J. Austral. Math. Soc. (Series A) 30 (1981), 366–368

ENTIRE SOLUTIONS OF THE DIFFERENTIAL EQUATION $\Delta u = f(u)$

WOLFGANG WALTER

(Received 25 April 1980)

Communicated by R. Vyborny

Abstract

The existence of spherically symmetric solutions of the equation $\Delta u = f(u)$ is proved for a large class of functions $f(z)$. Among others, functions satisfying an inequality $zf(z) < 0$ for $|z| > A$, and in particular the function $f(z) = -\sinh z$, belong to this class.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 35 J 15, 35 J 60.

0. Introduction

I learned from Dr. Norbert Steinmetz that the following problem was posed at the Oberwolfach conference on complex variables (17–23 February 1980): Are there solutions of $\Delta u + \sinh u = 0$ existing in the whole plane (other than functions of one variable $u(x, y) = v(x)$, $v'' + \sinh v = 0$)? The question has bearings on the minimal surface equation. It is of a quite different nature than the corresponding question regarding equations such as $\Delta u = e^u$, which have been investigated by many authors; see Walter and Rhee (1979) and the literature quoted there. The answer is yes. In fact, we will show that for a large class of functions $f(z)$, which includes all functions for which $zf(z) < 0$, there exist spherically symmetric entire solutions of the equation $\Delta u = f(u)$.

1. Some preliminary remarks

Let $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$ and $\mathbf{R}_+ = [0, \infty)$. A spherically symmetric function $u(|x|)$ is in $C^2(\mathbf{R}^n)$ if and only if $u(r)$ is in $C^2(\mathbf{R}_+)$

and $u'(0) = 0$. The problem of finding a spherically symmetric solution of

$$\Delta u = f(u) \quad \text{in } \mathbf{R}^n, u(0) = a \tag{1}$$

of class $C^2(\mathbf{R}^n)$ is therefore equivalent to the problem of finding a solution $u \in C^2(\mathbf{R}_+)$ of

$$u'' + \frac{n-1}{r} u' = f(u) \quad \text{for } 0 < r < \infty, u(0) = a, \quad u'(0) = 0. \tag{2}$$

Another equivalent formulation is given by the integral equation

$$u(r) = a + \int_0^r f(u(s)) s K\left(\frac{s}{r}\right) ds \quad \text{in } \mathbf{R}_+, \tag{3}$$

where

$$K(t) = \begin{cases} -\log t & \text{for } n = 2, \\ \dfrac{1}{n-2}(1 - t^{n-2}) & \text{for } n > 2 \end{cases}$$

is positive in (0, 1).

2. The main result

THEOREM. *Assume that* $f\colon \mathbf{R} \to \mathbf{R}$ *is continuous and that*

$$F(z) := \int_0^z f(t)\, dt$$

is bounded above, say, $F(z) \leqslant C$ *for* $z \in \mathbf{R}$. *Then, for any given* $a \in \mathbf{R}$, *there exists a spherically symmetric entire solution of* (1) *of class* $C^2(\mathbf{R}^n)$.

PROOF. It is well known that there exist "local" solutions of (2) or (3) existing in some interval $0 \leqslant r \leqslant R$ (this may be proved by replacing f by a bounded function, using a cut-off procedure, and employing Schauder's fixed point theorem). Now consider the expression

$$E(r) = u'^2 - 2F(u),$$

where u is a local solution. We have

$$E'(r) = 2u'u'' - 2u'f(u) = -\frac{2(n-1)}{r} u'^2 \leqslant 0.$$

Hence, E is decreasing, $E(r) \leqslant E(0) = -2F(a)$, or

$$u'^2(r) \leqslant 2F(u(r)) - 2F(a) \leqslant 2(C + |F(a)|).$$

https://doi.org/10.1017/S1446788700017249 Published online by Cambridge University Press

This inequality shows that u' remains bounded. By Peano's existence theorem the solution u can be continued to the right indefinitely.

COROLLARIES (a). *If* $F(z) \to -\infty$ *as* $|z| \to \infty$, *then every solution of* (2) *is bounded.*

(b) *If f is locally Lipschitz continuous, then* (2) *has exactly one entire solution.*

(c) *The theorem applies in particular, if there exists a constant A such that* $zf(z) \leqslant 0$ *for* $|z| \geqslant A$.

PROOF. (a) follows easily from the boundedness of $E(r)$. In the case (b) one may apply the contraction principle to the integral equation (3). Finally, (c) is evident.

References

W. Walter and H. Rhee (1979), 'Entire solutions of $\Delta^p u = f(r, u)$', *Proc. Roy. Soc. Edinburgh Sect. A* **82**, 189–192.

Mathematisches Institut I
der Universität Karlsruhe
Englerstrasse 2
7500 Karlsruhe
Deutschland

https://doi.org/10.1017/S1446788700017249 Published online by Cambridge University Press