 1.6, in the special case where φ is a cpp-word, only one initial subword of φ is mapped non-identically, and that subword is mapped to a product of cpp-words.

3.2 LEMMA

Let $\varphi = \varphi(\xi_0,\ \ldots,\ \xi_{m-1},\ \xi_m)$ *be a cpp-word in* B *such that* $\text{c-wt}(\varphi,\ \xi_m) = 1$. *Let* β *be an arbitrary cpp-word in* B , *and* $\{\alpha_l : l \in \underline{n}\}$ *a set of cpp-words such that* $\beta \leq' \alpha = \prod\{\alpha_l : l \in \underline{n}\}$. *Then every group-like variety* $\underline{V}$ *has a law of the form*

$$\varphi(\xi_0,\ \ldots,\ \xi_{m-1},\ \alpha) \overset{\underline{V}}{=} \prod\{\varphi(\xi_0,\ \ldots,\ \xi_{m-1},\ \alpha_l) : l \in \underline{n}\} \prod\{\delta_d : d \in \Delta\}\ ,$$

where for each d *in* Δ , δ_d *is a cpp-word such that*

$$\varphi(\xi_0,\ \ldots,\ \xi_{m-1},\ \beta) \leq' \delta_d\ ,$$

for h *in* $\underline{m}$,

$$\text{c-wt}(\delta_d,\ \xi_h) \geq \text{c-wt}(\varphi,\ \xi_h) + 2\ \text{c-wt}(\beta,\ \xi_h)\ ,$$

and

$$\text{c-wt}(\delta_d, \xi_m) \geq 2\ \text{c-wt}(\beta, \xi_m)\ .$$

Proof. Proceed by induction on the number of symbols $\underline{\gamma}$ or $\underline{\pi}$ in the word φ .

Case (i). If φ has no symbols $\underline{\gamma}$ or $\underline{\pi}$, then $\varphi = \xi_m$ and

$\varphi(\xi_0, \ldots, \xi_{m-1}, \alpha) = \alpha = \prod \{\alpha_l : l \in \underline{n}\}$. The statement of the lemma clearly holds true, with $\Delta = \emptyset$.

For the remaining cases, in which the word φ terminates in either $\underline{\gamma}$ or $\underline{\pi}$, assume inductively that the result is already established for all words with fewer symbols equal to $\underline{\gamma}$ or $\underline{\pi}$ than has φ .

Case (ii). If $\varphi = \psi_1\psi_2\underline{\gamma} = [\psi_1, \psi_2]$, then either $\text{c-wt}(\psi_1, \xi_m) = 1$ and $\text{c-wt}(\psi_2, \xi_m) = 0$ or the roles of ψ_1 and ψ_2 are reversed. Assume the former; the proof is essentially the same in either case. By the inductive hypothesis,

$$\psi_1(\xi_0, \ldots, \xi_{m-1}, \alpha) \underline{\underline{V}} \prod \{\psi_1(\xi_0, \ldots, \xi_{m-1}, \alpha_l) : l \in \underline{n}\} \prod \{\zeta_d : d \in \Delta_1\},$$

where for each d in Δ_1 ,

$$\psi_1(\xi_0, \ldots, \xi_{m-1}, \beta) \leq' \zeta_d\ ,$$

for all h in $\underline{m}$,

$$\text{c-wt}(\zeta_d, \xi_h) \geq \text{c-wt}(\psi_1, \xi_h) + 2\ \text{c-wt}(\beta, \xi_h)\ ,$$

and

$$\text{c-wt}(\zeta_d, \xi_m) \geq 2\ \text{c-wt}(\beta, \xi_m)\ .$$

Now

$$\begin{aligned}(\xi_0, \ldots, \xi_{m-1}, \alpha) &= [\psi_1(\xi_0, \ldots, \xi_{m-1}, \alpha), \psi_2(\xi_0, \ldots, \xi_{m-1}, \alpha)] \\ &\underline{\underline{V}} \left[\prod \{\psi_1(\xi_0, \ldots, \xi_{m-1}, \alpha_l) : l \in \underline{n}\} \prod \{\zeta_d : d \in \Delta_1\}, \psi_2\right] .\end{aligned}$$

Application of law (iv) to the last expression above gives

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$$\overset{\mathrm{V}}{=} \prod \{[\psi_1(\xi_0, \ldots, \xi_{m-1}, \alpha_l), \psi_2] : l \in \underline{n}\} \prod \{[\zeta_d, \psi_2] : d \in \Delta_1\} \prod \{\zeta_d : d \in \Delta_2\},$$

where for d in Δ_2, there exist elements ϑ_1 and ϑ_2 in the set $\{\psi_1(\xi_0, \ldots, \xi_{m-1}, \alpha_l) : l \in \underline{n}\} \cup \{\zeta_d : d \in \Delta_1\}$ such that

$$[\vartheta_1, \psi_2, \vartheta_2] \leq' \zeta_d .$$

For each l in $\underline{n}$,

$$[\psi_1(\xi_0, \ldots, \xi_{m-1}, \alpha_l), \psi_2] = \varphi(\xi_0, \ldots, \xi_{m-1}, \alpha_l) .$$

For each d in Δ_1, Lemma 1.6 shows that

$$\varphi(\xi_0, \ldots, \xi_{m-1}, \beta) = [\psi_1(\xi_0, \ldots, \xi_{m-1}, \beta), \psi_2] \leq' [\zeta_d, \psi_2]$$

and Lemma 1.4 that for h in $\underline{m}$,

$$\begin{aligned} \text{c-wt}([\zeta_d, \psi_2], \xi_h) &\geq \text{c-wt}(\psi_1, \xi_h) + 2\,\text{c-wt}(\beta, \xi_h) + \text{c-wt}(\psi_2, \xi_h) \\ &\geq \text{c-wt}(\varphi, \xi_h) + 2\,\text{c-wt}(\beta, \xi_h) \end{aligned}$$

and that

$$\text{c-wt}([\zeta_d, \psi_2], \xi_m) \geq 2\,\text{c-wt}(\beta, \xi_m) .$$

Thus each word of the form $[\xi_d, \psi_2]$ for d in Δ_1 is of the required form. If, for d in Δ_2, one of the corresponding words ϑ_1 or ϑ_2 is from the set $\{\zeta_d : d \in \Delta_1\}$, then *a fortiori* ζ_d satisfies the same conditions. For all other d in Δ_2, $\zeta_d = [\vartheta_1, \psi_2, \vartheta_2]$ where both ϑ_1 and ϑ_2 are in the set $\{\psi_1(\xi_0, \ldots, \xi_{m-1}, \alpha_l) : l \in \underline{n}\}$; and again it is routine to check that the appropriate conditions are satisfied.

Case (iii). If $\varphi = \psi\pi_{\underline{\ }}$, then $\text{c-wt}(\psi, \xi_m) = 1$. If for some non-negative integer n, $\psi = \xi_m\pi_{\underline{\ }}^{n}$, then the required result follows immediately from law (v). Otherwise, by the inductive hypothesis,

$$\psi(\xi_0, \ldots, \xi_{m-1}, \alpha) \overset{\mathrm{V}}{=} \prod \{\psi(\xi_0, \ldots, \xi_{m-1}, \alpha_l) : l \in \underline{n}\} \prod \{\zeta_d : d \in \Delta_3\} ,$$

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where for each d in Δ_3 ,

$$\psi(\xi_0, \ldots, \xi_{m-1}, \beta) <' \xi_d ,$$

for all $h \in \underline{m}$,

$$\text{c-wt}(\zeta_d, \xi_h) \geq \text{c-wt}(\psi, \xi_h) + 2 \text{ c-wt}(\beta, \xi_h) ,$$

and

$$\text{c-wt}(\zeta_d, \xi_m) \geq 2 \text{ c-wt}(\beta, \xi_m) .$$

Application of law (v) with $m = 1$ gives

$$\varphi(\xi_0, \ldots, \xi_{m-1}, \alpha) \stackrel{V}{=} \prod \left\{\zeta_{d\underline{\pi}}{}^{l(d)} : d \in \Delta_4\right\} ,$$

where for d in Δ_4 , ζ_d is a cpp-word; the set $\left\{\zeta_{d\underline{\pi}}{}^{l(d)} : d \in \Delta_4\right\}$ contains $\{\varphi(\xi_0, \ldots, \xi_{m-1}, \alpha_l) : l \in \underline{n}\}$ and $\{\zeta_{d\underline{\pi}} : d \in \Delta_3\}$ as subsets; and for those d in Δ_4 such that $\zeta_{d\underline{\pi}}{}^{l(d)}$ is not in one of the sets already mentioned, either $l(d) = 1$ and ζ_d has at least two distinct subwords equal to words from the set

$$\{\psi(\xi_0, \ldots, \xi_{m-1}, \alpha_l) : l \in \underline{n}\} \cup \{\zeta_d : d \in \Delta_3\} ,$$

or $l(d) = 0$ and ζ_d has at least p distinct subwords equal to words from the same set. In each of these cases it is easily checked that $\zeta_{d\underline{\pi}}{}^{l(d)}$ satisfies the required conditions. □

Lemma 3.4 will show that the variety of all groups is a group-like variety of $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebras. This proof involves the adaptation of some well-known results, gathered here for convenience as Lemma 3.3.

3.3 LEMMA

Let α, β , *and* γ *be elements of a group* G , *and* h *be a positive integer. Then*

(a) (see, for example, Huppert [5], Kapitel III, Hilfsatz 1.2, p. 253),

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$$[\alpha\beta,\ \gamma] = [\alpha,\ \gamma]^{\beta}[\beta,\ \gamma]$$

and

$$[\alpha,\ \beta\gamma] = [\alpha,\ \gamma][\alpha,\ \beta]^{\gamma}\ ;$$

(b) (*cf*. Hall [*4*], Theorem 3.2).

$$(\alpha\beta)\underline{\pi}^{h} = \alpha\underline{\pi}^{h}\beta\underline{\pi}^{h} \prod \left\{\kappa_{g}\underline{\pi}^{h(g)} : g \in \Gamma_1\right\}$$

where for g *in* Γ_1 *,* $0 \le h(g) \le h$ *and* κ_g *is a commutator in* G *with at least* $p^{h-h(g)}$ *entries from the set* $\{\alpha,\ \beta\}$;

(c) (*cf*. Hall [*4*], Theorem 4.14, where the same argument is used in a more specialised context. Haebich gives a more detailed result in terms of basic commutators in [3], Lemma 3.4.6)

$$[\alpha\underline{\pi}^{h},\ \beta] = [\alpha,\ \beta]\underline{\pi}^{h} \prod \left\{\kappa_{g}\underline{\pi}^{h(g)} : g \in \Gamma_2\right\}\ ,$$

$$[\alpha,\ \beta\underline{\pi}^{h}] = \prod \left\{\kappa_{g}\underline{\pi}^{h(g)} : g \in \Gamma_3\right\}[\alpha,\ \beta]\underline{\pi}^{h}\ ,$$

where for g *in* Γ_2 *,* $0 \le h(g) \le h$ *and* κ_g *is a commutator with at least* $p^{h-h(g)}$ *entries from the set* $\{\alpha,\ [\alpha,\ \beta]\}$ *, and hence may also be expressed as a commutator with at least* $\max\{2,\ p^{h-h(g)}\}$ *entries equal to* α *and at least one equal to* β *; and for* g *in* Γ_3 *,* $0 \le h(g) \le h$ *and* κ_g *may be expressed as a commutator with at least* $\max\{2,\ p^{h-h(g)}\}$ *entries equal to* β *and at least one equal to* α .

Proof. Statement *(a)* is so familiar as to require no proof. It is easily verified by expanding both sides.

Theorem 3.2 of [*4*] is stated as holding modulo a term of the lower dentral series. However, if statement *(b)* holds modulo the p^{h}th term, $\gamma_{p^h}(G)$, then it is true as stated, since an arbitrary element of $\gamma_{p^h}(G)$

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is, by definition, equal to a product of commutators of weight at least p^h.

(In terms of Hall's proof, the condition of nilpotency is required to ensure that his "commutator collecting process" terminates after finitely many steps. If this process is continued until all commutators of weight less than or equal to $p^h - 1$ are collected (a finite process) then the expression remaining, though not in "collected form", satisfies the requirements of the present lemma.)

Another point to be noted is that the terms of the final product are claimed to be of the form $\kappa_g \underline{\pi}^{h(g)} = \kappa_g^{p^{h(g)}}$ where Hall gives only $\kappa_g^{e(g)}$ where $p^{h(g)} | e(g)$. This change may be achieved simply by rewriting $\kappa_g^{e(g)}$ as a product of factors each equal to $\kappa_g \underline{\pi}^{h(g)}$ or its inverse.

To prove the first part of statement *(c)*, note that

$$\begin{aligned}[\alpha\underline{\pi}^h, \beta] &= \alpha^{-p^h}\left(\alpha^{p^h}\right)\beta \\ &= \alpha^{-p^h}\left(\alpha^\beta\right)^{p^h} \\ &= \alpha^{-p^h}(\alpha[\alpha, \beta])^{p^h},\end{aligned}$$

and then use part *(b)* to show that

$$(\alpha[\alpha, \beta])^{p^h} = \alpha^{p^h}[\alpha, \beta]^{p^h} \prod \left\{\kappa_g \underline{\pi}^{h(g)} : g \in \Gamma_2\right\},$$

which gives an expression of the required form.

The second part of statement *(c)* follows from the first by the observation that for all ξ and η in G, $[\xi, \eta] = [\eta, \xi]^{-1}$. □

3.4 LEMMA

The variety of all groups is a group-like variety of $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-*algebras.*

Proof. Laws (i), (ii), and (iii) clearly hold. What remains to be proved is that laws (iv), (v), and (vi) hold in an arbitrary group G.

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To show that law (iv) holds in G, let $\{\alpha_i : i \in \underline{m}\}$ and $\{\beta_j : j \in \underline{n}\}$ be non-empty sets of elements of G, and proceed by induction on $m + n$ to show that

$$\left[\prod\{\alpha_i : i \in \underline{m}\}, \prod\{\beta_j : j \in \underline{n}\}\right]$$
$$= \prod\{[\alpha_i, \beta_j] : (i, j) \in \underline{m} \times \underline{n}\} \prod\{\delta_d : d \in \Delta_1\},$$

where for each d in Δ_1 there exists a triple (i, j, k) either in $\underline{m} \times \underline{n} \times \underline{n}$ with $j \neq k$ such that $[\alpha_i, \beta_j, \beta_k] \leq' \delta_d$, or in $\underline{m} \times \underline{n} \times \underline{m}$ with $i \neq k$ such that $[\alpha_i, \beta_j, \alpha_k] \leq' \delta_d$, where of course the relation $\leq'$ is that described in 2.1.

When $m + n = 2$, the least possible value, the result is trivially true. When $m + n > 2$, either $m \geq 2$ or $n \geq 2$; suppose the former. From Lemma 3.3 *(a)* and then by the inductive hypothesis,

$$\left[\prod\{\alpha_i : i \in \underline{m}\}, \prod\{\beta_j : j \in \underline{n}\}\right]$$
$$= \left[\prod\{\alpha_i : i \in \underline{m-1}\}, \prod\{\beta_j : j \in \underline{n}\}\right]^{\alpha_{m-1}} \left[\alpha_{m-1}, \prod\{\beta_j : j \in \underline{n}\}\right]$$
$$= \left(\prod\{[\alpha_i, \beta_j] : (i, j) \in \underline{m-1} \times \underline{n}\} \prod\{\delta_g : g \in \Gamma_1\}\right)^{\alpha_{m-1}}$$
$$\prod\{[\alpha_{m-1}, \beta_j] : j \in \underline{n}\} \prod\{\delta_g : g \in \Gamma_2\},$$

where for each g in Γ_1 or Γ_2, δ_g satisfies the conditions required of elements of the set $\{\delta_d : d \in \Delta_1\}$. Since

$$\delta_g^{\alpha_{m-1}} = \delta_g[\delta_g, \alpha_{m-1}]$$

and

$$[\alpha_i, \beta_j]^{\alpha_{m-1}} = [\alpha_i, \beta_j][\alpha_i, \beta_j, \alpha_{m-1}],$$

and since all new commutators introduced by rearranging the order of the factors in the product above also satisfy the conditions required of elements of the set $\{\delta_d : d \in \Delta_1\}$, it is readily seen that the expression above is equal to

$$\prod\left\{[\alpha_i, \beta_j] : (i, j) \in \underline{m} \times \underline{n}\right\} \prod\left\{\delta_d : d \in \Delta_1\right\},$$

as required.

To show that law (v) holds in G, let $\{\alpha_i : i \in \underline{m}\}$ be an arbitrary non-empty set of elements of G. Consider the proposition $P(w)$: Let $\{\delta_d : d \in \underline{l}\}$ be a set of commutators, each with at least w entries from the set $\{\alpha_i : i \in \underline{m}\}$, and let k be the least non-negative integer such $w \geq p^{h-k}$; then

$$\left(\prod\{\delta_d : d \in \underline{l}\}\right)\underline{\pi}^k = \prod\left\{\kappa_g \underline{\pi}^{h(g)} : g \in \Gamma_3\right\},$$

where for g in Γ_3, $0 \leq h(g) \leq k \leq h$, the commutator κ_g has at least $p^{h-h(g)}$ entries from the set $\{\alpha_i : i \in \underline{m}\}$, there is a subset Γ_3^* of Γ_3 such that

$$\left\{\delta_d \underline{\pi}^k : d \in \underline{l}\right\} = \left\{\kappa_g \underline{\pi}^{h(g)} : g \in \Gamma_3^*\right\},$$

and for g in $\Gamma_3 \backslash \Gamma_3^*$, the commutator κ_g has at least two entries from the set $\{\alpha_i : i \in \underline{m}\}$.

The proposition $P(1)$ is the required result, that law (v) holds in G. The proposition $P\left(p^h\right)$ is clearly true, since in that proposition each κ_g may be taken to be one of the δ_d, and each $h(g)$ to be zero. For arbitrary w less than p^h, suppose that $P(v)$ is true for $v > w$.

Now proceed by a second induction, on l. When $l = 1$, the result is trivial. Suppose $l > 1$, and the result established for expressions with fewer than l factors. Lemma 3.3 *(b)* shows that

$$\left(\prod\{\delta_d : d \in \underline{l-1}\}.\delta_{l-1}\right)\underline{\pi}^k$$
$$= \left(\prod\{\delta_d : d \in \underline{l-1}\}\right)\underline{\pi}^k . \delta_{l-1}\underline{\pi}^k . \prod\left\{\delta_g \underline{\pi}^{k(g)} : g \in \Gamma_4\right\},$$

where for g in Γ_4, δ_g is a commutator with at least $\min\left\{2, p^{k-k(g)}\right\}$

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entries equal either to the product $\prod\{\delta_d : d \in \underline{l-1}\}$ or to the element δ_{l-1}. By law (iv) applied as often as required, each such δ_g for g in Γ_4 may be expressed as a product of commutators, each with at least $\min\{2, p^{k-k(d)}\}$ entries from the set $\{\delta_d : d \in \underline{l-1}\}$, and hence as a product of commutators each with at least $\min\{2w, p^{h-k(d)}\}$ entries from the set $\{\alpha_i : i \in \underline{m}\}$. The inductive hypothesis on w shows that each of these may be expressed in the required form. From the inductive hypothesis on l, the expression $\left(\prod\{\delta_d : d \in \underline{l-1}\}\right)\underline{\pi}^k$ may be expressed as the product of a set of powers of commutators of the required form which contains the subset $\left\{\delta_d\underline{\pi}^k : d \in \underline{l-1}\right\}$. This, together with the factor $\delta_{l-1}\underline{\pi}^k$ gives the distinguished subset required in the total product; and so the truth of $P(w)$ is proved.

By induction the truth of $P(1)$ follows, and law (v) holds in G.

That law (vi) holds in G is already shown in the proof of Lemma 3.3 *(c)*. □

4. Laws connecting a word with a product of cpp-words

The laws referred to in the heading of this section are the first step toward the aim of finding laws in a group-like variety which link an arbitrary given word with the product of a set of scpp-words which have appropriately heavy weights. Lemma 4.2 *(a)* is the central result; the others are technical lemmas useful in Section 6.

4.1 COMPLEXITY

A proper subword β of a word α in B is called a $\underline{\mu}$*'-subword* if and only if

(i) the last symbol in β is $\underline{\mu}$, and

(ii) the last symbol in the subword which immediately follows β in the maximal sequence of subwords linking β with α, is either $\underline{\gamma}$ or $\underline{\pi}$.

The *complexity* of a word α is now defined by

$$\mathrm{comp}(\alpha) = \sum \{k(\rho, \alpha)+l(\rho, \alpha) : \rho \text{ is a } \underline{\mu}'\text{-subword of } \alpha\} ,$$

where k and l are as described in 1.3.

A word has complexity zero if and only if it is a (possibly trivial) product of cpp-words.

4.2 LEMMA

Let $\underline{\underline{V}}$ *be a group-like variety. Then:*

(a) corresponding to an arbitrary word α *in* B *, there exists an ordered set* Δ *of cpp-words in* B *such that* $\alpha \leq' \Pi\Delta$ *and* $\alpha \overset{\underline{\underline{V}}}{=} \Pi\Delta$ *;*

(b) corresponding to a word α *in* B *whose last symbol is* γ *, there exists an ordered set* Δ *of cpp-words in* B *, all ending in* γ *, such that* $\alpha \leq' \Pi\Delta$ *and* $\alpha \overset{\underline{\underline{V}}}{=} \Pi\Delta$ *; and*

(c) corresponding to a word α *in* B *which does not contain the symbol* π *, there exists an ordered set* Δ *of c-words in* B *such that* $\alpha \leq' \Pi\Delta$ *and* $\alpha \overset{\underline{\underline{V}}}{=} \Pi\Delta$ *.*

Proof. Proceed by induction on $\mathrm{comp}(\alpha)$. If $\mathrm{comp}(\alpha) = 0$, then in each part of the lemma it is clear that the given word α already is in the required form. Otherwise, α has a subword β satisfying the conditions: either

(i) $\beta = \beta_1\beta_2\underline{\mu}\underline{\pi}$, or

(ii) $\beta = \beta_1\beta_2\underline{\mu}\beta_3\underline{\gamma}$ or $\beta = \beta_3\beta_1\beta_2\underline{\mu}\underline{\gamma}$;

where in either case β_1 and β_2 , and in case (ii), β_3 , have complexity zero.

(That a subword β of one of the forms (i) or (ii) exists follows from the fact that $\mathrm{comp}(\alpha) \neq 0$; that the subwords β_i of β have complexity zero can be arranged inductively, because if one of the β_i had non-zero complexity, it would itself have a subword β of strictly smaller complexity, of one of the forms (i) or (ii).)

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Since β_1 and β_2 are themselves products of cpp-words, there exists, by law (i), an ordered set Θ of cpp-words such that $\beta_1\beta_2\underline{\mu} \leq' \Pi\Theta$ and $\beta_1\beta_2\underline{\mu} \overset{\underline{V}}{=} \Pi\Theta$. By law (v) in case (i) and law (iv) in case (ii), it now follows that there exists an ordered set Γ of cpp-words such that $\beta \leq' \Pi\Gamma$ and $\beta \overset{\underline{V}}{=} \Pi\Gamma$. In the situation of part *(c)* of the lemma, no word in Γ contains a symbol $\underline{\pi}$; and if the word β ends in the symbol $\underline{\gamma}$ (that is, in case (ii)) then each word in Γ also ends in the symbol $\underline{\gamma}$.

Suppose that $\alpha = \alpha(\xi_0, \ldots, \xi_{z-1})$. Since β is a subword of α , there exists a word $\alpha' = \alpha'(\xi_0, \ldots, \xi_{z-1}, \xi_z)$ such that

$$\alpha = \alpha'(\xi_0, \ldots, \xi_{z-1}, \beta) \overset{\underline{V}}{=} \alpha'(\xi_0, \ldots, \xi_{z-1}, \Pi\Gamma) ;$$

and from Lemma 1.6, $\alpha \leq' \alpha'(\xi_0, \ldots, \xi_{z-1}, \Pi\Gamma)$. If the word α ends in the symbol $\underline{\gamma}$, then either α' also ends in $\underline{\gamma}$ or $\alpha' = \xi_z$ and $\alpha = \beta = \Pi\Gamma$. In the latter case, the proof of part *(b)* is complete. In the former case for part *(b)*, and in all cases for parts *(a)* and *(c)*, the expression $\alpha'(\xi_0, \ldots, \xi_{z-1}, \Pi\Gamma)$ satisfies all the conditions required of α in the statement of the lemma and has strictly lower complexity. From the inductive hypothesis then, there exists a set Δ of cpp-words satisfying the condition of the appropriate part of the lemma such that

$$\alpha \leq' \alpha'(\xi_0, \ldots, \xi_{z-1}, \Pi\Gamma) \leq' \Pi\Delta$$

and

$$\alpha \overset{\underline{V}}{=} \alpha'(\xi_0, \ldots, \xi_{z-1}, \Pi\Gamma) \overset{\underline{V}}{=} \Pi\Delta .$$

As both relations $\leq'$ and $\overset{\underline{V}}{=}$ are transitive, this completes the proof. □

4.3 LEMMA

Corresponding to a group-like variety $\underline{\underline{V}}$ *and an arbitrary word* $\alpha = \alpha(\xi_0, \ldots, \xi_{z-1})$ *in* B *, there exists an ordered set* Δ *of c-words in* B *such that for all* δ *in* Δ *and all* i *in* $\underline{z}$ *,*

$$\text{c-wt}(\delta, \xi_i) \geq \text{c-wt}(\alpha, \xi_i)$$

and $\alpha \overset{V}{=} \Pi\Delta$.

Proof. Proceed by induction on the number of symbols $\underline{\pi}$ in the word α . If there are none, then the result follows immediately from Lemma 4.2 *(c)*. Otherwise, α must have a subword $\beta\underline{\pi}$ where β does not contain the symbol $\underline{\pi}$. Law (iii) shows that

$$\beta\underline{\pi} \overset{V}{=} \beta(\beta\underline{\mu})^{p-1} = \beta^p ;$$

and although the weight functions $w^e_{a,b}$ with either $e = p$ or $b \geq 1$ take greater values at $\beta\underline{\pi}$ than at β^p , it remains true that, for all ξ in Ξ ,

$$\text{c-wt}(\beta\underline{\pi}, \xi) = \text{c-wt}\left(\beta^p, \xi\right) .$$

Let $\alpha' = \alpha'\left(\xi_0, \ldots, \xi_{z-1}, \xi_z\right)$ be the word in B with $\text{c-wt}\left(\alpha', \xi_z\right) = 1$ such that

$$\alpha = \alpha'\left(\xi_0, \ldots, \xi_{z-1}, \beta\underline{\pi}\right) \overset{V}{=} \alpha'\left(\xi_0, \ldots, \xi_{z-1}, \beta^p\right) .$$

The last word above has fewer symbols $\underline{\pi}$ then has α , but has the same c-weight in every element of Ξ . Hence the required result follows by induction. □

5. Subword arrays

In this section, laws are no longer the subject of direct attention, though the form of laws (v) and (vi) influences some definitions. Attention is focussed again on cpp-words in the $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-word algebra B ; and corresponding to each, various subword arrays are defined, with weights and a "power integer" attached to each. The definition of a subword array reflects the idea that, in the operation of laws (v) and (vi), a symbol $\underline{\pi}$ may either remain, on the right hand side, at least as effective as it was on the left or disappear; but in the latter case the c-weight of a subword on which it acted on the left is increased at least p-fold on the right.

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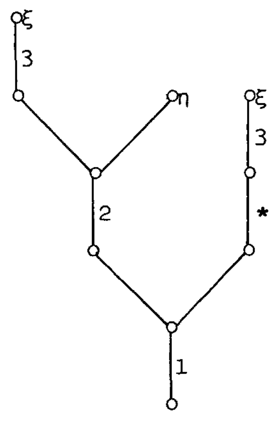
5.1 DEFINITIONS AND EXAMPLES

A subword array of a cpp-word φ is a family (constructed as described below) of subwords of φ whose symbols $\underline{\pi}$ are labelled in a suitable way.



A *$\underline{\pi}$-labelled cpp-word* is a cpp-word, φ say, to which has been attached, at each symbol $\underline{\pi}$, a symbol from the set $\{*\} \cup Z^+$ in accordance with the rule that for some fixed non-negative integer n, the final symbol $\underline{\pi}$ in each subword of the form $\rho\underline{\pi}$ has attached to it either the integer $l(\rho, \varphi) + n$ or the symbol $*$. A *basic* $\underline{\pi}$-labelling of φ is one in which $n = 0$. For example, $\xi\underline{\overset{3}{\pi}}\eta\underline{\gamma\overset{2}{\pi}}\xi\underline{\overset{3}{\pi}\overset{2}{\pi}\gamma\overset{1}{\pi}}$ and $\xi\underline{\overset{6}{\pi}}\eta\underline{\gamma\overset{*}{\pi}}\xi\underline{\overset{*}{\pi}\overset{*}{\pi}\gamma\overset{*}{\pi}}$ are $\underline{\pi}$-labelled cpp-words (the former being basic), but $\xi\underline{\overset{2}{\pi}}\eta\underline{\gamma\overset{*}{\pi}}\xi\underline{\overset{*}{\pi}\overset{*}{\pi}\gamma\overset{*}{\pi}}$ and $\xi\underline{\overset{4}{\pi}}\eta\underline{\gamma\overset{*}{\pi}}\xi\underline{\overset{*}{\pi}\overset{2}{\pi}\gamma\overset{*}{\pi}}$ are not.

Let φ be a $\underline{\pi}$-labelled cpp-word, and k a positive integer. Partition the set of symbols $\underline{\pi}$ in φ labelled with the integer k into two (possibly empty) subsets, S_1 and S_2. A *k-propagation* of φ corresponding to this partition is a family of $\underline{\pi}$-labelled words, whose elements are:

(i) the labelled word obtained from φ by replacing, for all $\underline{\pi}$ in S_1, the symbol k attached to $\underline{\pi}$ by a symbol $*$ (if S_1 is empty, then the labelled word φ is included without alteration); and

(ii) $p - 1$ distinctly-identified copies of each labelled subword ρ of φ on which a symbol $\underline{\pi}$ in S_2 acts.

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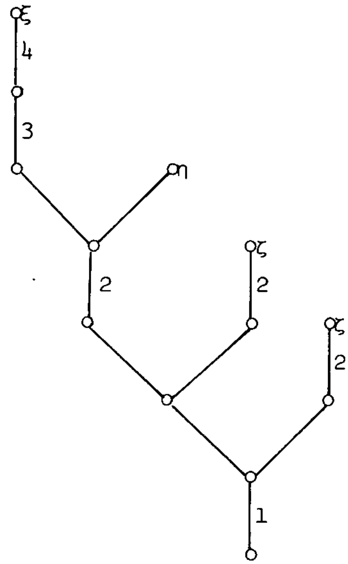
For example, let $\varphi = \xi\overset{4}{\underline{\pi}}\overset{3}{\underline{\pi}}\eta\gamma\overset{2}{\underline{\pi}}\zeta\overset{2}{\underline{\pi}}\gamma\zeta\overset{2}{\underline{\pi}}\gamma\overset{1}{\underline{\pi}}$, and $k = 2$. Consider first the partition such that S_1 contains all symbols $\overset{2}{\underline{\pi}}$ in φ, and S_2 is empty.

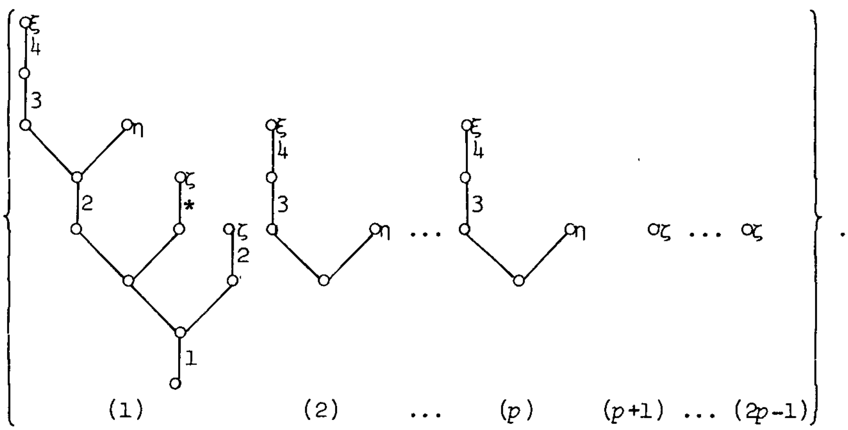


A corresponding 2-propagation of φ is simply:

$$\{\xi\overset{4}{\underline{\pi}}\overset{3}{\underline{\pi}}\eta\gamma\overset{*}{\underline{\pi}}\zeta\overset{*}{\underline{\pi}}\gamma\zeta\overset{*}{\underline{\pi}}\gamma\overset{1}{\underline{\pi}}\} .$$

Consider another partition of φ in which S_1 contains the second symbol $\overset{2}{\underline{\pi}}$, and S_2 contains the first and the third. A corresponding 2-propagation of φ is

$$\{\xi\overset{4}{\underline{\pi}}\overset{3}{\underline{\pi}}\eta\gamma\overset{2}{\underline{\pi}}\zeta\overset{*}{\underline{\pi}}\gamma\zeta\overset{2}{\underline{\pi}}\gamma\overset{1}{\underline{\pi}}(1)\} \cup \{\xi\overset{4}{\underline{\pi}}\overset{3}{\underline{\pi}}\eta\gamma(i) : 2 \le i \le p\} \cup \{\zeta(i) : p+1 \le i \le 2p-1\} ;$$



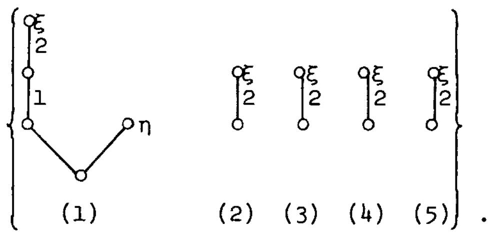
A *subword array* of a cpp-word φ is a family of $\underline{\pi}$-labelled cpp-words, constructed as follows. Let $h = \max\{l(\rho, \varphi) : \rho\underline{\pi} \le \varphi\}$. Let E_0 be a family whose only member is φ with a basic labelling not including any stars (that is, to each symbol $\underline{\pi}$ in φ is attached the integer $l(\rho, \varphi)$ where ρ is the subword on which the symbol $\underline{\pi}$ acts). For arbitrary k in $\underline{h}$, suppose that E_k has been constructed, and choose a $(k+1)$-propagation of each copy in E_k of a labelled word. Call the disjoint union of these families E_{k+1} . Finally, $E = E_h$.

As an example, suppose $p = 5$ and $\varphi = \xi\underline{\pi}\underline{\pi}\eta\underline{\gamma}$. Then

$$E_0 = \{\xi\overset{2}{\underline{\pi}}\overset{1}{\underline{\pi}}\eta\underline{\gamma}(1)\} \ .$$

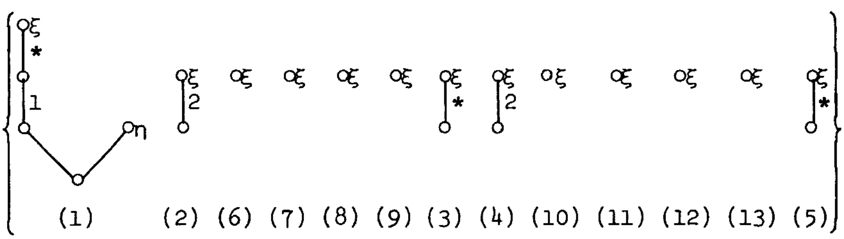
Now, for $k = 1$, choose $S_1 = \emptyset$ and S_2 to contain the only symbol $\overset{1}{\underline{\pi}}$ in the copy of φ in E_0 . Then

$$E_1 = \{\xi\overset{2}{\underline{\pi}}\overset{1}{\underline{\pi}}\eta\underline{\gamma}(1)\} \cup \{\xi\overset{2}{\underline{\pi}}(i) \ : \ 2 \le i \le p\} \ ,$$



Suppose that in each of the words (1), (3), and (5) the sole symbol $\overset{2}{\underline{\pi}}$ is allocated to S_1 , and that in words (2) and (4) it is allocated to S_2 . Then

$$E_2 = \{\xi\overset{*}{\underline{\pi}}\overset{1}{\underline{\pi}}\eta\underline{\gamma}(1), \ \xi\overset{2}{\underline{\pi}}(2), \ \xi\overset{*}{\underline{\pi}}(3), \ \xi\overset{2}{\underline{\pi}}(4), \ \xi\overset{*}{\underline{\pi}}(5)\} \cup \{\xi(i) \ : \ 6 \le i \le 13\} \ ,$$



□

If ψ is a subword of φ , and E is a subword array for φ , then E must contain at least one labelled word with a subword equal to ψ . Each such labelled word *induces* a subword array for ψ , in the natural sense that during the construction of the array for ψ , each symbol $\underline{\pi}$ occurring is treated in the same way (in allocation to an S_1 or S_2 subset) as was the corresponding symbol in the selected labelled word in E . Other generated subwords, and symbols $\underline{\pi}$ contained in them, remain in one-one correspondence throughout the construction.

Suppose $E = \{\rho(i) : i \in \underline{m}\}$ is a subword array for a cpp-word. For each ξ in Ξ, define

$$\text{c-wt}(E, \xi) = \sum \{\text{c-wt}(\rho(i), \xi) : i \in \underline{m}\}$$

and

$$\text{c-wt}(E) = \sum \{\text{c-wt}(\rho(i)) : i \in \underline{m}\} = \sum \{\text{c-wt}(E, \xi) : \xi \in \Xi\} .$$

Define $n(E)$ to be the total number of starred symbols $\underline{\pi}$ in labelled words $\rho(i)$ for i in $\underline{m}$.

Corresponding to a given cpp-word φ, let E_m be the subword array for φ in which every symbol $\underline{\pi}$ which occurs is starred; thus E_m contains only one $\underline{\pi}$-labelled word, a $\underline{\pi}$-labelling of φ itself. Let E_M be the subword array for φ such that no symbol $\underline{\pi}$ in a word in E_M is starred. Corresponding to each initial subword ρ of φ there are precisely $p^{l(\rho,\varphi)}$ distinct copies of words in E_M each containing a subword corresponding to ρ, and equal as a word to ρ. Hence, for all ξ in Ξ,

$$\text{c-wt}(E_m, \xi) = \text{c-wt}(\varphi, \xi)$$

and

$$\text{c-wt}(E_M, \xi) = \text{cpp-wt}(\varphi, \xi) .$$

It is easy to see from the construction process that if E is an arbitrary subword array of φ, then there exists a subfamily of E_M in one-one correspondence with E such that corresponding elements are $\underline{\pi}$-labellings of the same word. In the other direction, E contains a labelling of the word φ. Hence, for an arbitrary subword array E of φ and for all ξ in Ξ,

$$\text{c-wt}(E_m, \xi) \le \text{c-wt}(E, \xi) \le \text{c-wt}(E_M, \xi) .$$

The arrays E_m and E_M are called respectively the minimal and maximal arrays for φ.

5.2 LEMMA

Let φ *be a cpp-word and* E_m *the minimal subword array for* φ *. If* E *is an arbitrary subword array for* φ *, then*

$$\text{c-wt}(E) \geq \text{c-wt}\left(E_m\right) + (p-1)\left(n\left(E_m\right)-n(E)\right) .$$

Proof. If $n\left(E_m\right) \leq n(E)$, then the conclusion is obvious. Otherwise, in the construction of E , at least $n\left(E_m\right) - n(E)$ symbols $\underline{\pi}$ in the basically-labelled copy of φ are left unstarred; and correspondingly at least $(p-1)\left(n\left(E_m\right)-n(E)\right)$ copies of words, each of c-weight at least one, are included in E in addition to the basically-labelled copy of φ . □

Given a subword array other than the maximal one for a word, it is possible to construct for the same word another array with fewer stars and higher, but boundedly higher, c-weight.

5.3 LEMMA

Given a subword array E *for a cpp-word* φ *, and an integer* l *such that* $0 \leq l \leq n(E)$ *, there exists a subword array* E' *for* φ *such that*

$$n(E') = n(E) - l ,$$

$$\text{c-wt}(E) + l(p-1) \leq \text{c-wt}(E') \leq p^{l}\text{c-wt}(E) ,$$

and for all $\xi \in \Xi$ *,*

$$\text{c-wt}(E, \xi) \leq \text{c-wt}(E', \xi) \leq p^{l}(\text{c-wt } E, \xi) .$$

Proof. The case $l = 0$ is trivial. The form of the statement is such that it follows immediately by induction from its special case $l = 1$.

Suppose then that $l = 1$ and that the array E for φ contains at least one labelled word with a starred symbol $\underline{\pi}$. Consequently, that labelled word has a subword of the form $\rho\underline{\pi}^{*}$ where the array for ρ induced by E is the maximal one, F_M say, not containing a star. Note that for all ξ in Ξ ,

$$0 \leq \text{c-wt}\left(F_M, \xi\right) \leq \text{c-wt}(E, \xi) ;$$

and for at least one ξ in Ξ ,

$$1 \le \text{c-wt}\left(F_M, \xi\right) \le \text{c-wt}(E, \xi) .$$

Adding these inequalities gives

$$1 \le \text{c-wt}\left(F_M\right) \le \text{c-wt}(E) .$$

Now a subword array E' for φ is constructed in the same way as E , except that the final sumbol $\underline{\pi}$ of the copy of $\rho\underline{\pi}$ under consideration is allocated to the S_2 instead of the S_1 subset, and consequently E' contains $p - 1$ copies of F_M in addition to a complete copy of E lacking one star. Thus

$$n(E') = n(E) - 1$$

and for all $\xi \in \Xi$,

$$\text{c-wt}(E', \xi) = \text{c-wt}(E, \xi) + (p-1)\text{c-wt}\left(F_M, \xi\right) .$$

The inequalities for $\text{c-wt}\left(F_M, \xi\right)$ and $\text{c-wt}\left(F_M\right)$ written earlier now give the required result. □

The final result in this section links comparison of subword arrays with the earlier comparison of words, when the heavier word is an scpp-word.

5.4 LEMMA

Let E *be a subword array for a cpp-word* φ *, and let* κ *be a c-word and* h *a non-negative integer, such that for all* $\xi \in \Xi$ *,*

$$\text{c-wt}(E, \xi) \le \text{c-wt}(K, \xi)$$

and

$$n(E) \le h .$$

Then

$$\varphi \le' \kappa\underline{\pi}^h .$$

Proof. The required inequality in c-weights is immediate: for all $\xi \in \Xi$,

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$$\text{c-wt}(\varphi, \xi) \le \text{c-wt}(E, \xi) \le \text{c-wt}(\kappa\underline{\pi}^h, \xi) .$$

To deal with cpp-weights, set E_M to be the maximal subword array for φ, and apply Lemma 5.3 with $l = n(E)$: for all $\xi \in \Xi$,

$$\text{cpp-wt}(\varphi, \xi) = \text{c-wt}(E_M, \xi) \le p^{n(E)}\text{c-wt}(E, \xi)$$

$$\le p^h\text{c-wt}(\kappa, \xi) = \text{cpp-wt}(\kappa\underline{\pi}^h, \xi) .$$

From this it follows automatically that for $a \ge b \ge 0$,

$$w^p_{a,b}(\varphi) \le w^p_{a,b}(\kappa\underline{\pi}^h) .$$

Finally, let E_m be the minimal array for φ.

If $n(E_m) \le n(E)$, then for $a \ge b \ge 1$,

$$\begin{aligned} w^1_{a,b}(\varphi) &= a\ \text{c-wt}(E_m) + bn(E_m) \\ &\le a\ \text{c-wt}(E) + bn(E) \\ &\le a\ \text{c-wt}(\kappa) + bh = w^1_{a,b}(\kappa\underline{\pi}^h) . \end{aligned}$$

If $n(E_m) > n(E)$, then Lemma 5.2 is used:

$$\begin{aligned} w^1_{a,b}(\varphi) &= a\ \text{c-wt}(E_m) + bn(E_m) \\ &\le a\ \text{c-wt}(E) - \big(a(p-1)-b\big)\big(n(E_m)-n(E)\big) + bn(E) \\ &\le a\ \text{c-wt}(E) + bn(E) , \end{aligned}$$

as before. □

6. Laws linking a cpp-word with a product of scpp-words

In Lemma 4.2 *(a)* it was shown that the laws of a group-like variety link an arbitrary word with a product of cpp-words of greater or equal weight. Corollary 6.5 in turn links a cpp-word with a product of scpp-words of greater or equal weight; and the two results combine in Theorem 6.6 to complete the programme outlined at the beginning of Section 4.

The earlier lemmas in Section 6 lead up to Lemma 6.3, which is really the central result of the section. It gives more detailed information than its Corollary 6.5, and Subsection 6.4 illustrates the way in which this can be used. Finally, Lemma 6.7 applies Theorem 6.6 to the series described in Subsection 2.4; it in turn has applications in [6].

6.1 LEMMA

For arbitrary cpp-words θ *and* ρ *,*

(i) $[\rho\underline{\pi}, \theta] \overset{V}{=} [\rho, \theta]\underline{\pi} \prod \{\delta_g : g \in \Gamma_1\}$,

(ii) $[\theta, \rho\underline{\pi}] \overset{V}{=} \prod \{\delta_g : g \in \Gamma_2\}[\theta, \rho]\underline{\pi}$,

where for g *in* Γ_1 *or* Γ_2 *,* δ_g *is a cpp-word with at least two sub-words equal to* ρ *, and where, if* E *is a subword array for one of the factors on the right-hand side of one of the equations, then there exists a subword array* F *for the word on the left-hand-side such that for all* $\xi \in \Xi$ *,*

$$\text{c-wt}(F, \xi) \leq \text{c-wt}(E, \xi)$$

and

$$n(F) \leq n(E) .$$

Proof. Apply law (vi) (a) to the left-hand side of equation *(i)* to obtain the right-hand side. Let E be a subword array for either $[\rho, \theta]\underline{\pi}$ or a word δ_g ending in the symbol $\underline{\pi}$. If the final symbol $\underline{\pi}$ was allocated to S_1 (respectively S_2) in the construction of E , then there are at least two (respectively p) subwords equal to ρ and at least one equal to θ in E . In the construction of F , the final symbol $\underline{\pi}$ of $\rho\underline{\pi}$ is also allocated to S_1 (respectively S_2), and the subwords θ and ρ (respectively θ and p distinct copies of ρ) are placed in one-one correspondence with equal distinct subwords in E . The construction of F is continued in accordance with the corresponding induced arrays for θ and ρ , and clearly satisfies the required inequalities.

If E is a subword array for a word δ_g not ending in the symbol $\underline{\pi}$, then δ_g contains at least p subwords equal to ρ , and in the construction of F the final symbol $\underline{\pi}$ of $\rho\underline{\pi}$ is allocated to S_2 , and a correspondence is set up and the construction of F continued as before.

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The proof of *(ii)* is exactly similar. $\square$

6.2 LEMMA

Let $\eta = \eta(\xi_0, \ldots, \xi_{m-1}, \xi_m)$ *be a c-word with* $\text{c-wt}(\eta, \xi_m) = 1$ *, let* $\rho, \theta_0, \ldots, \theta_{m-1}$ *and* θ_m *be cpp-words, and set* $\psi = \eta(\theta_0, \ldots, \theta_{m-1}, [\rho\underline{\pi}, \theta_m])$. *Then:*

(i) $\psi \stackrel{V}{=} \eta(\theta_0, \ldots, \theta_{m-1}, [\rho, \theta]\underline{\pi}) \prod \{\eta_g(\theta_0, \ldots, \theta_{m-1}, \theta_m, \rho) : g \in \Gamma\}$ *where for* g *in* Γ , $\eta_g = \eta_g(\xi_0, \ldots, \xi_{m+1})$ *is a cpp-word such that* $\text{c-wt}(\eta_g, \xi_{m+1}) \geq 2$ *; and corresponding to a subword array* E *for an arbitrary factor on the right-hand side of equation (i), there exists a subword array* F *for* ψ *such that for all* $\xi \in \Xi$,

$$\text{c-wt}(F, \xi) \leq \text{c-wt}(E, \xi)$$

and

$$n(F) \leq n(E) \ ;$$

(ii) $\psi\pi^k \stackrel{V}{=} \eta(\theta_0, \ldots, \theta_{m-1}, [\rho, \theta]\underline{\pi})\underline{\pi}^k \prod \left\{\delta_d\underline{\pi}^{h(d)} : d \in \Delta\right\}$ *where for* d *in* Δ , δ_d *is a cpp-word with at least two distinct subwords equal to* ρ *; and corresponding to a subword array* E' *for an arbitrary factor on the right-hand side of equation (ii), there exists a subword array* F' *for* $\psi\underline{\pi}^k$ *such that for all* $\xi \in \Xi$,

$$\text{c-wt}(F', \xi) \leq \text{c-wt}(E', \xi)$$

and

$$n(F') \leq n(E') \ .$$

Proof. *(i)* In Lemma 3.2, set $\beta = [\rho\underline{\pi}, \theta]$ and $\alpha = [\rho, \theta]\underline{\pi} \prod \{\delta_g : g \in \Gamma_1\}$ where the latter product is as described in Lemma 6.1. It is easily checked that the conditions for Lemma 3.2 are satisfied, and that all resulting terms, except the first one on the right-hand side of *(i)*, have at least two distinct subwords equal to ρ . Let E be a subword array for an arbitrary factor, ζ say, on the right-hand side of equation *(i)*. Lemma 3.2 shows that the cpp-word ζ has at least one

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subword from the set $\{[\rho, \theta]\underline{\pi}\} \cup \{\delta_g : g \in \Gamma_1\}$, and for all i in $\underline{m}$ at least $\text{c-wt}(\eta, \xi_i)$ further distinct subwords equal to θ_i ; and E induces an array for each of these subwords. To the induced array for the first subword mentioned, Lemma 6.1 gives a corresponding array for $[\rho\underline{\pi}, \theta]$. This, together with precisely $\text{c-wt}(\eta, \xi_i)$ distinct arrays for θ_i for each i in $\underline{m}$, is used in the obvious way to construct an array F for ψ . Since η is a c-word, there are no symbols $\underline{\pi}$ in ψ outside the subwords already discussed. This construction clearly satisfies the required conditions.

(ii) Law (v) is applied to the result *(i)*. For each d in Δ , δ_d is a cpp-word with at least $p^{k-h(d)}$ distinct subwords equal to factors on the right-hand side of equation *(i)*, and these in turn contain at least two distinct subwords equal to ρ . The first $h(d)$ stages in the construction of F' exactly parallel those in the construction of E' for $\delta_{\underline{d}}\underline{\pi}^{h(d)}$, so that there is a one-one correspondence between subwords equal to $\psi\underline{\pi}^{k-h(d)}$ and subwords equal to δ_d . In the next $k - h(d)$ stages, all $\underline{\pi}$-edges are allocated to S_2 subsets; then there are $p^{k-h(d)}$ copies of ψ corresponding to each word δ_d . Each is placed in correspondence with a separate subword of δ_d equal to a factor on the right-hand side of equation *(i)*; to the array induced by E' for this subword corresponds, by part *(i)*, an array for ψ , and the construction of F' is completed in accordance with this correspondence. The required inequalities are again immediate. □

6.3 LEMMA

Given a group-like variety $\underline{\underline{V}}$ *and a cpp-word* φ *, there exist a finite ordered index set* Γ *and corresponding sets* $\{\kappa_g : g \in \Gamma\}$ *of c-words,* $\{h(g) : g \in \Gamma\}$ *of non-negative integers, and* $\{E_g : g \in \Gamma\}$ *of subword arrays for* φ *such that*

$$\varphi \overset{\underline{\underline{V}}}{=} \prod \left\{\kappa_{\underline{g}}\underline{\pi}^{h(g)} : g \in \Gamma\right\}$$

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and for all g *in* Γ *and all* ξ *in* Ξ *,*

$$(*)\qquad \begin{cases} \text{c-wt}\left(E_g, \xi\right) \le \text{c-wt}\left(\kappa_g, \xi\right) , \\ n\left(E_g\right) \le h(g) . \end{cases}$$

Proof. Consider two propositions:

(a) Let φ and ψ be cpp-words, h a non-negative integer, and E a subword array for φ such that for all ξ in Ξ,

$$\text{c-wt}(E, \xi) \le \text{c-wt}(\psi, \xi)$$

and

$$n(E) \le h .$$

Then there exist ordered sets $\left\{\kappa_g : g \in \Gamma_1\right\}$ of c-words, $\left\{h(g) : g \in \Gamma_1\right\}$ of non-negative integers, and $\left\{E_g : g \in \Gamma_1\right\}$ of subword arrays for φ such that

$$\psi\underline{\pi}^h \,\underline{\underline{\vee}}\, \left\{\kappa_g\underline{\pi}^{h(g)} : g \in \Gamma_1\right\}$$

and for all g in Γ_1 and all ξ in Ξ the conditions $(*)$ are satisfied.

(b) Let ψ be a cpp-word ending in the symbol $\underline{\gamma}$ and h a non-negative integer. Suppose that to every subword array F for $\psi\underline{\pi}^h$ there corresponds a subword array E for φ satisfying the conditions

$$\text{c-wt}(E, \xi) \le \text{c-wt}(F, \xi)$$

and

$$n(E) \le n(F) ,$$

for all $\xi \in \Xi$. Then there exist sets $\left\{\kappa_g : g \in \Gamma\right\}$ of c-words, $\{h(g) : g \in \Gamma\}$ of non-negative integers, and $\left\{E_g : g \in \Gamma\right\}$ of subword arrays for φ such that

$$\psi\underline{\pi}^h \,\underline{\underline{\vee}}\, \left\{\kappa_g\underline{\pi}^{h(g)} : g \in \Gamma\right\}$$

and for all g in Γ and all ξ in Ξ, conditions $(*)$

are satisfied.

It is not hard to see that the lemma follows from proposition (b), since the given cpp-word φ may be expressed in the form $\varphi'\underline{\pi}^{h'}$ where either $\varphi' \in \Xi$ or φ' ends in the symbol $\underline{\gamma}$. In the former case, φ is already in the form required by the conclusion of the lemma; in the latter, the assumptions of proposition (b) are satisfied with $\psi = \varphi'$ and $h = h'$.

Proposition (a) is used at some points in the proof of proposition (b), and is proved first.

Proof of Proposition (a). By Lemma 4.3,

$$\psi \stackrel{\mathrm{V}}{=} \prod \{\lambda_d : d \in \Delta_1\} ,$$

where each λ_d is a c-word such that for all $\xi \in \Xi$,

$$\text{c-wt}(E, \xi) \le \text{c-wt}(\psi, \xi) \le \text{c-wt}(\lambda_d, \xi) .$$

Hence, by law (v),

$$\psi\underline{\pi}^h \stackrel{\mathrm{V}}{=} \prod \left\{\kappa_g \underline{\pi}^{h(g)} : g \in \Gamma_1\right\} ,$$

where for g in Γ_1, κ_g is a c-word with at least $p^{h-h(g)}$ distinct subwords (not necessarily unequal as words) from the set $\{\lambda_d : d \in \Delta_1\}$, whence, for all $\xi \in \Xi$,

$$p^{h-h(g)}\text{c-wt}(E, \xi) \le \text{c-wt}(\kappa_g, \xi) .$$

For g in Γ_1 such that $h - h(g) \ge n(E)$, Lemma 5.3 is used with $l = n(E)$ to show the existence of a subword array E_g such that for all $\xi \in \Xi$,

$$\text{c-wt}(E_g, \xi) \le p^{h-h(g)}\text{c-wt}(E, \xi) \le \text{c-wt}(\kappa_g, \xi)$$

and

$$n(E_g) = 0 \le h(g) ,$$

so that conditions (*) are satisfied. For g in Γ_1 such that

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$h - h(g) \le n(E)$, Lemma 5.3 is used with $l = h - h(g)$ to construct E_g such that for all $\xi \in \Xi$,

$$\text{c-wt}\left(E_g, \xi\right) \le p^{h-h(g)}\text{c-wt}(E, \xi) \le \text{c-wt}\left(\kappa_g, \xi\right)$$

and

$$n\left(E_g\right) = n(E) - \left(h-h(g)\right) \le h - \left(h-h(g)\right) = h(g) .$$

This completes the proof of proposition (a).

Proof of Proposition (b). The proof of proposition (b) is unfortunately rather more complicated, proceeding by a sequence of four nested induction arguments. As a preliminary, let F_M and F_m be the maximal and minimal subword arrays respectively for ψ ; and let E'_M and E'_m be the subword arrays for φ corresponding respectively to F_M and F_m under the hypotheses of proposition (b). Let E_M and E_m be the maximal and minimal arrays for φ . Note that E_m, E_M, E'_m , and E'_M are all subword arrays for φ ; the first two depend on φ only, and the last two depend also on the word ψ .

The first induction is in the reverse direction on h . If $h \ge n\left(E_m\right)$, then the assumptions of proposition (a) are satisfied by the choice $E = E_m$, and the conclusion of porposition (a) gives the required result. From now on, only words of the form $\psi\underline{\pi}^h$ with $h < n\left(E_m\right)$ need be considered. The first inductive hypothesis will be that the result is established for all words $\psi'\underline{\pi}^{h'}$ satisfying the assumptions of the proposition and the condition that $h' > h$.

Suppose $\psi = \psi\left(\xi_0, \ldots, \xi_{z-1}\right)$. The second and third inductive arguments between them cover only a finite number $\left(\text{at most } \text{c-wt}\left(E_M\right)\right)$ of steps. Note that if, for all i in $\underline{z}$,

$$\text{c-wt}\left(\psi, \xi_i\right) \ge \text{c-wt}\left(E_M, \xi_i\right)$$

then *a fortiori*, for all i in $\underline{z}$,

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$$\text{c-wt}(\psi,\ \xi_i) \geq \text{c-wt}(E'_M,\ \xi_i)\ .$$

If the latter condition is satisfied, then the assumptions of proposition (a) are satisfied by the choice $E = E'_M$, and again the required result follows. Thus only words $\psi\underline{\pi}^h$ such that for some i in $\underline{z}$, $\text{c-wt}(\psi,\ \xi_i) < \text{c-wt}(E'_M,\ \xi_j)$ need henceforth be considered. Let j be the least integer in $\underline{z}$ such that $\text{c-wt}(\psi,\ \xi_j) < \text{c-wt}(E'_M,\ \xi_j)$. Assume that the result is established for all words $\psi'\underline{\pi}^{h'}$ such that either (second inductive hypothesis, on j):

$$\text{c-wt}(\psi',\ \xi_i) \geq \text{c-wt}(E_M,\ \xi_i) \quad \text{for} \quad 0 \leq i \leq j$$

(note the final equality) or (third inductive hypothesis, in the reverse direction on $\text{c-wt}(\psi,\ \xi_j)$)

$$\text{c-wt}(\psi,\ \xi_j) < \text{c-wt}(\psi',\ \xi_j)\ .$$

For every word $\psi\underline{\pi}^h$ satisfying the conditions remaining to be considered,

$$\text{c-wt}(\psi,\ \xi_j) < \text{c-wt}(E'_M,\ \xi_j) \leq \text{c-wt}(F_M,\ \xi_j) = \text{cpp-wt}(\psi,\ \xi_j)\ .$$

This strict inequality shows that there exists a subword of ψ ending in the symbol $\underline{\pi}$ and having c-weight in ξ_j at least 1 . Among such subwords, let $\rho\underline{\pi}$ be one such that the value of $k(\rho,\ \psi)$ is minimal. The fourth induction is on this value $k(\rho,\ \psi)$. Since ψ itself ends in the symbol $\underline{\gamma}$, it follows that ρ is a proper subword of ψ and $k(\rho,\ \psi) \geq 1$.

Thus ψ has a subword χ with form either $[\rho\underline{\pi},\ \theta]$ or $[\theta,\ \rho\underline{\pi}]$ (for convenience, assume the former) with $\text{c-wt}(\rho,\ \xi_j) \geq 1$. The minimality of $k(\rho,\ \psi)$ shows that $l(\chi,\ \psi) = 0$, and hence that there exists a non-negative integer m , a c-word $\eta = \eta(\xi_0,\ \ldots,\ \xi_{m-1},\ \xi_m)$ with $\text{c-wt}(\eta,\ \xi_m) = 1$, and cpp-words $\theta_0,\ \ldots,\ \theta_m$ such that

$$\psi = \eta(\theta_0,\ \ldots,\ \theta_{m-1},\ [\rho\underline{\pi},\ \theta_m])\ .$$

Lemma 6.2 *(ii)* now gives

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$$\psi\underline{\pi}^{k} \overset{\underline{\underline{V}}}{=} \eta\left(\theta_0, \ldots, \theta_{m-1}, [\rho, \theta_m]\underline{\pi}\right)\underline{\pi}^{k} \prod\left\{\delta_{d}\underline{\pi}^{h(d)} : d \in \Delta\right\} ,$$

where for all d in Δ, $\text{c-wt}\left(\delta_d, \xi_j\right) > \text{c-wt}\left(\psi, \xi_j\right)$, and if G is a subword array for $\delta_d\underline{\pi}^{h(d)}$ then there exist corresponding arrays F for $\psi\underline{\pi}^{k}$ and hence E for φ such that for all $\xi \in \Xi$,

$$\text{c-wt}(E, \xi) \le \text{c-wt}(F, \xi) \le \text{c-wt}(G, \xi)$$

and

$$n(E) \le n(F) \le n(G) .$$

By the second or third inductive hypothesis, then, according as $\text{c-wt}\left(\delta_d, \xi_j\right)$ is or is not at least $\text{c-wt}\left(E'_M, \xi_j\right)$, each word $\delta_d\underline{\pi}^{h(d)}$ may be expressed in the required form.

The word $\psi' = \eta\left(\theta_0, \ldots, \theta_{m-1}, [\rho, \theta_m]\underline{\pi}\right)$ also is shown by Lemma 6.2 to satisfy the hypotheses of proposition (b). Further, ψ' has a subword $\rho'\underline{\pi} = [\rho, \theta_m]\underline{\pi}$ such that

$$k(\rho', \psi') = k(\rho, \psi) - 1 .$$

In the initial case, $k(\rho, \psi) = 1$; this means that $\psi' = \rho'\underline{\pi}$, whence $\psi'\underline{\pi}^{k} = \rho'\underline{\pi}^{k+1}$, which by the first inductive hypothesis may be expressed in the required form. Otherwise, ψ' ends in the symbol $\underline{\gamma}$, and the result is given by the fourth inductive hypothesis on $k(\rho, \psi)$.

This completes the proof of proposition (b), and hence of the lemma. $\square$

6.4 APPLICATION OF LEMMA 6.3 - COMMENT AND EXAMPLE

An arbitrary cpp-word φ may be written $\varphi = \varphi'\underline{\pi}^{n}$ where $n \ge 0$ and φ' is a cpp-word ending in the symbol $\underline{\gamma}$. The word φ' is then called the *crown* of φ (because the tree of φ' is the crown of the tree of φ). Let $\underline{\underline{V}}$ be a group-like variety of $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$-algebras, and

$$\varphi \overset{\underline{\underline{V}}}{=} \prod\left\{\kappa_g\underline{\pi}^{h(g)} : g \in \Gamma\right\}$$

the law linking φ with a product of scpp-words, given by Lemma 6.3.

One word $\kappa_g\underline{\pi}^{h(g)}$ is obtained from φ simply by moving all symbols

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$\underline{\pi}$ from their positions in φ to the end of the word; this is the only word in the product whose c-weight has the minimal value equal to $\text{c-wt}(\varphi)$; and it corresponds to the minimal subword array for φ . If φ is not already an scpp-word, then this word has strictly greater cpp-weight than has φ .

Those words $\kappa_g \underline{\pi}^{h(g)}$ whose cpp-weight is minimal, that is, equal to $\text{cpp-wt}(\varphi)$ are at another extreme. Each of them has $\text{c-wt}\left(\kappa_g\right) \geq \text{cpp-wt}(\varphi')$ and corresponds to a subword array for φ which, while not itself necessarily maximal, induces only maximal arrays of φ' on all labelled copies of φ' which it contains.

Between these extremes, Lemma 6.3 gives information about the minimum value of $\text{c-wt}\left(\kappa_g\right)$ - or values of $\text{c-wt}\left(\kappa_g, \xi\right)$ - associated with each possible value of $h(g)$, as is shown in the following example. Note that for each integer h , $0 \leq h \leq n\left(E_m\right)$, the subword array of minimal c-weight (or minimal c-weight in a specified generator) does not contain a labelled tree in which a path from a starred $\underline{\pi}$-edge to the root passes through an unstarred $\underline{\pi}$-edge.

As an example, let $\varphi = \xi\underline{\pi}\eta\underline{\gamma}\xi\underline{\pi}\underline{\pi}\underline{\gamma}\underline{\pi}$, and as usual let E_m and E_M be minimal and maximal arrays for φ . If E is an arbitrary array,

$$3 = \text{c-wt}\left(E_m\right) \leq \text{c-wt}(E) \leq \text{c-wt}\left(E_M\right) = 2p^3 + p^2 .$$

Since $n\left(E_m\right) = 5$, values $0 \leq h \leq 5$ are of interest.

When $h(g) = 5$, clearly E_m itself is the minimal array, so $\text{c-wt}\left(\kappa_g\right) \geq \text{c-wt}\left(E_m\right) = 3$. In this case, $\text{cpp-wt}\left(\kappa_g \underline{\pi}^{h(g)}\right) \geq 3p^5$. There are two candidates for a minimal array with $n\left(E_g\right) = 4$; they are not essentially different, each containing one word equal to φ and $(p-1)$ equal to ξ , and having

$$\text{c-wt}\left(E_g\right) = 3 + p - 1 = p + 2 .$$

Hence for $h(g) = 4$, $\text{c-wt}\left(\kappa_g\right) \geq p + 2$ and $\text{cpp-wt}\left(\kappa_g \underline{\pi}^{h(g)}\right) \geq p^5 + 2p^4$.

Among arrays E_g for φ such that $n\left(E_g\right) = 3$, it is easy to see

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that the one with the least value of c-wt$\left(E_g\right)$ is

$$\left\{\xi\underline{\pi}^{3}\eta\underline{\gamma}\underline{\pi}^{*}\xi\underline{\pi}^{3}\underline{\pi}^{*}\underline{\gamma}\underline{\pi}^{*}(1)\right\} \cup \{\xi(i) \;:\; 2 \le i \le 2p-1\} .$$

Hence for those g in Γ with $h(g) = 3$, it follows that c-wt$\left(\kappa_g\right) \ge 2p + 1$ and cpp-wt$\left(\kappa_{g\underline{\pi}}{}^{h}\right) \ge 2p^4 + p^3$.

Among arrays E_g for ϕ such that $n\left(E_g\right) = 2$, it is not obvious at first sight whether that containing $\xi\underline{\pi}^{3}\eta\underline{\gamma}\underline{\pi}^{*}\xi\underline{\pi}^{3}\underline{\pi}^{2}\underline{\gamma}\underline{\pi}^{*}$ or that containing $\xi\underline{\pi}^{3}\eta\underline{\gamma}\underline{\pi}^{2}\xi\underline{\pi}^{3}\underline{\pi}^{*}\underline{\gamma}\underline{\pi}^{*}$ has less c-weight; the values turn out to be $p^2 + p + 1$ and $p^2 + 2p$ respectively. Hence for each g in Γ with $h(g) = 2$, c-wt$\left(\kappa_g\right) \ge p^2 + p + 1$ and cpp-wt$\left(\kappa_{g\underline{\pi}}{}^{h(g)}\right) \ge p^4 + p^3 + p^2$.

The lightest array E_g with $n\left(E_g\right) = 1$ is clearly that containing $\xi\underline{\pi}^{3}\eta\underline{\gamma}\underline{\pi}^{2}\xi\underline{\pi}^{3}\underline{\pi}^{2}\underline{\gamma}\underline{\pi}^{*}$, which has c-weight $2p^2 + p$; so for each g in Γ with $h(g) = 1$, c-wt$\left(\kappa_g\right) \ge 2p^2 + p$ and cpp-wt$\left(\kappa_{g\underline{\pi}}{}^{h(g)}\right) \ge 2p^3 + p^2$. Note that $2p^2 + p$ is the cpp-weight of the crown of φ , and $2p^3 + p^2$ is the cpp-weight of φ itself, and consequently the minimal possible cpp-weight.

Finally, if $n\left(E_g\right) = 0$ then $E_g = E_M$; and for each g in Γ with $h(g) = 0$,

$$\text{c-wt}\left(\kappa_g\right) = \text{cpp-wt}\left(\kappa_{g\underline{\pi}}{}^{h(g)}\right) \ge 2p^3 + p^2 .$$

The information given in this way by Lemma 6.3 appears to be the best possible. However, it is very complicated, and for many purposes the simpler result obtained by combining it with Lemma 5.4 is sufficient.

6.5 COROLLARY

Corresponding to a group-like variety $\underline{\underline{V}}$ *and a cpp-word* φ *, there exists an ordered set* Δ *of scpp-words such that* $\varphi \overset{\underline{\underline{V}}}{=} \Pi\Delta$ *and* $\varphi \le' \Pi\Delta$. □

Combining this in turn with Lemma 4.2 *(a)* gives the main result.

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6.6 THEOREM

Corresponding to a group-like variety $\underline{\underline{V}}$ *and an arbitrary word* φ *in* B *, there exists an ordered set* Δ *of scpp-words such that* $\varphi \overset{\underline{V}}{=} \Pi\Delta$ *and* $\varphi \leq' \Pi\Delta$. □

Return now to the descending central series described in 2.4. Theorem 6.6, together with the definitions there, gives:

6.7 LEMMA

In an arbitrary $\{\underline{\gamma}, \underline{\pi}, \underline{\mu}\}$*-word algebra* D *, the ideal* $\gamma_i^{a,b,e}(D)$ *is generated, modulo* $\gamma_{i+1}^{a,b,e}(D)$ *by the set of homomorphic images in* D *of scpp-words* φ *such that* $w_{a,b}^{e}(\varphi) = i$.

In particular, in a group G *,*

$\gamma_i(G)$ *is generated modulo* $\gamma_{i+1}(G)$ *by the set of commutators of weight* i *;*

$\varepsilon_i(G)$ *is generated modulo* $\varepsilon_{i+1}(G)$ *by the set of* p^j*th powers of commutators of weight* w *, where* $w + j = i$ *; and*

$\pi_i(G)$ *is generated modulo* $\pi_{i+1}(G)$ *by the set of* p^j*th powers of commutators of weight* w *, where* $wp^j = i$. □

The well-known result that, in fact, the commutators of weight i generating $\gamma_i(G)$ may be restricted either to being basic or to being left-normed carries across also to the other series. This fact is not needed in [6], and since the proofs are rather tedious, they are omitted.

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