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DIRAC SYSTEMS WITH DISCRETE SPECTRA

D. B. HINTON AND J. K. SHAW

1. Introduction. In this paper we consider the one dimensional Dirac system

$$(1.1)\quad y' = \begin{pmatrix} p(x) & \lambda\alpha_2(x) + p_2(x) \\ -\lambda\alpha_1(x) - p_1(x) & -p(x) \end{pmatrix} y$$

$$a \leqq x < b \leqq \infty, y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

where $\alpha_k(x) > 0$, λ is a complex spectral parameter, and the remaining coefficients are suitably smooth and real valued. We regard (1.1) as regular at $x = a$ but singular at $x = b$; in Section 4 we extend our result to problems having two singular endpoints.

Equation (1.1) arises from the three dimensional Dirac equation with spherically symmetric potential, following a separation of variables. For the choices $p(x) = k/x$, $\alpha_k(x) = 1$, $p_2(x) = (z/x) + c$, $p_1(x) = (z/x) - c$, and appropriate values of the constants, (1.1) is the radial wave equation in relativistic quantum mechanics for a particle in a field of potential $V = z/x$ [**17**]. Such an equation was studied by Kalf [**11**] in the context of limit point-limit circle criteria, which is one of the matters we consider here.

We will be concerned here generally with the bound energy states of (1.1). Specifically, our objective is to give sufficient conditions under which the selfadjoint operators associated with (1.1) have purely discrete spectra; i.e., spectra which consist solely of isolated eigenvalues. In so doing, we will extend the results of Roos and Sangren [**15**] and the account of their work in the book of Levitan and Sargsjan [**13**]. These articles treat a simpler system

$$(1.2)\quad y' = \begin{pmatrix} 0 & \lambda + q_2(x) \\ -\lambda - q_1(x) & 0 \end{pmatrix} y(x),\ 0 \leqq x < \infty,$$

in which the potential terms $q_k(x)$ are smooth and regularly growing. Their method is analogous to one of Titchmarsh [**18**, **19**, **20**] and involves the asymptotic form of solutions of (1.2) under hypotheses which guarantee absolute integrability of the quantity

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$$(1.3)\qquad P(x) = \left(\frac{\lambda + q_2(x)}{\lambda + q_1(x)}\right)^{1/4} \times \frac{d}{dx}\left\{\frac{(\lambda + q_2(x))q_1'(x) - (\lambda + q_1(x))q_2'(x)}{(\lambda + q_1(x))^{5/4}(\lambda + q_2(x))^{7/4}}\right\}.$$

Levitan and Sargsjan assume conditions which yield $P \in L^1[0, \infty)$, while [**15**] takes the condition $P \in L^1[0, \infty)$ as a direct hypothesis. The most crucial conditions are that $q_1(x) \to \pm\infty$ and $q_2(x) \to \mp\infty$, as $x \to \infty$, and when these and the other hypotheses are satisfied the spectrum is purely discrete. The additional hypotheses are essentially that $q_k(x)$ and $q_k'(x)$ be of one sign and satisfy

$$|q_k(x)| = 0\{|q_k(x)|^c\}, \; 0 < c < (3/2).$$

Our results extend those of [**13**] and [**15**] to the extent that we allow main diagonal and weight terms in (1.1), we treat both one and two singular endpoint problems, and we require only that certain components of the potentials be of smoothly regular growth; our specific smoothness assumptions are much weaker than those of [**13**, **15**]. We break down our presentation into two cases: (i) $p(x)$ is the dominant term of (1.1), or (ii) the $p_k(x)$ are the dominant terms of (1.1). These cases occupy Sections 3 and 2, respectively. Our basic hypothesis is that when the dominant term is factored out of the matrix in (1.1), the resulting entries are decomposable into long range, short range, and oscillating terms; we define these expressions precisely below.

The authors used the same decomposition of potential terms in [**8**], where we studied smoothness of the spectral function throughout the continuous spectrum. But in the present paper the continuous spectrum is, of course, empty.

When coupled with [**8**], our results here extend further the duality which exists between discrete and continuous spectral criteria of (1.1) when the terms $p_k(x)$ are sufficiently dominant. The basic principle is that if $p_1(x)$ and $p_2(x)$ have the same sign the spectrum is absolutely continuous on the whole λ-axis, and if their signs are opposite the spectrum is purely discrete; compare [**13**, pp. 228-230].

To study the spectrum of (1.1) we will derive the asymptotic form of its solutions. We generalize the technique used in [**9**], which dealt with a special case of (1.1) with singularity only at $x = 0$.

In order to state our theorems, we require some terminology. Introducing the operator T, whose domain is defined below, by

$$(1.4)\qquad Ty = \begin{pmatrix} \alpha_1^{-1} & 0 \\ 0 & \alpha_2^{-1} \end{pmatrix}\left\{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}y' - \begin{pmatrix} p_1 & p \\ p & p_2 \end{pmatrix}y\right\},$$

we may express (1.1) in operator form $Ty = \lambda y$. We employ a boundary condition at $x = a$ in the usual way by writing

$$B(y) = \sin \beta y_1(a) + \cos \beta y_2(a),$$

and we let $L^2_\alpha[a, b)$ denote the Hilbert space

$$\left\{ y \,|\, y \text{ is measurable and } \int_a^b (\alpha_1 |y_1|^2 + \alpha_2 |y_2|^2) < \infty \right\}.$$

Now let

$$D(T) = \{ y \in L^2_\alpha[a, b) \,|\, y \text{ is locally absolutely continuous,}$$
$$B(y) = 0 \text{ and } Ty \in L^2_\alpha[a, b) \}.$$

We know that Weyl's limit point-limit circle classification holds for Dirac systems [**12**]; i.e., either there is exactly one independent solution of (1.1) in $L^2_\alpha[a, b)$ for all non-real λ (limit point case) or every solution lies in $L^2_\alpha[a, b)$ for every λ (limit circle case). If (1.1) lies in the limit point case, then $T:D(T) \to L^2_\alpha[a, b)$ is a selfadjoint operator; in the limit circle case a boundary condition must also be imposed at $x = b$ [**16**, **21**]. In the limit circle case, the spectrum of each selfadjoint operator associated with (1.1) has discrete spectrum; see [**10**, **16**] and remarks below linking the spectrum to the Titchmarsh-Weyl function. Hence there are two routes to proving discreteness of the spectrum of an operator arising from (1.1). We may either establish that (1.1) is of limit circle type, or we may prove directly that the spectrum of T is discrete if the limit point case holds.

We say that the complex valued function $f(x)$ is of short range type if $f \in L^1[a, b)$; i.e., f is measurable and

$$\int_a^b |f(x)| dx < \infty.$$

A differentiable function $f(x)$ with $f' \in L^1[a, b)$ and $f(x) \to 0$ $(x \to b)$ will be called a long range term. If $f(x)$ is conditionally integrable on $[a, b)$ and if the function

$$V(x) = \int_x^b f(s) ds$$

lies in $L^1[a, b)$, then we say that $f(x)$ is of oscillatory type; a typical example is

$$f(x) = x^{-1} \sin(x^2) \quad \text{on } [1, \infty).$$

We now state our principle result for the case in which the terms $p_k(x)$ are the dominant ones in (1.1).

THEOREM 1. *Suppose that* $p_k(x) = p_{k1}(x) + p_{k2}(x)$, *where* $p_{11}(x) > 0$ *and* $p_{21}(x) < 0$. *Assume that*

$$(-\alpha_k(x)/p_{k2}(x)) = r_{k1}(x) + r_{k2}(x) + r_{k3}(x) \quad \textit{and}$$
$$(-p_{k2}(x)/p_{k1}(x)) = s_{k1}(x) + s_{k2}(x) + s_{k3}(x),$$

where r_{k1} and s_{k1} are long range, $Q(x)r_{k2}(x)$ and $Q(x)s_{k2}(x)$ are oscillatory, and where $Q(x)r_{k3}(x)$ and $Q(x)s_{k3}(x)$ are short range, with

$$Q(x) = (-p_{11}(x)p_{21}(x))^{1/2}.$$

Let

$$\begin{aligned}\Delta &= (p_{11}^{-1}(x)p_{11}'(x) - p_{21}^{-1}(x)p_{21}'(x) - p(x))/Q(x)\\ &= \Delta_1(x) + \Delta_3(x),\end{aligned}$$

where Δ_1 is long range and $Q(x)\Delta_3(x)$ is short range. Letting

$$R_k(x) = \int_x^b Q(t)r_{k2}(t)dt \quad \text{and} \quad S_k(x) = \int_x^b Q(t)s_{k2}(t)dt$$

(R_k, $S_k \in L^1[a, b)$ by definition) assume that all of the functions

$$\int_x^b |K(t)u'(t)|dt, \ K(x)v(x) \quad \text{and} \quad v(x)\int_x^b |K(t)u'(t)|dt$$

lie in $L^1[a, b)$, where K stands for either R_k or S_k, v stands for either Qr_{j2} or Qs_{j2}, and where u stands for any one of r_{ml}, s_{ml} or Δ_1. Let

$$\mu_0(x, \lambda) = [(1 - r_{11}(x) - \lambda s_{11}(x))(1 - r_{21}(x) - \lambda s_{21}(x)) + \Delta_1^2(x)]^{1/2}$$

and

$$E(x, \lambda) = \exp\left\{\int_a^x \mu_0(t, \lambda)Q(t)dt\right\}.$$

Then the condition

$$\int_a^b (E^2(x, 0) + E^{-2}(x, 0))\left(\alpha_1(x)\left|\frac{p_{21}(x)}{p_{11}(x)}\right|^{1/2} + \alpha_2(x)\left|\frac{p_{11}(x)}{p_{21}(x)}\right|^{1/2}\right)dx = \infty \tag{1.5}$$

is necessary and sufficient for (1.1) *to be of limit point type. If the limit point case holds and*

$$\int_a^b [-p_{11}(x)p_{21}(x)]^{1/2} = \infty,$$

then T has discrete spectrum.

The main discrete spectrum result of [**13**, p. 230] is the special case of Theorem 1, in which $\alpha_k = 1$, $p_{k2} = 0$, $s_{kj} = 0$, $r_{k2} = r_{k3} = 0$, $r_{k1} = -1/p_{k1}$, $\Delta_1 = \Delta_3 = 0$. The hypotheses of [**13**] are easily seen to imply that r_{k1} is a long range term.

An interesting special case of Theorem 1 is the one in which (α_k/p_{k1}) is bounded for each k. In this situation, the condition

$$\int_a^b [-p_{11}(x)p_{21}(x)]^{1/2}dx < \infty$$

implies finiteness of (1.5) and so becomes a limit circle criterion.

The dominance of the $p_k(x)$ terms over $p(x)$ and the $\alpha_k(x)$ is reflected in the assumptions on $(\alpha_k(x)/p_{k1}(x))$ and $\Delta(x)$.

The theorem may be illustrated using powers of x on $1 \leqq x < \infty$ if we take $p(x) = x^n$, $\alpha_1(x) = x^\gamma$, $\alpha_2(x) = x^\delta$, $p_{11}(x) = x^k$, $p_{21}(x) = -x^m$, $p_{22}(x) = 0$, $p_{12}(x) = x^{k+\alpha}\sin(x^4)$. Then for the hypotheses to be satisfied it is sufficient that $\gamma < k$, $\delta < m$, $n < (m+k)/2$, $-1 < (m+k)/2$, and $k + \alpha < 1$.

We now require a brief discussion of the Titchmarsh-Weyl m-coefficient for (1.1). Let $\Theta(x, \lambda)$ and $\Phi(x, \lambda)$ denote the solutions of (1.1) with initial values

$$\Theta(a, \lambda) = \begin{pmatrix} \sin\beta \\ \cos\beta \end{pmatrix}, \quad \Phi(a, \lambda) = \begin{pmatrix} -\cos\beta \\ \sin\beta \end{pmatrix}, \quad (0 \leqq \beta < 2\pi)$$

with the same β as introduced below (1.4). Since $B(\Phi) = 0$, then the eigenvalues of T are those real values of λ for which $\Phi \in L^2_\alpha[a, b)$. In the limit point case for (1.1), the limits

$$(1.6) \qquad m(\lambda) = -\lim_{x \to b} \frac{\Theta_1(x, \lambda)}{\Phi_1(x, \lambda)} = -\lim_{x \to b} \frac{\Theta_2(x, \lambda)}{\Phi_2(x, \lambda)}$$

exist for $\operatorname{Im}(\lambda) \neq 0$, and define a function which is analytic in each half plane; see [**4**, **5**]. The function $m(\lambda)$ is known as the Titchmarsh-Weyl coefficient. The real axis may comprise regular points, poles or other types of singularities of $m(\lambda)$. A complete characterization of the spectrum of T in terms of $m(\lambda)$ was given in [**6**], and in [**7**] for two singular endpoint problems. The only results we need here assert that a point λ_0 is an isolated eigenvalue of T if and only if λ_0 is a simple pole of the m-coefficient, and λ_0 is in the resolvent set of T (i.e., is not in the spectrum) if and only if $m(\lambda)$ is analytic at λ_0. Consequently the spectrum of T is discrete if and only if $m(\lambda)$ is a meromorphic function, and the latter is what we will actually prove in all cases in this paper.

Finally, we require a theorem on the asymptotic behavior of solutions of (1.1). Consider a system of the form

$$(1.7) \qquad z'(x) = [\Omega'(x)\Omega^{-1}(x) + B(x) + C(x)]z(x), \quad a \leqq x < b,$$

where Ω is an $n \times n$ diagonal matrix whose entries are complex valued, locally absolutely continuous, nonvanishing functions, and $B(x)$ and $C(x)$ are $n \times n$ complex, locally integrable matrix functions. Assume that each term $e_{ik}(x) = \Omega_{ii}(x)\Omega^{-1}{}_{kk}(x)$ satisfies either

$$(1.8) \qquad |e_{ik}(x)/e_{ik}(s)| \leqq M \quad \text{for } x_0 \leqq s \leqq x < b, \ \lim_{x \to b} e_{ik}(x) = 0$$

or

(1.9) $|e_{ik}(x)/e_{ik}(s)| \leqq M$ for $x_0 \leqq x \leqq s < b$,

for constants x_0 and $M > 0$. In [3], condition (1.8) is called "essentially decreasing", (1.9) is called "essentially increasing", and the following theorem is proved.

THEOREM A ([3]). *Suppose in addition to* (1.8) *and* (1.9) *that*

$$B_0(x) = \int_x^b B(s)ds \text{ exists,}$$

$$\int_a^b \|C(s)\|ds < \infty$$

(*where* $\|\cdot\|$ *is the matrix operator norm*) *and*

$$\int_a^b \|G(s)\|ds < \infty,$$

where $G = -\Omega'\Omega^{-1}B_0 + B_0\Omega'\Omega^{-1} + B_0B$. *Then there is a fundamental matrix* Z *of* (1.7) *such that*

(1.10) $\lim_{x \to b} Z(x)\Omega^{-1}(x) = I.$

We are now ready to prove Theorem 1, the p_k-dominant case of (1.1). The $p(x)$-dominant case will be considered in Section 3.

2. Proof of theorem 1. We begin by removing the $p(x)$ terms from (1.1) by letting

$$F(x) = \exp \int_a^x p(s)ds$$

and introducing a new dependent variable $w(x)$, where

$$y(x) = \begin{pmatrix} F(x) & 0 \\ 0 & F^{-1}(x) \end{pmatrix} w(x),$$

so that $w(x)$ satisfies

(2.1) $$w'(x) = \begin{pmatrix} 0 & F^{-2}(\lambda\alpha_2 + p_2) \\ -F^2(\lambda\alpha_1 + p_1) & 0 \end{pmatrix} w(x), \quad a \leqq x < b.$$

A further change of dependent variable will "equalize" the magnitudes of the dominant terms in (2.1). Let

$$\eta_1(x) = F^{-1}(x)(-p_{21}(x)/p_{11}(x))^{1/4}, \quad \eta_2(x) = 1/\eta_1(x)$$

and define $z(x)$ by

$$w(x) = \begin{pmatrix} \eta_1(x) & 0 \\ 0 & \eta_2(x) \end{pmatrix} z(x).$$

Then a calculation shows that $z(x)$ satisfies

$$(2.2)\qquad z'(x) = \begin{pmatrix} -\eta_1'/\eta_1 & (\eta_1 F)^{-2}(\lambda\alpha_2 + p_2) \\ -(\eta_1 F)^2(\lambda\alpha_1 + p_1) & -\eta_2'/\eta_2 \end{pmatrix} z(x),$$

$$a \leqq x < b.$$

Recall that $p_k(x) = p_{k1}(x) + p_{k2}(x)$ and note that

$$(-p_{21}/\eta_1^2F^2) = \eta_1^2F^2p_{11} = (-p_{11}p_{21})^{1/2} = Q.$$

Substituting this into (2.2) and factoring out the term $Q(x)$, we obtain

$$(2.3)\qquad z'(x) = Q(x)\begin{pmatrix} -\eta_1'/(\eta_1 Q) & -1 - \dfrac{\lambda\alpha_2}{p_{21}} - \dfrac{p_{22}}{p_{21}} \\ -1 - \dfrac{\lambda\alpha_1}{p_{11}} - \dfrac{p_{12}}{p_{11}} & -\eta_2/(\eta_2 Q) \end{pmatrix} z(x).$$

Recalling the notation from the statement of Theorem 1, and introducing

$$a_{kj}(x, \lambda) = r_{kj}(x) + \lambda s_{kj}(x),$$

(2.3) may be written as

$$z'(x) = Q(x)\begin{pmatrix} \Delta_1 + \Delta_3 & -1 + a_{21} + a_{22} + a_{23} \\ -1 + a_{11} + a_{12} + a_{13} & -\Delta_1 - \Delta_3 \end{pmatrix}(x),$$

which we decompose as

$$(2.4)\qquad z'(x) = \left\{Q(x)\begin{pmatrix} \Delta_1 & -1 + a_{21} \\ -1 + a_{11} & -\Delta_1 \end{pmatrix} + Q\begin{pmatrix} 0 & a_{22} \\ a_{12} & 0 \end{pmatrix} + Q\begin{pmatrix} \Delta_3 & a_{23} \\ a_{13} & -\Delta_3 \end{pmatrix}\right\} z(x)$$

$$= \{Q(x)D_1(x, \lambda) + Q(x)D_2(x, \lambda) + Q(x)D_3(x, \lambda)\}z(x),$$

with obvious notation.

Our approach will be to diagonalize the leading term in (2.4) and apply Theorem A to the resulting system. To this end we compute the eigenvalues of the first matrix in (2.4) by solving the determinant equation

$$\det\begin{pmatrix} \mu - \Delta_1 & 1 - a_{21} \\ 1 - a_{11} & \mu + \Delta_1 \end{pmatrix} = \mu^2 - \Delta_1^2 - (1 - a_{11})(1 - a_{21}).$$

The roots are $\mu = \pm\mu_0$, where

$$(2.5)\qquad \mu_0(x, \lambda) = [\,(1 - a_{11}(x, \lambda))(1 - a_{21}(x, \lambda)) + \Delta_1(x)\,]^{1/2},$$

taking the principal branch of the square root. Note that $\mu_0(x, \lambda) \to 1$ as

$x \to b$, uniformly on compact λ sets, since a_{k1} and Δ_1 are long range terms. By redefining, if necessary, the long and short range parts of (2.4), we may assume without loss of generality that $\mu_0 \neq 0$ and $a_{21} \neq 1$. The matrix of eigenvectors

$$S = \begin{pmatrix} 1 - a_{21} & 1 - a_{21} \\ -\mu_0 + \Delta_1 & \mu_0 + \Delta_1 \end{pmatrix},$$

$$S^{-1} = \begin{pmatrix} \mu_0 + \Delta_1 & -1 + a_{21} \\ \mu_0 - \Delta_1 & 1 - a_{21} \end{pmatrix}[2\mu_0(1 - a_{21}]^{-1},$$

diagonalizes D_1 in (2.4) in that

$$S^{-1}D_1S = \begin{pmatrix} \mu_0 & 0 \\ 0 & -\mu_0 \end{pmatrix}.$$

Following the usual procedure let $\xi = S^{-1}z$, so that (2.4) is equivalent to

$$\xi'(x) = \left\{Q\begin{pmatrix} \mu_0 & 0 \\ 0 & -\mu_0 \end{pmatrix} + S^{-1}QD_2S + S^{-1}QD_3S - S^{-1}S'\right\}\xi(x), \quad a \leqq x < b. \tag{2.6}$$

Let

$$\Omega(x, \lambda) = \begin{pmatrix} E(x, \lambda) & 0 \\ 0 & 1/E(x, \lambda) \end{pmatrix}$$

where $E(x, \lambda)$ is defined above (1.5), so that

$$\Omega'\Omega^{-1} = Q\begin{pmatrix} \mu_0 & 0 \\ 0 & -\mu_0 \end{pmatrix},$$

and put

$$B = S^{-1}QD_2S, \ C = S^{-1}QD_3S - S^{-1}S'.$$

We wish to apply the asymptotics of Theorem A to (2.6), and so we must show that its hypotheses are satisfied.

Starting with the term $S^{-1}S'$, note that

$$2S^{-1}(x, \lambda) \to \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad x \to b,$$

uniformly on compact λ sets. Now S' contains the terms a_{21}' and Δ_1' which are long range by assumption. As for the term μ_0, we have

$$\mu_0^2 = (1 - a_{11})(1 - a_{21}) + \Delta_1^2,$$

which implies

$$2\mu_0\mu_0' = (1 - a_{21})(-a_{21}') + (1 - a_{21})(-a_{11}') + 2\Delta_1\Delta_1',$$

and so it follows that $\mu_0' \in L^1[a, b)$ and then that $S^{-1}S' \in L^1[a, b)$. Since S^{-1} and S approach constant matrices, $S^{-1}QD_3S$ consists entirely of $L^1[a, b)$ terms by assumption. Therefore the matrix $C(x, \lambda)$ has absolutely integrable entries, so

$$\int_a^b \|C(s)\|ds < \infty.$$

Turning now to conditions (1.8) and (1.9) we have $e_{11} = e_{22} = 1$ and $e_{12} = E^2$, $e_{21} = E^{-2}$. Considering e_{21}, note that

$$|e_{12}(x)| = \exp\left(2 \operatorname{Re} \int_a^x \mu_0(t)Q(t)dt\right),$$

so that $|e_{12}(x)|$ is actually monotone for sufficiently large x since $\mu_0 \to 1$ and $Q(t) > 0$, and similarly for $e_{21}(x)$.

Next we will show that

$$B_0(x) = \int_x^b B(t)dt$$

satisfies the hypotheses of Theorem A. By a routine calculation,

$$(2.7) \quad B = \frac{Qa_{12}(1 - a_{21})}{2\mu_0}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} + \frac{Qa_{22}}{2\mu_0(1 - a_{21})}\begin{pmatrix} -\mu_0^2 + \Delta_1^2 & (\mu_0 + \Delta_1)^2 \\ -(\mu_0 - \Delta_1)^2 & \mu_0^2 - \Delta_1^2 \end{pmatrix}$$

and this term is required to be conditionally integrable. Looking at the first scalar term on the right of (2.7), let us introduce $V_k(x) = R_k(x) + \lambda S_k(x)$ and then write

$$(2.8) \quad \begin{aligned} \int_a^x \frac{Qa_{12}(1 - a_{21})}{\mu_0} &= -\int_a^x V_1'\left(\frac{(1 - a_{21})}{\mu_0}\right) \\ &= \left[-\frac{V_1(1 - a_{21})}{\mu_0}\right]_a^x \\ &\quad + \int_a^x V_1\left(\frac{\mu_0(-a_{21}') - (1 - a_{21})\mu_0'}{\mu_0^2}\right). \end{aligned}$$

Since V_1, $a_{21} \to 0$ and $\mu_0 \to 1$ the first term on the right of (2.8) converges (to 0) as $x \to b$. As for the integral on the right of (2.8), we know a_{21} and μ_0 are long range terms and so the integral converges absolutely as $x \to b$. Hence the first term on the right of (2.7) is conditionally integrable. Turning to the second term in (2.7) consider first the expression

$$Qa_{22}(\mu_0^2 - \Delta_1^2)/(\mu_0(1 - a_{21})),$$

which occurs on the main diagonal; proofs for the other entries will be similar. Proceeding as above,

$$(2.9)\quad \int_a^T \frac{Qa_{22}(\mu_0^2 - \Delta_1^2)}{\mu_0(1 - a_{21})} = -\int_a^T V_2'\left(\frac{(\mu_0^2 - \Delta_1^2)}{\mu_0(1 - a_{21})}\right)$$
$$= \left[-V_2\left(\frac{\mu_0^2 - \Delta_1^2}{\mu_0(1 - a_{21})}\right)\right]_a^T + \int_a^T V_2\left(\frac{(\mu_0^2 - \Delta_1^2)}{\mu_0(1 - a_{21})}\right)',$$

where the differentiated expression contains in each term a long range function. Thus the right side of (2.9) converges as $x \to \infty$, and this completes the proof that $B(t)$ is conditionally integrable.

Lastly, we come to the function G of Theorem A, where we are required to prove $G \in L^1(a, \infty)$. Write $B_0(x) = U(x) + W(x)$, where U and W are integrals of the respective terms in (2.7). Then by definition of G,

$$G = 2\mu_0\begin{pmatrix} 0 & -U_{21} - W_{21} \\ U_{21} + W_{21} & 0 \end{pmatrix} + B_0B,$$

and we will now show that U_{21}, U_{12}, W_{21}, W_{12}, $B_0B \in L^1[a, b)$. Working first with $U_{21} = -U_{12}$, by (2.8) we see that

$$U_{21} = \frac{V_1(1 - a_{21})}{2\mu_0} + \int_x^b V_1\left(\frac{\mu_0(-a_{21}') - (1 - a_{21})\mu_0'}{2\mu_0}\right).$$

By hypothesis, $V_1 = R_1 + \lambda S_1 \in L^1[a, b)$, and furthermore

$$\int_x^\infty V_1a_{21}' = \int_x^\infty (R_1 + \lambda S_1)(r_{21}' + \lambda s_{21}') \in L^1[a, b).$$

Since

$$2\mu_0\mu_0' = (1 - a_{11})(-a_{21}') + (1 - a_{21})(-a_{11}') + 2\Delta_1\Delta_1',$$

the hypotheses also imply

$$\int_x^b V_2\mu_0' \in L^1[a, b),$$

and so we now know that U_{12} and U_{21} are in $L^1[a, b)$. For W_{12} (and similarly for W_{21}) a calculation such as (2.9) brings

$$W_{12} = V_2\frac{(\mu_0 + \Delta_1)^2}{2\mu_0(1 - a_{21})} + \int_x^b V_2\left(\frac{(\mu_0 + \Delta_1)^2}{2\mu_0(1 - a_{21})}\right)'.$$

To begin with, $V_2 \in L^1[a, b)$ by hypothesis. After taking the indicated derivative in the integral and expanding, we find that the crucial terms are

$$\int_x^b V_2a_{11}',\quad \int_x^b V_2a_{21}' \quad\text{and}\quad \int_x^b V_2\Delta_1',$$

all of which belong to $L^1[a, b)$ because $V_2 = R_2 + \lambda S_2$ and $a_{k1} = r_{k1} + \lambda s_{k1}$. Hence $W_{12} \in L^1[a, b)$.

To show $G \in L^1[a, b)$ it remains now only to prove $B_0 B \in L^1[a, b)$. However, $B_0' = -B = U' + W'$, and so it is sufficient to prove that

$$(2.10) \quad (U + W)(U' + W') \in L^1[a, b).$$

We will look at the four resulting products, but the first one, $UU' = 0$ (zero matrix) because the matrix

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}^2 = 0.$$

Looking next at UW', this involves apart from terms which tend to constants simply

$$Qa_{22} \int_x^b Qa_{12}(1 - a_{21})/\mu_0$$

or by (2.8)

$$[Qa_{22}V_2(1 - a_{21})/\mu_0] + Qa_{22} \int_x^b V_1\left(\frac{\mu_0(-a_{21}') - (1 - a_{21})\mu_0'}{\mu_0^2}\right).$$

Now

$$Qa_{22}V_2 = Q(r_{22} + \lambda s_{22})(R_2 + \lambda S_2),$$

and these cross products are absolutely integrable by hypothesis. The crucial terms in the integral expression are Qa_{22} multiplied by either

$$\int_x^b V_1 a_{21}', \quad \int_x^b V_1 a_{11}', \quad \text{or} \quad \int_x^b V_1 \Delta_1',$$

and these products likewise are assumed to lie in $L^1[a, b)$. Consequently $UW' \in L^1$. The proofs for the other products in (2.10) are similar and will be omitted. This now completes the proof that system (2.6) satisfies the hypotheses of Theorem A.

By Theorem A there is a fundamental matrix solution $\Gamma(x)$ of (2.6) such that

$$\Gamma(x)\Omega^{-1}(x) = \begin{pmatrix} 1 + \epsilon_1 & \epsilon_3 \\ \epsilon_4 & 1 + \epsilon_2 \end{pmatrix},$$

where $\epsilon_k(x) \to 0$ as $x \to b$. Tracing back through the substitutions, it follows that there is a fundamental matrix solution $Y(x, \lambda)$, an entire function of λ, of (1.1) such that

$$(2.11) \quad S^{-1}\begin{pmatrix} (\eta_1 F)^{-1} & 0 \\ 0 & \eta_1 F \end{pmatrix} Y\Omega^{-1} = \begin{pmatrix} 1 + \epsilon_1 & \epsilon_3 \\ \epsilon_4 & 1 + \epsilon_2 \end{pmatrix}$$

recalling that $\eta_2 = \eta_1^{-1}$. But $\eta_1 F = (-p_{21}/p_{11})^{1/4}$, and so (2.11) is equivalent to

$$(2.12)\quad Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}$$
$$= \begin{pmatrix} (-p_{21}/p_{11})^{1/4}E(1 - a_{21})(1 + \epsilon_5) & (-p_{21}/p_{11})^{1/4}E^{-1}(1 - a_{21})(1 + \epsilon_6) \\ (-p_{11}/p_{21})^{1/4}E(-\mu_0 + \Delta_1)(1 + \epsilon_7) & (-p_{11}/p_{21})^{1/4}E^{-1}(\mu_0 + \Delta_1)(1 + \epsilon_8) \end{pmatrix},$$

where the $\epsilon_k \to 0$. Therefore

$$(2.13)\quad (\alpha_1|Y_{11}|^2 + \alpha_2|Y_{21}|^2) + (\alpha_1|Y_{12}|^2 + \alpha_2|Y_{22}|^2)$$
$$= \alpha_1(-p_{21}/p_{11})^{1/2}E^2(1 - a_{21})^2(1 + \epsilon_5)^2$$
$$+ \alpha_2(-p_{11}/p_{21})^{1/2}E^2(-\mu_0 + \Delta_1)^2(1 + \epsilon_7)^2$$
$$+ \alpha_1(-p_{21}/p_{11})^{1/2}E^{-2}(1 - a_{21})^2(1 + \epsilon_6)^2$$
$$+ \alpha_2(-p_{11}/p_{21})^{1/2}E^{-2}(\mu_0 + \Delta_1)^2(1 + \epsilon_8)^2.$$

Since $1 - a_{21}$, $1 + \epsilon_k$, and $\mu_0 \pm \Delta_1$ all approach 1 as $x \to b$ (for all λ) then the right side of (2.13) for $\lambda = 0$ is bounded both above and below by constant multiples of the expression under the integral in (1.5). Consequently, the integral

$$\int_a^b \{ (\alpha_1|Y_{11}|^2 + \alpha_2|Y_{21}|^2) + (\alpha_1|Y_{12}|^2 + \alpha_2|Y_{22}|^2) \}$$

is infinite for $\lambda = 0$ if and only if the integral in (1.5) is infinite, and obviously these statements are equivalent to both columns of $Y(x, 0)$ belonging to $L^2_\alpha[a, b]$. Since the limit point-limit circle classification is independent of λ, (1.5) is necessary and sufficient for (1.1) to be in the limit point case.

To complete the proof of Theorem 1, there remains only to establish discreteness of the spectrum in the limit point case under the condition that

$$\int_a^b [-p_{11}(t)p_{21}(t)]^{1/2}dt = \infty.$$

The fundamental matrix $[\Theta, \Phi]$ defined above (1.6) is related to $Y(x)$ in (2.12) by

$$[\Theta, \Phi](x, \lambda) = Y(x, \lambda)C(\lambda),$$

where $C(\lambda)$ is a 2×2 matrix with entries which are entire functions, so that

$$\Theta_1 = Y_{11}C_{11} + Y_{12}C_{21}, \quad \Phi_1 = Y_{11}C_{12} + Y_{12}C_{22}.$$

Forming the quotient in (1.6) from the terms in (2.12), we see that the factor $(-p_{21}/p_{11})^{1/4}$ cancels out. Also $E(x, \lambda) \to \infty$ as $x \to b$, for each λ, since

$$\mu_0(x, \lambda) \to 1 \quad \text{and} \quad \int_a^b Q(t)dt = \infty.$$

Therefore we have $m(\lambda) = -C_{11}(\lambda)/C_{12}(\lambda)$, $\text{Im}(\lambda) \neq 0$ (in Section 4 below we give a proof that C_{12} does not vanish identically). But this formula provides the analytic continuation of $m(\lambda)$ onto the real axis save for zeros of $C_{12}(\lambda)$; i.e., $m(\lambda)$ must be meromorphic. By the result of **[6]** quoted after (1.6), the spectrum is discrete. This completes the proof of Theorem 1.

3. The diagonally dominant case. This section takes up the case of (1.1) in which no larger terms appear than $p(x)$. Specifically we suppose that $p(x) \neq 0$ and $p_k(x) = a_k p(x) + q_k(x)$, where the a_k are real constants, $a_1 a_2 \neq 1$,

$$(q_k(x)/p(x)) = q_{k1}(x) + q_{k2}(x) + q_{k3}(x)$$

is a resolution of $q_k p^{-1}$ such that q_{k1} is long range, pq_{k2} is oscillatory and pq_{k3} is short range (using the same definitions as in Section 2) and

$$(\alpha_k(x)/p(x)) = \alpha_{k1}(x) + \alpha_{k3}(x)$$

with α_{k1} long range and $p\alpha_{k3}$ short range. Additional assumptions will be made concerning the coefficients.

In its presently constituted form (1.1) may be written

$$\text{(3.1)} \qquad y' = p(x)\begin{pmatrix} 1 & a_2 + \left(\dfrac{\lambda\alpha_2 + q_2}{p}\right) \\ -a_1 - \left(\dfrac{\lambda\alpha_1 + q_2}{p}\right) & -1 \end{pmatrix} y,$$

$$a \leqq x < b \leqq \infty.$$

Introducing the notation $\Gamma_k(x) = \lambda a_{k1}(x) + q_{k1}(x)$ for the combined long range parts, (3.1) may be written

$$\begin{aligned} \text{(3.2)} \qquad y' &= p(x)\left\{\begin{pmatrix} 1 & a_2 + \Gamma_2 \\ -a_1 - \Gamma_1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & q_{22} \\ -q_{12} & 0 \end{pmatrix}\right. \\ &\quad \left. + \begin{pmatrix} 0 & q_{23} + \lambda\alpha_{23} \\ -q_{13} - \lambda\alpha_{13} & 0 \end{pmatrix}\right\} \\ &= \{p(x)D_1(x) + p(x)D_2(x) + p(x)D_3(x)\}y, \end{aligned}$$

with obvious notation.

We cannot proceed as in Section 2 with diagonalizing (3.2) unless $a_1 a_2 \neq 1$, for a degeneracy arises in the excluded case $a_1 a_2 = 1$ inasmuch as the lead term in (3.2) approaches a singular matrix as $x \to \infty$. In the following we state and prove an analogue of Theorem 1, for $a_1 a_2 < 1$. Then we give some partial results for the degenerate case. If $a_1 a_2 > 1$, then the asymptotic solutions of (3.1) are oscillatory, which typically leads to continuous spectrum.

Thus assume $a_1a_2 < 1$, let

$$\mu_0(x, \lambda) = [1 - a_1a_2 - (a_2\Gamma_1 + a_1\Gamma_2 + \Gamma_1\Gamma_2)]^{1/2},$$

let

$$\mu_\infty = (1 - a_1a_2)^{1/2} = \lim_{x\to b} \mu_0 \neq 0,$$

and define

$$S = \begin{pmatrix} 1 & -\dfrac{a_2 + \Gamma_2}{1 + \mu_0} \\ -\dfrac{\alpha_1 + \Gamma_1}{1 + \mu_0} & 1 \end{pmatrix};$$

we may assume without loss of generality that $1 + \mu_0 \neq 0$, $a \leqq x < b$. The square root defining μ_0 will be one with $0 \leqq \arg \mu_0 < \pi$. Putting $y(x) = S(x)z(x)$, we may calculate that

$$(3.3)\quad z' = \left\{p\begin{pmatrix} \mu_0 & 0 \\ 0 & -\mu_0 \end{pmatrix} + pS^{-1}D_2S + pS^{-1}D_3S - S^{-1}S'\right\}z,$$

and if $E(x, \lambda) = \exp \int_a^x \mu_0(t)p(t)dt$,

$$\Omega(x, \lambda) = \begin{pmatrix} E(x, \lambda) & 0 \\ 0 & 1/E(x, \lambda) \end{pmatrix},$$

$B(x) = pS^{-1}D_2S$, $C(x) = pS^{-1}D_3S - S^{-1}S'$, then (3.3) is of the form (1.7). Conditional integrability of $B(x)$ and absolute integrability of $C(x)$ follow from the long and short range hypotheses given at the beginning of this section. To apply Theorem A, we need that

$$G = -\Omega'\Omega^{-1}B_0 + B_0\Omega'\Omega^{-1} + B_0B$$

be absolutely integrable. Following the lead of Theorem 1, we could write down conditions on the a_{kj} and q_{kj} which would insure that $G \in L^1$, but it will be much simpler to take this as a direct hypothesis.

THEOREM 2. *Suppose for system* (3.1) *that the integrability conditions of Theorem* A *hold, where B, C, and G are as above. Then* (3.1) *is limit point at $x = b$ if*

$$(3.4)\quad \int_b^a \left\{E^2(x, 0)\left(\alpha_1(x) + \alpha_2(x)\frac{|a_1|}{|1 + \mu_\infty|}\right) + E^{-2}(x, 0)\left(\alpha_1(x)\frac{|a_2|}{|1 + \mu_\infty|} + \alpha_2(x)\right)\right\}dx = \infty,$$

and is limit circle at $x = b$ *if*

$$(3.5) \qquad \int_a^b (\alpha_1(x) + \alpha_2(x))(E^2(x, 0) + E^{-2}(x, 0))dx < \infty.$$

If the limit point case holds and

$$\int_a^b |p(x)|dx = \infty,$$

then T *has discrete spectrum.*

Proof. The proof that the entries $e_{ik} = \Omega_{ii}\Omega_{kk}^{-1}$ satisfy either (1.8) or (1.9) is similar to the corresponding one for Theorem 1 above (2.7). Then for (3.3), Theorem A asserts that there is a fundamental matrix $Z(x, \lambda)$ such that $Z\Omega^{-1} \to 1$, $x \to b$. The corresponding fundamental matrix $Y = SZ$ of (3.1) then satisfies

$$(3.6) \qquad \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}$$
$$= \begin{pmatrix} E(1 + \epsilon_1) & E^{-1} \dfrac{a_2}{1 + \mu_\infty}(-1 + \epsilon_2) \\ E^{-1}\left(\dfrac{a_1}{1 + \mu_\infty}\right)(-1 + \epsilon_3) & E^{-1}(1 + \epsilon_4) \end{pmatrix}$$

where $\epsilon_k(x, \lambda) \to 0$ as $x \to b$; λ is any complex number. Summing down the columns of Y against the weights $a_k(x)$ and taking $\lambda = 0$, the quantity

$$\alpha_1(x)|Y_{11}(x, 0)|^2 + \alpha_2(x)|Y_{21}(x, 0)|^2 + \alpha_1(x)|Y_{12}(x, 0)|^2 + \alpha_2(x)|Y_{22}(x, 0)|^2$$

is bounded below by a multiple of the integrand in (3.4), and above by a multiple of that of (3.5). Then both columns of $Y(x, 0)$ belong to $L^2_\alpha[a, b)$ if (3.5) holds, and at most one lies in $L^2_\alpha[a, b)$ in the case of (3.4). Since the limit point-limit circle classification is independent of λ, the first conclusion of the theorem follows.

As for discreteness of the spectrum, we begin by noting that either $E(x, \lambda) \to \infty$ or $E^{-1}(x, \lambda) \to \infty$, $x \to b$, for each λ. The first alternative holds if $p(x) > 0$, and the second if $p(x) < 0$; recall that $p(x) \neq 0$. Suppose $E(x, \lambda) \to \infty$. Writing $[\Theta, \Phi](x) = Y(x)C(\lambda)$ as in the proof of Theorem 1 and computing $m(\lambda)$ by (1.6), we have by (3.6)

$$-\frac{\Theta_1}{\Phi_1} = -\frac{Y_{11}C_{11} + Y_{12}C_{21}}{Y_{11}C_{12} + Y_{12}C_{21}} \to -\frac{C_{11}(\lambda)}{C_{12}(\lambda)}.$$

If $E^{-1}(x, \lambda) \to \infty$ instead, we use the second components,

$$-\frac{\Theta_2}{\Phi_2} = -\frac{Y_{21}C_{11} + Y_{22}C_{21}}{Y_{21}C_{21} + Y_{22}C_{22}} \to -\frac{C_{21}(\lambda)}{C_{22}(\lambda)}$$

by (3.6) again. In either case $m(\lambda)$ is meromorphic, being the quotient of two entire functions. See Section 4 for a proof that C_{12} and C_{22} do not vanish identically. This completes the proof of Theorem 2.

To illustrate the theorem, let us take $p(x) = x^n$, $p_1(x) = \pm x^k$, $p_2(x) = \pm x^m$, $\alpha_1(x) = x^\gamma$ and $\alpha_2(x) = x^\delta$. Then if $k = n$ we take $a_1 = \pm 1$ and $q_1 = 0$; if $k = n - 1$ we have $a_1 = 0$, $q_1 = p_1$ and $q_{11} = x^{-1}$, $q_{12} = q_{13} = 0$; if $k < n - 1$ then $a_1 = 0$ and we could have either $q_{11} = x^{k-n}$ or $q_{13} = x^{k-n}$. Similar decompositions can be worked out for $p_2(x)$, $\alpha_1(x)$, and $\alpha_2(x)$. If $n > -1$ then

$$|E(x, 0)| \geqq \epsilon \exp(x^{n+1}/(n + 1)) \quad \text{for some } \epsilon > 0,$$

and thus

$$\int^\infty E^2\alpha_1 = \infty$$

for any choice of γ; i.e., we are in the limit point case, by (3.4). When $n = -1$, $|E(x, 0)| \geqq \epsilon x$ and so the limit point prevails if, say, $\gamma \geqq -3$, but by (3.5) $\max\{\gamma, \sigma\} < -3$ implies the limit circle case. Finally, $n < -1$ implies that $E(x, 0)$ tends to a constant, and so we have the limit circle case if $\max\{\gamma, \sigma\} < -1$.

Theorem 2 can be extended to the point where $|p(x)|$ dominates instead the geometric mean of the terms $|p_k(x)|$; we will briefly sketch the idea and illustrate it with an example. Suppose $p_k(x) \neq 0$, let $\eta = |p_2/p_1|^{1/4}$ and change dependent variables in (3.1) by the formula

$$y(x) = \begin{pmatrix} \eta(x) & 0 \\ 0 & -\eta^{-1}(x) \end{pmatrix} z(x).$$

Then $z(x)$ satisfies the system

$$(3.7) \qquad z'(x) = \begin{pmatrix} p - (\eta'/\eta) & (\operatorname{sign} p_2)|p_1p_2|^{1/2} + \lambda\alpha_2\eta^{-2} \\ (-\operatorname{sign} p_1)|p_1p_2|^{1/2} - \lambda\alpha_1\eta^2 & -p + (\eta'/\eta) \end{pmatrix} z(x).$$

Now (3.1) is limit point if and only if (3.7) is limit point, relative to the weights $\alpha_1\eta^2$ and $\alpha_2\eta^{-2}$. Furthermore, the $m(\lambda)$ function for (3.1) agrees with the one for (3.7) because the common term $\eta(x)$ cancels when the quotient (1.6) is formed. Thus we may apply Theorem 2 to system (3.7) to obtain limit point-limit circle and discrete spectrum criteria for (3.1). As an example let $p(x) = x^n$, $p_1(x) = \pm x^k$, $p_2(x) = \pm x^m$, $\alpha_1(x) = x^\gamma$ and $\alpha_2(x) = x^\delta$ as above and assume $n > -1$, $\gamma < k$, $\sigma < m$ but $n > (m + k)/2$. Then (3.1) is limit point and the spectrum is discrete.

As a final example, consider the example (3.1) where

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$$p(x) = ax^{-1} + bx^{-2},$$
$$p_1(x) = p_2(x) = c + dx^{-1}(d \neq 0),$$
$$\alpha_1(x) = \alpha_2(x),$$
$$\int_0^1 \alpha_1(t)dt < \infty \quad \text{and } 0 < x \leqq 1.$$

Replacing x by $-x$ through the transformation $\xi(X) = y(-x)$, we obtain a system

$$(3.8) \qquad \xi'(x) = \begin{pmatrix} ax^{-1} - bx^{-2} & \lambda\tilde{\alpha}_2(x) - c + dx^{-1} \\ -\lambda\tilde{\alpha}_1(x) + c - dx^{-1} & -ax^{-1} + bx^{-2} \end{pmatrix}, \quad -1 \leqq x < 0,$$

which is regular at $x = -1$ and singular at the right endpoint $x = 0$ and where $\tilde{\alpha}_k(x) = -\alpha_k(-x)$. We assume that the α_k can be so defined. This system is a generalization of one used in [**1**] to model a relativistic electron in a Coulomb field with anomalous magnetic moment; in [**1**] the weights were set to unity. For (3.8) we see that

$$p_k(-x)/p(-x) = q_{k1}(x) = O(x), \quad x \to 0,$$

is purely long range and that $\alpha_k(-x)/p(-x) = \alpha_{k3}(x)$ is purely short range. In the notation of (3.2), for (3.8)

$$a_1 = a_2 = 0, \quad \Gamma_1^2(x) = \Gamma_2^2(x)$$
$$= (-x)^2[(d - cx)/(b - ax)]^2,$$
$$\mu_0(x) = [1 - \Gamma_1^2(x)]^{1/2} = 1 + \Gamma_1^2(x) \cdot O(1), \quad \text{and}$$
$$\int_{-1}^{x} (-as^{-1} + bs^{-2})(1 + \Gamma_1^2(s))ds$$
$$= -a\ln(-x) - bx^{-1} + C + O(1),$$

where C is constant. Therefore the E function for (3.8) satisfies

$$E(x, \lambda) = (-x)^{-a}\exp(-b/x)[K + o(1)],$$

where K is a constant. Replacing x by $-x$, it follows via (1.10) that (3.1) for this example (in which $x = 0$ is the singular point) has a fundamental matrix

$$(3.9) \qquad Y(x, \lambda) = \left\{ \begin{bmatrix} 1 & -\dfrac{\Gamma_1(-x)}{1 + \mu_0(-x)} \\ -\dfrac{\Gamma_1(-x)}{1 + \mu_0(-x)} & 1 \end{bmatrix} + o(1) \right\}$$
$$\times \begin{pmatrix} E(-x, \lambda) & 0 \\ 0 & E^{-1}(-x, \lambda) \end{pmatrix}$$

$$= \begin{pmatrix} 1 + o(1) & [-2 + o(1)]x^2(d/b)^2 \\ [-2 + o(1)]x^2(d/b)^2 & 1 + o(1) \end{pmatrix}$$

$$\times \begin{pmatrix} (K + o(1))x^{-a}e^{b/x} & 0 \\ 0 & (K + o(1))^{-1}x^a e^{-b/x} \end{pmatrix}$$

as $x \to 0$; compare equation (2.11). One of the exponential terms $e^{\pm b/x}$ will dominate (3.9), depending on the sign of b, as $x \to 0$. Clearly, the limit point case holds if, say

$$\int_0^1 \alpha(s)s^{-2a}e^{2b/s}ds = \infty$$

or

$$\int_0^1 \alpha(s)s^{2a}e^{-2b/s}ds = \infty,$$

and the spectrum is discrete in any event.

Finally, consider the degenerate case $a_1 a_2 = 1$. We will give only a brief discussion of this case, stopping short of the most general results. Thus let $a_1 = a$, $a_2 = a^{-1}$ and write (3.1) as

$$(3.10) \quad y'(x) = \left\{ p(x)\begin{pmatrix} 1 & a \\ -a^{-1} & -1 \end{pmatrix} + \begin{pmatrix} 0 & \lambda\alpha_2 + q_2 \\ -\lambda\alpha_1 - q_1 & 0 \end{pmatrix} \right\} y(x).$$

For definiteness we will suppose $p(x) > 0$, so that its antiderivative

$$P(x) = \int_a^x p(t)dt$$

is strictly increasing. Let us further assume that $p \in L^1[a, b)$, and also the strong conditions $P\alpha_k$, $Pq_k \in L^1[a, b)$.

THEOREM 3. *Under the above assumptions*, (3.10) *is limit point if and only if*

$$(3.11) \quad \int_a^b (\alpha_1(t) + \alpha_2(t))P^2(t)dt = \infty,$$

and the spectrum of T *is discrete in this case.*

Proof. First let

$$T = \begin{pmatrix} a & 0 \\ -1 & 1 \end{pmatrix}$$

and $y = Tz$, so that $z(x)$ is a solution of

$$z'(x) = \left(p\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + T^{-1}QT \right) z(x)$$

where $Q(x)$ stands for the second matrix in (3.10). Following a variable change suggested in [**2**, p. 91] let

$$z(x) = \begin{pmatrix} 1 & 1 \\ 0 & P^{-1}(x) \end{pmatrix} w(x),$$

and note that $w(x)$ satisfies

$$(3.12) \quad w'(x) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & P'/P \end{pmatrix} + M^{-1}QM \right\} w(x),$$

where

$$M = T\begin{pmatrix} 1 & 1 \\ 0 & p^{-1} \end{pmatrix}.$$

If

$$\Omega(x) = \begin{pmatrix} 1 & 0 \\ 0 & p(x) \end{pmatrix},$$

then (3.12) becomes

$$w'(x) = [\Omega'\Omega^{-1} + C]w,$$

where $C = M^{-1}QM$. The entries of $C(x)$ are all bounded above by constant multiples of either $P|\lambda\alpha_2 + q_2|$ or $P|\lambda\alpha_1 + q_1|$, and these lie in $L^1[a, b)$ by hypothesis. Hence so does $C(x)$, and this means that Theorem A may be brought to bear on (3.13). In fact, there is a fundamental matrix $W(x, \lambda)$ such that

$$W(x, \lambda) = [1 + O(\lambda)]\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix},$$

and tracing back to (3.10) we find a fundamental matrix $Y(x, \lambda)$ such that

$$Y(x, \lambda) = \begin{pmatrix} a(1 + \epsilon_1) & aP(1 + \epsilon_2) \\ -1 + \epsilon_3 & P(1 + \epsilon_4) \end{pmatrix}$$

where $\epsilon_k(X) \to 0$, $x \to b$. Thus

$$\int_a^b (\alpha_1|Y_{11}|^2 + \alpha_2|Y_{12}|^2) < \infty$$

because $\alpha_k \in L^1$, but as for the second column

$$\int_a^b (\alpha_1|Y_{12}|^2 + \alpha_2|Y_{22}|^2) = \infty$$

if and only if (3.11) holds. Hence (3.11) is necessary and sufficient for the limit point case.

Writing $[\Theta, \Phi] = YC(\lambda)$ as in the proof of Theorem 1, and looking at the quotient in (1.6),

$$-\frac{\Theta_1(x, \lambda)}{\Phi_1(x, \lambda)} = -\frac{a(1 + \epsilon_1)C_{11} + aP(1 + \epsilon_2)C_{21}}{a(1 + \epsilon_1)C_{12} + aP(1 + \epsilon_2)C_{22}}$$

and the condition $P(x) \to \infty$ (which follows from $p \notin L^1[a, b)$) implies that $m(\lambda) = -C_{21}(\lambda)/C_{22}(\lambda)$. This represents $m(\lambda)$ as a quotient of entire functions, meaning that $m(\lambda)$ is meromorphic and the spectrum is discrete. See Section 4 for the proof that $C_{22}(\lambda)$ does not vanish identically. This completes the proof.

4. Remarks. We comment now on various ways in which some of the foregoing results may be extended.

(1) The assumption that the coefficients in (1.1) be real is to assure that the operator T be selfadjoint. However, our asymptotic form of solutions persists, with some qualifications, even in the presence of complex valued coefficients. If the p_{k1} are real, then the estimate (2.12) is still valid with complex coefficients. The factors $\Omega_{ii}\Omega_{kk}^{-1}$ are essentially increasing or essentially decreasing because $\mu_0 \to 1$, even though it is complex valued. The situation is different with the corresponding μ_0 of Section 3 for

$$\mu_0 \to (1 - a_1a_2)^{1/2}$$

which could be purely imaginary. It would be possible even with real $p(x)$ for

$$\operatorname{Re} \int_a^x \mu_0 p$$

to oscillate in such a way that neither (1.8) or (1.9) could hold.

The conditions

$$\int_a^b Q = \infty \quad \text{and} \quad \int_b^a |p| = \infty$$

in Theorems 1 and 2, respectively, are needed to guarantee that either $E \to \infty$ or $E^{-1} \to \infty$; these conditions are used to determine $m(\lambda)$ from (1.6). But even if neither of E, $E^{-1} \to \infty$ holds, the asymptotic estimates in the theorems are still valid.

(2) Clearly the conditions $p_{11} > 0$ and $p_{21} < 0$ in Theorem 1 can be reversed without altering its conclusions. We have said that continuous spectrum is to be expected in the event of same signs, $p_{11}p_{21} > 0$, but there are exceptions. If in the proof of Theorem 1 we assume $p_{11} > 0$, $p_{21} > 0$ and replace $-p_{21}$ by p_{21}, then we find that the same type of asymptotic estimate (2.12) holds. As the proof shows, suitably small α_k then leads to the limit circle case, and hence a discrete spectrum.

(3) All the above results may be placed in a two singular endpoint context. If (1.1) is singular and of limit point type at each end of (a, b), then a selfadjoint operator T_0 may be introduced as in (1.4) except that no boundary condition at $x = a$ is needed [**16**, **21**]. Letting $m_a(\lambda)$ and $m_b(\lambda)$ denote the Titchmarsh-Weyl coefficients for the two endpoints, the corresponding m-function for T_0 is [**5**, **7**]

$$(4.1) \qquad M_0(\lambda) = (m_a - m_b)^{-1}\begin{pmatrix} 1 & (m_a + m_b)/2 \\ (m_a + m_b)/2 & m_a m_b \end{pmatrix}.$$

This matrix Titchmarsh-Weyl coefficient bears the same relation to T_0 as $m(\lambda)$ does to T [7]. Obviously M_0 is meromorphic if m_a and m_b are, and so if we impose discrete spectrum criteria at both endpoints of (a, b) then the spectrum of T_0 will be discrete. If one of the endpoints, say $x = b$, gives rise to a λ-interval I of continuous spectrum of T, then $m_b(\lambda)$ is not meromorphic; in fact it will have nonreal limits at each point $\lambda_0 \in I$. Now $m_a(\lambda_0)$ is either real-valued or infinite, and if infinite then the pole at λ_0 cancels in (4.1). Either way, $M_0(\lambda)$ persists in having nonreal limits over I, so that T_0 has continuous spectrum there; see **[8]** for a fuller discussion of this phenomenon. In summary, the discrete spectrum criteria given above may be viewed as conditions at an endpoint which insures that singular behavior of (1.1) at that endpoint does not contribute to the continuous spectrum of T_0.

(4) It is possible to weaken the assumptions of **[3]**, and thus Theorem A, by iterating the method through integration by parts of an oscillatory term; see **[3]**. What we can do is assume only that $G(x)$ in Theorem A is conditionally integrable, but let

$$G_0(x) = \int_x^\infty G(t)dt$$

and then assume that the matrix function

$$H = -\Omega'\Omega^{-1}G_0 + G_0\Omega'\Omega^{-1} + G_0G$$

belongs to $L^1[a, b)$. Then the conclusion of Theorem A continues to hold. In certain examples, this allows one to relax growth conditions on the oscillatory components of α_k/p_{k1} and p_{k2}/p_{k1}. In the example cited after Theorem 1, if we suppose that $k = m$, $\gamma = \sigma$ and $\gamma < k - 1$, then we can replace the requirement $k + \alpha < 1$ by $k + \alpha < 2$.

(5) Now we turn to the proof that the matrix $C(\lambda)$, in the relation

$$[\Theta, \Phi](x, \lambda) = Y(x, \lambda)C(\lambda)$$

in all three theorems above, has entries C_{12} and C_{22} not identically vanishing. Since $[\Theta, \Phi]$ and Y are fundamental matrices, the determinant $C_{11}C_{22} - C_{21}C_{12}$ is never zero. Suppose, by way of contradiction, that $C_{12}(\lambda)$ is identically zero. Then neither $C_{11}(\lambda)$ nor $C_{22}(\lambda)$ can be identically zero. We have

$$(4.2) \qquad \begin{aligned} \Theta_1 &= Y_{11}C_{11} + Y_{12}C_{21}, & \Phi_1 &= Y_{11}C_{12} + Y_{12}C_{22} \\ \Theta_2 &= Y_{21}C_{11} + Y_{22}C_{21}, & \Phi_2 &= Y_{21}C_{12} + Y_{22}C_{22} \end{aligned}$$

and in all the cases considered we had either $Y_{11} \to \infty$ and $Y_{22} \to \infty$. Also the limit point case prevailed in all relevant cases. With β fixed, we consider $\tilde{\beta} = \beta - (\pi/2)$ and construct the functions Θ and Φ with initial data governed by $\tilde{\beta}$. Calling these $\tilde{\Theta}$ and $\tilde{\Phi}$, clearly we have $\tilde{\Theta} = \Phi$ and $\tilde{\Phi} = \Theta$. If $\tilde{m}(\lambda)$ is the m-coefficient corresponding to $\tilde{\beta}$, then $\tilde{m}(\lambda) = 1/m(\lambda)$; we know that $m(\lambda)$ is zero-free for $\operatorname{Im} \lambda \neq 0$, and in fact

$$\operatorname{Im} m(\lambda) \cdot \operatorname{Im} \lambda > 0 \quad [\mathbf{5}].$$

If it is Y_{11} that becomes unbounded we compute $\tilde{m}(\lambda)$ by (1.6) and (4.2) to be

$$\tilde{m}(\lambda) = -C_{12}(\lambda)/C_{11}(\lambda)$$

at any λ for which $C_{11} \neq 0$. If instead Y_{12} and Y_{22} are unbounded, we use the second components in (1.6) and (4.2) to compute $m(\lambda)$, not $\tilde{m}(\lambda)$, to be

$$m(\lambda) = -C_{21}(\lambda)/C_{22}(\lambda).$$

Since m and $\tilde{m}$ are analytic and zero-free for $\operatorname{Im}(\lambda) \neq 0$, it cannot be that any entry of $C(\lambda)$ can vanish identically, contradicting $C_{12} \equiv 0$. Similarly $C_{22} \not\equiv 0$.

(6) Our theorems may be viewed as limit point-limit circle criteria, and in fact Theorem 2 includes the result of Kalf in [**11**]. Setting $\alpha_2 = 0$ and $p = 0$ makes (1.1) equivalent to

$$-((1/p_2)u')' - p_1 u = \lambda \alpha_1 u,$$

and since the asymptotic formulas of Theorem 1 continue to hold in this case, we obtain limit point-limit circle for Sturm-Liouville equations. For example, (1.5) is analogous to a condition of T. T. Read [**14**].

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University of Tennessee,
Knoxville, Tennessee;
Virginia Tech,
Blacksburg, Virginia