Topological Representation of Sheaf Cohomology of Sites

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Abstract. For a site $\mathcal{S}$ (with enough points), we construct a topological space $X_{(\mathcal{S})}$ and a full embedding φ^* of the category of sheaves on $\mathcal{S}$ into those on $X_{(\mathcal{S})}$ (i.e., a morphism of toposes $\varphi: \mathrm{Sh}(X_{(\mathcal{S})}) \to \mathrm{Sh}(\mathcal{S})$). The embedding will be shown to induce a full embedding of derived categories, hence isomorphisms $\mathrm{H}^*(\mathcal{S}, A) = \mathrm{H}^*(X_{(\mathcal{S})}, \varphi^* A)$ for any Abelian sheaf A on $\mathcal{S}$. As a particular case, this will give for any scheme Y a topological space $X_{(Y)}$ and a functorial isomorphism between the étale cohomology $\mathrm{H}^*(Y_{\text{ét}}, A)$ and the ordinary sheaf cohomology $\mathrm{H}^*(X_{(Y)}, \varphi^* A)$, for any sheaf A for the étale topology on Y.

1. Introduction and Statement of the Theorem

Many cohomology groups arising in geometry and topology are (or can be) defined as the cohomology groups of some topos; that is, as the sheaf cohomology groups of some site. This applies directly to étale and other cohomologies of schemes [1, 10], but also to many others such as Galois cohomology [12] and cyclic cohomology [2].

The purpose of this paper is to give a general construction which shows that all these cohomology groups are isomorphic to the ordinary sheaf cohomology groups of a topological space associated to the site or the topos. When the site is a group G (with associated topos of G-sets), our construction gives a model for the classifying space BG. In general, our result can be interpreted as the construction of a 'classifying space' for any site (satisfying the following technical condition).

Our construction applies to topoi *with enough points*. We recall that a point p of a topos $\mathcal{T}$ is a topos morphism $p: \mathcal{S} \to \mathcal{T}$, from the topos $\mathcal{S}$ of sets into $\mathcal{T}$. Such a morphism can equivalently be described as a functor $p^*: \mathcal{T} \to \mathcal{S}$ which preserves colimits and finite limits, or as a morphism of sites $\mathbb{F}: \mathbb{C} \to \mathcal{S}$, where $\mathbb{C}$ is any site of definition for $\mathcal{T}$. The topos $\mathcal{T}$ is said to have enough points if for any sequence

$A \rightarrow B \rightarrow C$ of Abelian groups in $\mathcal{T}$ (i.e., sheaves of Abelian groups on $\mathbb{C}$), the sequence is exact whenever for each point p of $\mathcal{T}$ the associated sequence $p^*A \rightarrow p^*B \rightarrow p^*C$ is an exact sequence of Abelian groups. We hasten to point out that virtually all topoi arising in geometric practice have enough points. This applies, for example, to the presheaf topos $\hat{\mathbb{C}}$ on an arbitrary small category $\mathbb{C}$, and to the étale topos associated to a scheme. In fact, any 'coherent' topos has enough points (Deligne, Appendix to Exposé VI in [1]).

For any topological space X, the category $\mathrm{Sh}(X)$ of sheaves on X is a topos (with enough points), whose cohomology groups are the ordinary sheaf cohomology groups of X [3, 6]. We will prove the following result

THEOREM. *Let $\mathcal{T}$ be a topos with enough points. There exists a topological space $X_{\mathcal{T}}$ and a topos morphism $\varphi \colon \mathrm{Sh}(X_{\mathcal{T}}) \rightarrow \mathcal{T}$, such that*

 (i) *φ^* is a full and faithful embedding of $\mathcal{T}$ into $\mathrm{Sh}(X_{\mathcal{T}})$;*
(ii) *for any Abelian group A in $\mathcal{T}$, the morphism φ induces isomorphisms*

$$\mathrm{H}^n(\mathcal{T}, A) \xrightarrow{\widetilde{=}} \mathrm{H}^n(X_{\mathcal{T}}, \varphi^*(A)), \quad n \geqslant 0.$$

Here $\mathrm{H}^*(X_{\mathcal{T}}, \varphi^*(A))$ denotes the sheaf cohomology of the space $X_{\mathcal{T}}$ with the sheaf $\varphi^*(A)$ as coefficients. We will give an explicit construction of this space $X_{\mathcal{T}}$ from $\mathcal{T}$, which depends not only on $\mathcal{T}$, but also on the choice of a site for $\mathcal{T}$. For this reason, the construction $\mathcal{T} \mapsto X_{\mathcal{T}}$ is only functorial in $\mathcal{T}$ in a weak sense (see Remark 2.4 below).

Note that, since the topos $\mathrm{Sh}(X_{\mathcal{T}})$ always has enough points, the (mild) assumption that $\mathcal{T}$ has enough points is a necessary one, being implied by part (i) of the theorem. For part (ii) of the theorem, we will actually prove that the derived functors $\mathrm{R}^q\varphi_*$ of the direct image functor $\varphi_* \colon \mathrm{Sh}(X_{\mathcal{T}}) \rightarrow \mathcal{T}$ have the property that

$$\mathrm{R}^q\varphi_*(\varphi^*A) = \begin{cases} A, & q = 0, \\ 0, & q > 0, \end{cases}$$

for any Abelian group A in $\mathcal{T}$. This property states that $\varphi \colon \mathrm{Sh}(X_{\mathcal{T}}) \rightarrow \mathcal{T}$ is an *acyclic morphism*. It implies in particular that φ^* induces a full and faithful embedding of derived categories $D^+(\mathcal{T}) \hookrightarrow D^+(X_{\mathcal{T}})$. The same argument applies to ringed topoi: if $\mathcal{O}_{\mathcal{T}}$ is any ring in $\mathcal{T}$ and $D^+(\mathcal{T}, \mathcal{O}_{\mathcal{T}})$ is the associated derived category of complexes of $\mathcal{O}_{\mathcal{T}}$-modules [1], then φ^* induces a full and faithful embedding $D^+(\mathcal{T}, \mathcal{O}_{\mathcal{T}}) \hookrightarrow D^+(X_{\mathcal{T}}, \varphi^*(\mathcal{O}_{\mathcal{T}}))$.

The theorem, as well as the construction of the space $X_{\mathcal{T}}$, have been inspired by [8], where it is proved that any topos (not necessarily with enough points) is cohomologically equivalent to the topos of sheaves on a 'locale'. (A locale is an abstract notion of 'topological space without points'.) However, our theorem is not a consequence of this result of [8]. Furthermore, our proof is different. The

proof in [8] made essential use of the 'internal logic' of a topos and its behaviour under change-of-base. These methods cannot be applied to the topological space $X_{\mathcal{T}}$ constructed here.

2. Construction of the Space $X_{\mathcal{T}}$ and of the Map φ

In this section, $\mathcal{T}$ denotes a fixed topos with enough points. Recall [1, 9] that the latter means that the functors $p^*\colon \mathcal{T} \to \mathscr{S}$, for all points $p\colon \mathscr{S} \to \mathcal{T}$, are jointly conservative. Although the collection of all such points p is in general a proper class rather than a set, there will always be a *set $\mathscr{P}$* of points p for which the functors p^*, for $p \in \mathscr{P}$, are already jointly conservative. We will fix such a set $\mathscr{P}$, and henceforth refer to points in this set as *small* points of $\mathcal{T}$. For a point p of $\mathcal{T}$ and an object (sheaf) E in $\mathcal{T}$, we will also use the common notation E_p for the set $p^*(E)$, and refer to E_p as 'the stalk of E at p'.

Next, we fix a sheaf G in $\mathcal{T}$ so that the collection of all subsheaves $C \subset G^n$, $n \geqslant 0$, generates $\mathcal{T}$. For example, G can be the disjoint sum (coproduct) of all the objects in some site of definition for $\mathcal{T}$. But often, there is a smaller and much more natural choice for G: the topos $\mathcal{T}$ will generally contain some 'universal' structure U of a certain kind. For example, in the case of the étale topos, U is the universal strictly local ring [5]. More generally, if $\mathcal{T}$ is a classifying topos, U is the universal model for the theory classified by $\mathcal{T}$ (see [9], Chapter VIII). This object U will have the property required for G, namely that the subsheaves of finite products $U \times \cdots \times U$ generate $\mathcal{T}$.

Finally, we fix an infinite set I, which is big enough so that it surjects onto all the stalks G_p, for all small points p of $\mathcal{T}$; in other words, $\mathrm{card}(G_p) \leqslant \mathrm{card}(I)$.

The construction of the space $X_{\mathcal{T}}$ will depend on these choices, of the set $\mathscr{P}$ of points, of the sheaf G, and of the set I. (We come back to this dependence in Remark 2.4 below.)

The points of the space $X = X_{\mathcal{T}}$ are now defined to be equivalence classes of pairs (p, α), where p is a small point of $\mathcal{T}$ and α is a function from a subset of I to G_p, $I \supset \mathrm{dom}(\alpha) \xrightarrow{\alpha} G_p$, with the property that $\alpha^{-1}(g)$ is infinite, for each $g \in G_p$. Two such pairs (p, α) and (q, β) are *equivalent* (i.e., define the same point $x \in X$), if there exists a natural isomorphism of functors $\theta\colon p^* \to q^*$ so that $\beta = \theta_G \circ \alpha$. We will often write $x = (p, \alpha)$ for a point $x \in X$, and not distinguish explicitly between such pairs (p, α) and their equivalence classes.

The topology on this set X of points is defined as follows: For any $n \geqslant 0$ and any subsheaf $C \subset G^n$, and any $i_1, \ldots, i_n \in I$, the set

$$U_{i_1, \ldots, i_n, C} = \{(p, \alpha) \mid i_1, \ldots, i_n \in \mathrm{dom}(\alpha) \text{ and } (\alpha(i_1), \ldots, \alpha(i_n)) \in C_p\} \quad (1)$$

is to be a basic open set. Note that this set is well-defined on equivalence classes, i.e., $(p, \alpha) \in U_{i_1, \ldots, i_n, C}$ iff $(q, \beta) \in U_{i_1, \ldots, i_n, C}$. In the sequel, we will usually write i for $i_1, \ldots, i_n$ and $\alpha(i)$ for $(\alpha(i_1), \ldots, \alpha(i_n))$, so that

$$U_{i, C} = \{(p, \alpha) \mid i \in \mathrm{dom}(\alpha) \text{ and } \alpha(i) \in C_p\}. \quad (2)$$

We remark that, by changing C, we can always assume that the sequence $i = (i_1, \ldots, i_n)$ does not contain repetitions. For example, $U_{i,i,C}$ for $C \subset G^2$ is equal to $U_{i,C'}$ for C' the pullback of C along the diagonal $\Delta: G \to G^2$. In the sequel we will often tacitly assume that a sequence i does not contain repetitions.

LEMMA 2.1. *The sets $U_{i,C}$ form a basis for a topology on X.*

Proof. This is clear from the formula $U_{i,C} \cap U_{j,D} = U_{i,j,C \times D}$, for any $C \subset G^n$, $D \subset G^m$, $i = (i_1, \ldots, i_n)$, $j = (j_1, \ldots, j_m)$, and i, j the concatenation of these two sequences. $\square$

It can be shown that the space X thus defined is always a sober topological space ([1], IV.4.2.1), although it is not a Hausdorff space.

Next, we describe the morphism $\varphi: \mathrm{Sh}(X) \to \mathcal{T}$ occurring in the statement of the theorem. Recall that such a morphism of topoi is given by an inverse image functor $\varphi^*: \mathcal{T} \to \mathrm{Sh}(X)$ and a direct image functor $\varphi_*: \mathrm{Sh}(X) \to \mathcal{T}$, right adjoint to φ^*. The functor φ^* preserves colimits and finite limits, and these properties imply that φ^* has a right adjoint, unique up to isomorphism. So, to define φ, it suffices to define such a functor $\varphi^*: \mathcal{T} \to \mathrm{Sh}(X)$. For any sheaf E in $\mathcal{T}$, consider the set $\varphi^*(E) = \{(p, \alpha, e) \mid (p, \alpha) \in X, e \in E_p\}$, with obvious projection $\pi: \varphi^*(E) \to X$. (Again, being more precise we should speak about equivalence classes of such triples, where (p, α, e) is equivalent to (q, β, g) if there exists a natural isomorphism of functors $\theta: p^* \to q^*$ so that $\beta = \theta_G \circ \alpha$ and $\theta_E(e) = g$.) The set $\varphi^*(E)$ carries a natural topology, with basic open sets $V_{i,C,f} = \{(p, \alpha, e) \mid (p, \alpha) \in U_{i,C} \text{ and } e = f(\alpha(i))\}$, for any $i = (i_1, \ldots, i_n)$ and $C \subset G^n$ as above, and any morphism $f: C \to E$ in $\mathcal{T}$.

LEMMA 2.2. *These sets $V_{i,C,f}$ form the basis for a topology on $\varphi^*(E)$, which makes the projection $\pi: \varphi^*(E) \to X$ into a local homeomorphism.*

Proof. Consider two such basic open sets $V_{i,C,f}$ and $V_{j,D,g}$. Let $h: C \times_E D \to E$ be the map from the pullback, $h = f \circ \pi_1 = g \circ \pi_2$. Then $V_{i,C,f} \cap V_{j,D,g} = V_{i,j,C \times_E D,h}$.

Thus the sets $V_{i,C,f}$ form a basis for a well-defined topology on $\varphi^*(E)$. Furthermore, the sections $\sigma: U_{i,C} \to V_{i,C,f}$, and $\sigma(p, \alpha) = f_p(\alpha(i))$ are well-defined on equivalence classes and show that the projection $\pi: \varphi^*(E) \to X$ restricts to a homeomorphism $V_{i,C,f} \to U_{i,C}$. $\square$

Thus $\pi: \varphi^*(E) \to X$ is a sheaf on X. Note that for the stalk of this sheaf at a point (p, α) of X we have

$$\varphi^*(E)_{(p,\alpha)} = E_p. \tag{3}$$

PROPOSITION 2.3. *The construction $E \mapsto \varphi^*(E)$ defines the inverse image functor of a topos morphism $\varphi: \mathrm{Sh}(X) \to \mathcal{T}$.*

Proof. We observe first that the construction is functorial in E. If $h: E \to F$ is a morphism in $\mathcal{T}$, the induced map

$$\varphi^*(h): \varphi^*(E) \to \varphi^*(F), \qquad (p, \alpha, e) \mapsto (p, \alpha, h_p(e))$$

is continuous for the topologies just defined. To see this, take any point (p, α, e) of $\varphi^*(E)$, and let $V_{i,C,f}$ be a basic open neighbourhood of $(p, \alpha, h_p(e))$ in $\varphi^*(F)$, where $f: C \to F$. Since the subsheaves of G^n generate $\mathcal{T}$, it follows that there is a $B \subset G^m$ and a map $u: B \to C \times_F E$ so that, for $c = \alpha(i)$, there exists a point $b \in B_p$ with $u_p(b) = (c, e) \in (C \times_F E)_p$. Choose $j = (j_1, \ldots, j_m)$ with $j_k \in I$, so that $b = \alpha(j) = (\alpha(j_1), \ldots, \alpha(j_m))$. Let $v = \pi_1 \circ u: B \to C$, and let $D = \mathrm{graph}(v) \subset B \times C \subset G^m \times G^n$. Then $W = V_{j,i,D,\pi_2 \circ u}$ is a basic open set in $\varphi^*(E)$, such that $(p, \alpha, e) \in W$ and $\varphi^*(h)$ maps W into $V_{i,C,f}$.

This shows that φ^* is a functor. It remains to verify that φ^* preserves colimits and finite limits. But it suffices to show that this holds at the level of the stalks, where it is obvious from the identity (3). $\qquad \square$

Remark 2.4. The construction of $X = X_{\mathcal{T}}$ depends on $\mathcal{P}$, G and I, in a functorial way. Clearly, for a larger set $\mathcal{P}' \supset \mathcal{P}$ of small points, there is a map $X(\mathcal{P}) \to X(\mathcal{P}')$ over $\mathcal{T}$. Similarly, it will be clear from Section 3 that a surjection $s: J \twoheadrightarrow I$ induces a map $s^*: X(I) \to X(J)$, while if $G' \supset G$ is a larger choice of an object so that the subsheaves of its finite powers generate, there is a restriction map $X(G') \to X(G)$. It is a consequence of our theorem that all these comparison maps induce isomorphisms in cohomology for Abelian coefficients which come from $\mathcal{T}$, so that the dependence of X on $\mathcal{P}$, G and I is inessential in this sense.

If $f: \mathcal{T}_1 \to \mathcal{T}_2$ is a topos morphism, we can fix first the parameters $\mathcal{P}_1$ and I_1 for $\mathcal{T}_1$ and G_2 for $\mathcal{T}_2$, and then choose $\mathcal{P}_2$ large enough to include all composites $f \circ p$ for $p \in \mathcal{P}_1$, and $G_1 \supset f^*(G_2)$, and finally I_2 so large that there exists a surjection $I_2 \to I_1$. Then the constructed spaces X_1 and X_2 fit into a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Sh}(X_1) & \xrightarrow{\ \tilde{f}\ } & \mathrm{Sh}(X_2) \\
{\scriptstyle \varphi_1}\downarrow & & \downarrow{\scriptstyle \varphi_2} \\
\mathcal{T}_1 & \xrightarrow[\ f\ ]{} & \mathcal{T}_2.
\end{array}
$$

3. Enumeration Spaces

The fibres of the morphism $\varphi: \mathrm{Sh}(X_{\mathcal{T}}) \to \mathcal{T}$ will turn out to be (approximated by) certain acyclic topological spaces, which we will discuss separately in this section.

Let I be a fixed infinite index set. For any set S, with cardinality $\mathrm{card}(S) \leqslant \mathrm{card}(I)$, the enumeration space $\mathrm{En}(S)$ (or $\mathrm{En}_I(S)$) has as points all functions

$\alpha\colon D \to S$ defined on some subset $D = \mathrm{dom}(\alpha) \subset I$, and with the property that $\alpha^{-1}(s) \subset D$ is infinite for each $s \in S$. The basic open sets of $\mathrm{En}(S)$ are the sets of the form $V_{i_1,\ldots,i_n,s_1,\ldots,s_n} = \{\alpha \mid \alpha(i_k) = s_k, \text{ for } k = 1,\ldots,n\}$, for any $i_1,\ldots,i_n \in I$ and $s_1,\ldots,s_n \in S$. It will be convenient to use a shorter notation, and write u for the finite partial function from I to S defined by $u(i_k) = s_k$ ($k = 1,\ldots,n$), and write $V_u = \{\alpha \in \mathrm{En}(S) \mid u \subset \alpha\}$ for the same basic open set. Note that for $n = 0$ (i.e., $u = \emptyset$) the entire space $\mathrm{En}(S)$ occurs among these basic open sets.

NOTATION 3.1. These finite partial functions u induce various continuous operations on $\mathrm{En}(S)$, which will be used in the sequel. For $\alpha \in \mathrm{En}(S)$, denote by $\alpha - u$ the restriction of α to $\mathrm{dom}(\alpha) - \mathrm{dom}(u)$. Furthermore, denote by $\alpha \cup u$ the union of these partial functions, defined only in case $\mathrm{dom}(\alpha) \cap \mathrm{dom}(u) = \emptyset$. Finally, we will use the notation (u/α) for $(\alpha - u) \cup u$, which is the function obtained by 'writing u over α'.

Remark 3.2. In relation to Remark 2.4, we note that if $S' \subset S$ is a subset, the restriction of $\alpha\colon D \to S$ to $\{i \in D \mid \alpha(i) \in S'\}$ defines a continuous map $\mathrm{res}\colon \mathrm{En}(S) \to \mathrm{En}(S')$. Furthermore, any surjection $t\colon J \to I$ defines by composition an obvious continuous map $t^*\colon \mathrm{En}_I(S) \to \mathrm{En}_J(S)$.

LEMMA 3.3. *Each enumeration space* $\mathrm{En}(S)$ *is connected and locally connected; in fact, each basic open set* V_u *is connected.*

Proof. Fix an open set V_u, and let $V_u = O_1 \cup O_2$ be a cover by two nonempty open sets. Choose points $\alpha_1 \in O_1$ and $\alpha_2 \in O_2$, and basic open sets V_{u_1} and V_{u_2} with $\alpha_1 \in V_{u_1} \subset O_1$ and $\alpha_2 \in V_{u_2} \subset O_2$. These are given by finite partial functions u_1, u_2 with $u \subset u_1 \subset \alpha_1$ and $u \subset u_2 \subset \alpha_2$. Let $\beta = u_2/\alpha_1 \in O_2$ and $\gamma = (\alpha_1 - u_2) \cup u$. Thus, $\gamma \subset \beta$ and $\gamma \subset \alpha_1$, hence β and α_1 belong to every open neighbourhood of γ in $\mathrm{En}(S)$. Now $\gamma \in V_u$, so $\gamma \in O_1$ or $\gamma \in O_2$. But if $\gamma \in O_1$, then $\beta \in O_1 \cap O_2$, and if $\gamma \in O_2$ then $\alpha_1 \in O_1 \cap O_2$. Thus $O_1 \cap O_2 \neq \emptyset$, showing that V_u is connected. $\qquad\square$

Next we consider Čech homology of $\mathrm{En}(S)$. The following proposition forms the crucial part of the proof of our theorem.

PROPOSITION 3.4. *For any cover* $\mathcal{U}$ *of* $\mathrm{En}(S)$ *by basic open sets, we have* $\mathrm{H}_n(\mathcal{U}, \mathbb{Z}) = 0$ $(n > 0)$.

Proof. Let $\mathcal{U} = \{V_{u_\sigma} \mid \sigma \in \Sigma\}$ be such an open cover, indexed by a set Σ. To avoid too many indices, we will in this proof write σ for u_σ, and V_σ for V_{u_σ}. Let $C_\bullet(\mathcal{U})$ be the usual Čech complex, i.e., $C_n(\mathcal{U})$ is the free Abelian group on the set $N_n(\mathcal{U}) = \{(\sigma_0,\ldots,\sigma_n) \mid V_{\sigma_0} \cap \cdots \cap V_{\sigma_n} \neq \emptyset\}$. Note that $(\sigma_0,\ldots,\sigma_n) \in N_n(\mathcal{U})$ iff the finite partial functions $\sigma_0,\ldots,\sigma_n$ are compatible, in the sense that their union

$\sigma_0 \cup \cdots \cup \sigma_n$ (short for $u_{\sigma_0} \cup \cdots \cup u_{\sigma_n}$) is well-defined. We will show that this complex is chain contractible, by exhibiting an explicit chain homotopy h:

$$0 \leftarrow \mathbb{Z} \underset{h_{-1}}{\overset{\partial}{\leftrightarrows}} C_0(\mathcal{U}) \underset{h_0}{\overset{\partial}{\leftrightarrows}} C_1(\mathcal{U}) \underset{h_1}{\overset{\partial}{\leftrightarrows}} C_2(\mathcal{U}) \underset{h_2}{\overset{\partial}{\leftrightarrows}} \cdots$$

$$\partial \circ h_{-1} = \mathrm{id}, \qquad \partial h_n + h_{n-1}\partial = \mathrm{id}. \tag{4}$$

To define h, we fix a point $\alpha \in \mathrm{En}(S)$ and an index $\tau \in \Sigma$ with $\alpha \in V_{u_\tau}$. Furthermore, for each sequence $\boldsymbol{\sigma} = (\sigma_0, \ldots, \sigma_n) \in N_n(\mathcal{U})$, we choose an index $f(\boldsymbol{\sigma})$ so that

$$\alpha - (\sigma_0 \cup \cdots \cup \sigma_n \cup \tau) \in V_{f(\boldsymbol{\sigma})}. \tag{5}$$

The h_n are now defined by induction, by

$$\begin{aligned} h_{-1}(1) &= \tau, \\ h_n(\boldsymbol{\sigma}) &= (-1)^{n+1}[\boldsymbol{\sigma} f(\boldsymbol{\sigma}) - h_{n-1}(\partial\boldsymbol{\sigma}) f(\boldsymbol{\sigma})]. \end{aligned} \tag{6}$$

Here $\boldsymbol{\sigma}$ is the tuple $(\sigma_0, \ldots, \sigma_n)$, $\boldsymbol{\sigma} f(\boldsymbol{\sigma}) = (\sigma_0, \ldots, \sigma_n, f(\boldsymbol{\sigma}))$, and $h_{n-1}(\partial\boldsymbol{\sigma}) f(\boldsymbol{\sigma})$ is the sum $\sum (-1)^i h_{n-1}(\sigma_0 \ldots \hat{\sigma}_i \ldots \sigma_n) f(\boldsymbol{\sigma})$ obtained by adding $f(\boldsymbol{\sigma})$ to the end of every term in $h_{n-1}(\partial\boldsymbol{\sigma})$. For example,

$$\begin{aligned} h_0(\sigma_0) \ &= \ -(\sigma_0 f(\sigma_0) - \tau f(\sigma_0)), \\ h_1(\sigma_0\sigma_1) \ &= \ \sigma_0\sigma_1 f(\sigma_0\sigma_1) + \sigma_1 f(\sigma_1) f(\sigma_0\sigma_1) - \tau f(\sigma_1) f(\sigma_0\sigma_1) - \\ &\quad - \sigma_0 f(\sigma_0) f(\sigma_0\sigma_1) + \tau f(\sigma_0) f(\sigma_0\sigma_1), \end{aligned}$$

etc. Let us observe first that $h_n(\boldsymbol{\sigma})$ is a well-defined element of $C_{n+1}(\mathcal{U})$; i.e., that for any sequence $\boldsymbol{\mu} = (\mu_0, \ldots, \mu_{n+1})$ occurring in $h_n(\boldsymbol{\sigma})$, the corresponding basic open $V_{\boldsymbol{\mu}} = V_{\mu_0} \cap \cdots \cap V_{\mu_{n+1}}$ is nonempty. We will show by induction on n that for any generator $\boldsymbol{\mu}$ occurring in $h_n(\boldsymbol{\sigma})$, there exists a point $\beta = \beta_{\boldsymbol{\sigma}}(\boldsymbol{\mu})$ in $\mathrm{En}(S)$ such that

$$\beta \supset \alpha - (\sigma_0 \cup \cdots \cup \sigma_n \cup \tau) \quad \text{and} \quad \beta \in V_{\boldsymbol{\mu}} = V_{\mu_0} \cap \cdots \cap V_{\mu_{n+1}}. \tag{7}$$

For $n = 0$, the two generators occurring in $h_0(\sigma_0)$ are $\sigma_0 f(\sigma_0)$ and $\tau f(\sigma_0)$ and, by (5), we can choose

$$\begin{aligned} \beta(\sigma_0 f(\sigma_0)) &= \alpha - (\sigma_0 \cup \tau) \cup \sigma_0 \in V_{\sigma_0 f(\sigma_0)}, \\ \beta(\tau f(\sigma_0)) &= \alpha - (\sigma_0 \cup \tau) \cup \tau \in V_{\tau f(\sigma_0)}. \end{aligned}$$

Suppose, then, that we have found a point β as in (7) for each $(\sigma_0, \ldots, \sigma_n)$ and each generator $\boldsymbol{\mu}$ in $h_n(\boldsymbol{\sigma})$. Now consider a sequence $\boldsymbol{\sigma} = (\sigma_0, \ldots, \sigma_{n+1}) \in N_{n+1}(\mathcal{U})$, with

$$h_n(\boldsymbol{\sigma}) = (-1)^{n+1}[\boldsymbol{\sigma} f(\boldsymbol{\sigma}) - h_{n-1}(\partial\boldsymbol{\sigma}) f(\boldsymbol{\sigma})] \tag{8}$$

as in (6). For the generator $\boldsymbol{\sigma} f(\boldsymbol{\sigma})$, we can take $\beta = (\alpha - (\sigma_0 \cup \cdots \cup \sigma_{n+1} \cup \tau)) \cup (\sigma_0 \cup \cdots \cup \sigma_{n+1}) = (\sigma_0 \cup \cdots \cup \sigma_{n+1})/(\alpha - \tau)$, since by (5), this β will satisfy $\beta \in V_{\boldsymbol{\sigma} f(\boldsymbol{\sigma})}$. Next consider $h_{n-1}(\partial \boldsymbol{\sigma}) f(\boldsymbol{\sigma})$. For a generator $\boldsymbol{\mu} = (\mu_0, \ldots, \mu_{n+1})$ in $h(\sigma_0 \ldots \hat{\sigma}_i \ldots \sigma_n)$, we have by induction found a β_0 so that

$$\beta_0 \supset \alpha - (\sigma_0 \cup \ldots \hat{\sigma}_i \ldots \cup \sigma_n \cup \tau) \quad \text{and} \quad \beta_0 \in V_{\boldsymbol{\mu}}.$$

Also, $f(\boldsymbol{\sigma}) \subset \alpha - (\sigma_0 \cup \cdots \cup \sigma_n \cup \tau) \subset \alpha - (\sigma_0 \cup \cdots \hat{\sigma}_i \cdots \cup \sigma_n \cup \tau)$, so $\beta_0 \in V_{\boldsymbol{\mu} f(\boldsymbol{\sigma})}$. Thus β_0 is also a witness for the fact that the part $\boldsymbol{\mu} f(\boldsymbol{\sigma})$ occurring in $h_{n-1}(\partial \boldsymbol{\sigma}) f(\boldsymbol{\sigma})$ corresponds to a nonempty intersection of basic open sets.

It remains to prove the identities (4) for a chain homotopy. Clearly, $\partial h_{-1} = \text{id}$, while for $\sigma_0 \in C_0(\mathcal{U})$,

$$\begin{aligned}
\partial h_0(\sigma_0) + h_{-1}(\partial \sigma_0) &= -\partial(\sigma_0 f(\sigma_0)) + \partial(\tau f(\sigma_0)) + \tau \\
&= -(f(\sigma_0) + \sigma_0) + (f(\sigma_0) - \tau) + \tau \\
&= \sigma_0.
\end{aligned}$$

We proceed by induction, and suppose the identity $\partial h_n + h_{n-1} \partial = \text{id}$ has been proved. Consider, then, any generator $\sigma_0 \ldots \sigma_{n+1} \in C_{n+1}(\mathcal{U})$. The induction hypothesis implies that $\partial h_n(\partial \boldsymbol{\sigma}) = \partial \boldsymbol{\sigma} - h_{n-1}(\partial^2 \boldsymbol{\sigma}) = \partial \boldsymbol{\sigma}$, whence

$$\partial h_n(\partial \boldsymbol{\sigma}) f(\boldsymbol{\sigma}) = (\partial \boldsymbol{\sigma}) f(\boldsymbol{\sigma}). \tag{9}$$

Thus, using the general identity

$$\partial(\mu_0 \ldots \mu_n \rho) = \partial(\mu_0 \ldots \mu_n)\rho + (-1)^{n+1} \mu_0 \ldots \mu_n \tag{10}$$

we find

$$\begin{aligned}
\partial h_{n+1}(\boldsymbol{\sigma}) &= \partial(-1)^n[\boldsymbol{\sigma} f(\boldsymbol{\sigma}) - h_n(\partial \boldsymbol{\sigma}) f(\boldsymbol{\sigma})] \quad \text{(by definition)} \\
&= (-1)^n[(\partial \boldsymbol{\sigma}) f(\boldsymbol{\sigma}) + (-1)^n \boldsymbol{\sigma} - \partial(h_n(\partial \boldsymbol{\sigma}) f(\boldsymbol{\sigma}))] \quad \text{(by (10))} \\
&= \boldsymbol{\sigma} + (-1)^n[(\partial \boldsymbol{\sigma}) f(\boldsymbol{\sigma}) - (\partial h_n(\partial \boldsymbol{\sigma})) f(\boldsymbol{\sigma}) - \\
&\qquad\qquad -(-1)^{n+2} h_n(\partial \boldsymbol{\sigma})] \quad \text{(by (10))} \\
&= \boldsymbol{\sigma} - h_n(\partial \boldsymbol{\sigma}) + (-1)^n[(\partial \boldsymbol{\sigma}) f(\boldsymbol{\sigma}) - \partial h_n(\partial \boldsymbol{\sigma}) f(\boldsymbol{\sigma})] \\
&= \boldsymbol{\sigma} - h_n(\partial \boldsymbol{\sigma}).
\end{aligned}$$

This completes the proof of the proposition. $\square$

PROPOSITION 3.5. *Let V be a basic open set in $\text{En}(S)$, and let $\mathcal{U}$ be a cover of V by basic open sets. Then $\text{H}_n(\mathcal{U}, \mathbb{Z}) = 0$ ($n > 0$).*
    Proof. This is proved in exactly the same way as the previous proposition. If $V = V_u$, then one modifies the proof by restricting all constructions to finite sequences v or points α with $u \subset v, \alpha$. $\square$

4. Construction of $\varphi_!$ and a Projection Formula

The enumeration spaces $En(S)$ are related to the space $X = X_{\mathcal{T}}$, constructed for a topos above, in the following way. For each small point $p: \mathscr{S} \to \mathcal{T}$, with stalk G_p of the special sheaf G, there is a continuous map $i_p: En(G_p) \to X$, $i_p(\alpha) = (p, \alpha)$.

Denote by $\pi: En(G_p) \to$ pt the unique map into the one-point space. These two maps induce topos morphisms $\mathscr{S} \xleftarrow{\pi} Sh(En(G_p)) \xrightarrow{i_p} Sh(X)$, which relate to the map $\varphi: Sh(X) \to \mathcal{T}$ in the following way.

LEMMA 4.1. *The square*

$$
\begin{array}{ccc}
Sh(En(G_p)) & \xrightarrow{\ i_p\ } & Sh(X) \\
{\scriptstyle \pi}\Big\downarrow & & \Big\downarrow{\scriptstyle \varphi} \\
\mathscr{S} & \xrightarrow{\ \ p\ \ } & \mathcal{T}
\end{array}
\qquad (11)
$$

commutes up to natural isomorphism.

Proof. Let E be an object in $\mathcal{T}$, with sheaf $\varphi^*(E)$ on X as constructed in Section 2. Using the notation of the proof of Lemma 2.2, consider a canonical section $\sigma: U_{i,C} \to V_{i,C,f} \subset \varphi^*(E)$, $\sigma(p, \alpha) = f_p(\alpha(i))$, of the sheaf $\varphi^*(E)$. The connected components of $i_p^{-1}(U_{i,C})$ are the basic open sets $V_g = \{\alpha \mid \alpha(i_1) = g_1, \ldots, \alpha(i_n) = g_n\}$, for all $g = (g_1, \ldots, g_n) \in C_p \subset G_p^n$. The section σ is constant on V_g, with value $f_p(g_1, \ldots, g_n)$. This shows that $i_p^* \varphi^*(E)$ is a constant sheaf, with stalk E_p since $i_p^* \varphi^*(E)_{(p,\alpha)} = \varphi^*(E)_{(p,\alpha)} = E_p$. $\qquad \square$

We note that the square (11) need not be a pullback of topoi, although it is very close to being one: $En(G_p)$ is the space of points of the topos theoretic pullback.

COROLLARY 4.2. *Let $\sigma: U_{i,C} \to \varphi^*(E)$ be any section of the sheaf $\varphi^*(E)$, defined on the basic open set $U_{i,C}$. Then for any two points (p, α) and (p, β) in $U_{i,C}$,*

$$
\alpha(i) = \beta(i) \Rightarrow \sigma(p, \alpha) = \sigma(p, \beta). \qquad (12)
$$

Proof. The section σ restricts along $i_p: En(G_p) \to X$ to a section on $i_p^{-1}(U_{i,C})$ of the constant sheaf with stalk E_p. This section is constant on the connected components $V_g = \{\alpha \mid \alpha(i) = g\}$ of $i_p^{-1}(U_{i,C})$ already occurring in the proof of Lemma 4.1. Formula (12) follows. $\qquad \square$

Recall that a topos morphism $\varphi: \mathcal{T}' \to \mathcal{T}$ consists of two particular functors φ^* and φ_*, with φ^* left exact and left adjoint to φ_*. The particular morphism $\varphi: Sh(X) \to \mathcal{T}$ constructed above, has the following additional property.

PROPOSITION 4.3. *There exists a functor $\varphi_! \colon \mathrm{Sh}(X) \to \mathcal{T}$ which is left adjoint to $\varphi^* \colon \mathcal{T} \to \mathrm{Sh}(X)$, i.e.,*

$$\mathrm{Hom}_{\mathcal{T}}(\varphi_!(F), E) \cong \mathrm{Hom}_{\mathrm{Sh}(X)}(F, \varphi^*(E)) \tag{13}$$

for any sheaf F on X and any object E of the topos $\mathcal{T}$.

Proof. For the proof of this proposition, we will construct for each sheaf F on X an object $\varphi_!(F)$ of the topos $\mathcal{T}$. Note that each basic open set $U_{i,C} \subset X$ can be viewed as a sheaf on X (where the sheaf projection is the inclusion $U_{i,C} \hookrightarrow X$). Furthermore, an arbitrary sheaf F is the colimit of such sheaves $U_{i,C}$ (the colimit being taken over the poset of sections of F defined on basic open sets). Thus, since the desired left adjoined $\varphi_!$ must necessarily commute with colimits, it suffices to construct $\varphi_!(U_{i,C})$ for each basic open set $U_{i,C}$ and prove the natural bijective correspondence of (13) in this special case

$$\mathrm{Hom}(\varphi_!(U_{i,C}), E) \cong \Gamma(U_{i,C}, \varphi^*(E)). \tag{14}$$

We define

$$\varphi_!(U_{i,C}) =_{\mathrm{def}} C. \tag{15}$$

To prove (14) for this definition, we shall use the following two lemmas.

LEMMA 4.4. *Let $U_{i,C}$ and $U_{j,B}$ be two basic open sets in X, and suppose that $U_{j,B} \neq \emptyset$. Then $U_{j,B} \subset U_{i,C}$ iff the sequence $i = (i_1, \ldots, i_n)$ is a subsequence of $j = (j_1, \ldots, j_m)$, and the corresponding projection $G^m \to G^n$ maps B into C.*

Proof. The implication ($\Leftarrow$) is clear. For ($\Rightarrow$), choose a point $(p, \alpha) \in U_{j,B}$. If i_k is any index in i which does not occur among $(j_1, \ldots, j_m)$, let α' be the restriction of α to $\mathrm{dom}(\alpha) - \{i_k\}$. Then $(p, \alpha') \in U_{j,B}$ but $(p, \alpha') \notin U_{i,C}$. This shows that if $U_{j,B} \subset U_{i,C}$ then i must be a subsequence of j. Now consider the projection $\pi \colon G^m \to G^n$ coming from the fact that i is a subsequence of j. (Here we use that we can assume that both i and j do not contain repetitions, as explained just below (2).) To prove $\pi(B) \subset C$, it suffices to prove that, for each small point p, $\pi_p(B_p) \subset C_p$, (because the stalks at the small points are jointly conservative, by assumption).

Take $(g_1, \ldots, g_m) \in B_p$, and let $\alpha \in \mathrm{En}(G_p)$ be any enumeration with $\alpha(j_k) = g_k$ ($k = 1, \ldots, m$). Then $(p, \alpha) \in U_{j,B} \subset U_{i,C}$, so $\pi_p(g_1, \ldots, g_m) = (\alpha(i_1), \ldots, \alpha(i_n)) = \alpha(i) \in C_p$. $\square$

LEMMA 4.5. *Let $U_{i,C}$ be a basic open set. Let $\{U_{j_\xi, B_\xi}\}$ be a family of nonempty basic open subsets of $U_{i,C}$, with associated projections $\pi_\xi \colon B_\xi \to C$ as in Lemma 4.4. Then $U_{i,C}$ is covered by $\{U_{j_\xi, B_\xi}\}$ in the space X iff $\{\pi_\xi \colon B_\xi \to C\}$ is an epimorphic family in $\mathcal{T}$.*

Proof. To simplify notation, we just treat the case where $i = i_1$ and $C \subset G$, while $j = (i_1, j_\xi)$ is a sequence of length 2 and $B_\xi \subset G^2$. By Lemma 4.3, the projection $\pi_2 \colon G^2 \to G$ maps each B_ξ into C, giving a map $\pi_\xi \colon B_\xi \to C$.

Suppose now that $U_{i,C} = \bigcup U_{j_\xi, B_\xi}$. To show that $\{\pi_\xi \colon B_\xi \to C\}$ is an epimorphic family, it suffices to prove, for each small point p, $C_p = \bigcup_\xi \pi_\xi(B_\xi)_p$.

Take any $c \in C_p$, and choose an enumeration $\alpha \in \mathrm{En}(G_p)$ with $\alpha(i) = c$. Then $(p, \alpha) \in U_{i,C}$, hence for some ξ also $(p, \alpha) \in U_{j_\xi, B_\xi}$. Thus, $j_\xi \in \mathrm{dom}(\alpha)$ and $b = (\alpha(i), \alpha(j_\xi)) \in (B_\xi)_p$, whence $c = \pi_\xi(b) \in \pi_\xi(B_\xi)_p$, as desired.

The converse is similar. $\qquad\square$

We now continue the proof of Proposition 4.3, and show the isomorphism (13) for $\varphi_!(U_{i,C}) = C$. In one direction, any map $f \colon C \to E$ in $\mathcal{T}$ defines a canonical section

$$\sigma_f \colon U_{i,C} \to \varphi^*(E), \qquad \sigma_f(p, \alpha) = f_p(\alpha(i)), \tag{16}$$

(as in the proof of Lemma 2.2).

In the other direction, suppose $\sigma \colon U_{i,C} \to \varphi^*(E)$ is an arbitrary section of $\varphi^*(E)$. Locally, σ must be a canonical section as described in Section 2. Thus, there is a cover

$$U_{i,C} = \bigcup_\xi U_{j_\xi, B_\xi} \tag{17}$$

and for each ξ a map $f_\xi \colon B_\xi \to E$ so that

$$\sigma(p, \alpha) = (f_\xi)_p(\alpha(j_\xi)), \quad \text{for } (p, \alpha) \in U_{j_\xi, B_\xi}. \tag{18}$$

By Lemma 4.5, the identity (17) implies that the B_ξ form a cover of C in the topos $\mathcal{T}$. Let us simplify the notation as in the proof of Lemma 4.5, and write $i = i_1$, $j = (i_1, j_\xi)$, $C \subset G$, $B_\xi \subset G^2$, and $\pi_\xi \colon B_\xi \to C$ for the restriction of the first projection $G^2 \to G$. We claim that the maps $f_\xi \colon B_\xi \to E$ form a compatible family for this cover $\{B_\xi \to C\}$, hence define a unique map $f \colon C \to E$ with $f \circ \pi_\xi = f_\xi$. For this, it needs to be shown, for any two indices ξ and ζ, that the square

$$\begin{array}{ccc}
B_\xi \times_C B_\zeta & \xrightarrow{\;\pi_2\;} & B_\zeta \\
{\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle f_\zeta} \\
B_\xi & \xrightarrow[\;f_\xi\;]{} & E
\end{array} \tag{19}$$

commutes in $\mathcal{T}$. It suffices to check that the corresponding diagram of stalks commutes for every small point p. Choose such a point p, and consider an element

$b \in (B_\xi \times_C B_\zeta)_p$. Write $\pi_1(b) = (c, b_\xi) \in (B_\xi)_p$ and $\pi_2(b) = (c, b_\zeta) \in (B_\zeta)_p$. Choose now two enumerations $\alpha, \beta \in \mathrm{En}(G_p)$, such that

$$\alpha(i) = c, \quad \alpha(j_\xi) = b_\xi,$$
$$\beta(i) = c, \quad \beta(j_\zeta) = b_\zeta.$$

Then $(p, \alpha) \in U_{j_\xi, B_\xi}$ and $(p, \beta) \in U_{j_\zeta, B_\zeta}$, so

$$
\begin{aligned}
(f_\xi \circ \pi_1)_p(b) &= (f_\xi)_p(c, b_\xi) \\
&= (f_\xi)_p(\alpha(i), \alpha(j_\xi)) \\
&= \sigma(p, \alpha) \quad \text{(by (18))},
\end{aligned}
$$

and similarly $(f_\zeta \circ \pi_2)_p(b) = \sigma(p, \beta)$. But $(p, \alpha), (p, \beta) \in U_{i,C}$, while $\alpha(i) = \beta(i)$, so $\sigma(p, \alpha) = \sigma(p, \beta)$ by Corollary 4.2. This proves that $(f_\xi \circ \pi_1)_p(b) = (f_\zeta \circ \pi_2)_p(b)$ for any $b \in (B_\xi \times_C B_\zeta)_p$, and hence that (19) commutes. Thus, the f_ξ together uniquely determine a map $f = f_\sigma \colon C \to E$.

It is now straightforward to check that these constructions, of σ_f from f and of f_σ from σ, are mutually inverse, and prove the required bijection (14).

This completes the proof of Proposition 4.3. $\square$

Let us reconsider the square (11) at the beginning of this section. Since $\mathrm{En}(G_p)$ is a locally connected space (Lemma 3.3) the inverse image functor $\pi^* \colon \mathcal{S} \to \mathrm{Sh}(\mathrm{En}(G_p))$, which sends a set to the constant sheaf, has a left adjoint $\pi_! \colon \mathrm{Sh}(\mathrm{En}(G_p)) \to \mathcal{S}$. For a sheaf F on $\mathrm{En}(G_p)$, $\pi_!(F)$ is simply the set of connected components of F, where F is viewed as an étale space over $\mathrm{En}(G_p)$.

COROLLARY 4.6. *For the square (11), the projection formula $\pi_!(i_p)^* = p^*\varphi_!$ holds.*

Proof. First, a more precise formulation of this corollary should state that the canonical natural transformation

$$\pi_!(i_p)^*(F) \to p^*\varphi_!(F), \tag{20}$$

obtained from the isomorphism $i_p^*\varphi^* \cong \pi^*p^*$ and the adjunctions, is an isomorphism. Since the functors in (20) all preserve colimits, it suffices to check that (20) is an isomorphism in case F is (the sheaf corresponding to) a basic open set $U_{i,C}$. But $\pi_! i_p^*(U_{i,C})$ is the set of connected components of $i_p^{-1}(U_{i,C})$, and these are exactly the basic open sets $V_g = \{\alpha \mid \alpha(i_1) = g_1, \ldots, \alpha(i_n) = g_n\}$, for $g = (g_1, \ldots, g_n) \in C_p \subset G_p^n$, hence are in bijective correspondence with elements of $C_p = p^*(C) = p^*\varphi_!(U_{i,C})$ by (15). $\square$

5. Proof of the Theorem

We will now prove the theorem, stated in the introduction and repeated here

THEOREM 5.1. *For any sheaf of Abelian groups A in $\mathcal{T}$, the morphism φ:* $\mathrm{Sh}(X_{\mathcal{T}}) \to \mathcal{T}$ *induces an isomorphism $\varphi^*: H^n(\mathcal{T}, A) \to H^n(X_{\mathcal{T}}, \varphi^*A)$, for any* $n \geqslant 0$.

For $n = 0$, this follows from

LEMMA 5.2. *The inverse image functor $\varphi^*: \mathcal{T} \to \mathrm{Sh}(X_{\mathcal{T}})$ is full and faithful.*

Proof. The statement of the lemma is equivalent to the assertion that the counit of the adjunction $\varphi_! \varphi^*(E) \to E$ is an isomorphism, for every sheaf E on $\mathcal{T}$. It suffices to check this for the stalks at each small point p. But there we have

$$
\begin{aligned}
\varphi_! \varphi^*(E)_p &= p^* \varphi_! \varphi^*(E) \\
&= \pi_!(i_p)^* \varphi^*(E) \quad \text{(by Corollary 4.6)} \\
&= \pi_! \pi^*(E_p) \quad \text{(by Lemma 4.1)} \\
&= E_p,
\end{aligned}
$$

the latter since $\mathrm{En}(G_p)$ is connected (Lemma 3.3). $\qquad\square$

Later, we will have to compare the Čech complex of an open cover in X to its inverse image along the map $i_p: \mathrm{En}(G_p) \to X$, where p is any small point of the topos $\mathcal{T}$. We will use the following simple observation

LEMMA 5.3. *Let $U_1, \ldots, U_n \subset U \subset X$ be basic open sets, and let $V \subset i_p^{-1}(U)$ be a connected component. Then the connected components of $i_p^{-1}(U_1 \cap \cdots \cap U_n)$ contained in V are the nonempty intersections $V_1 \cap \cdots \cap V_n$, where $V_i \subset V$ is a component of $i_p^{-1}(U_i)$.*

Proof. We already observed (e.g. in the proofs of 4.1 and 4.5) that for any basic open set $U \subset X$, the connected components of $i_p^{-1}(U)$ are basic open sets V in $\mathrm{En}(G_p)$. These basic open sets in $\mathrm{En}(G_p)$ are all connected (Lemma 3.3) and closed under intersection. The lemma follows immediately from this. $\qquad\square$

LEMMA 5.4. *Let I be any injective Abelian sheaf in $\mathcal{T}$. Let $U \subset X$ be a basic open set, and let $\mathcal{U}$ be a cover of U by basic open sets. Then $H^n(\mathcal{U}, \varphi^*(I) \restriction U) = 0$ for $n > 0$.*

Proof. Write $\mathcal{U} = \{U_\sigma \mid \sigma \in \Sigma\}$, and $N_n(\mathcal{U}) = \sum_{\sigma_0 \ldots \sigma_n} U_{\sigma_0 \ldots \sigma_n}$ where the sum is over all $(n+1)$-tuples of indices, and $U_{\sigma_0 \ldots \sigma_n} = U_{\sigma_0} \cap \cdots \cap U_{\sigma_n}$. Viewing each $U_{\sigma_0 \ldots \sigma_n}$ as an object of $\mathrm{Sh}(X)$, we see that $N_\bullet(\mathcal{U})$ is a simplicial object in $\mathrm{Sh}(X)$. The Čech complex $C^n(\mathcal{U}, \varphi^*(I) \restriction U)$ computing $H^*(\mathcal{U}, \varphi^*(I) \restriction U)$ can now be described as

$$
\begin{aligned}
C^n(\mathcal{U}, \varphi^*(I) \restriction U) &= \mathrm{Hom}_{\mathrm{Sh}(X)}(N_n(\mathcal{U}), \varphi^*(I)) \\
&= \mathrm{Hom}_{\mathcal{T}}(\varphi_! N_n(\mathcal{U}), I),
\end{aligned}
$$

the latter by the adjunction of 4.3. To prove the lemma, it thus suffices to show that the associated chain complex $\mathbb{Z}[\varphi_! N_\bullet(\mathcal{U})]$ of Abelian groups in $\mathcal{T}$ is exact

at $n > 0$. It is enough to check this for the stalk at each small point p. But

$$\mathbb{Z}[\varphi_! N_n(\mathcal{U})]_p = \mathbb{Z}[\varphi_!(N_n(\mathcal{U}))_p]$$

$$= \mathbb{Z}[\pi_!(i_p)^* N_n(\mathcal{U})], \quad \text{(by Corollary 4),}$$

which is the chain complex of the simplicial set $\pi_! i_p^*(N_\bullet(\mathcal{U}))$. Now

$$\pi_! i_p^*(N_n(\mathcal{U}))$$
$$= \{(\sigma_0 \ldots \sigma_n, W) \mid W \text{ a connected component of } i_p^{-1}(U_{\sigma_0 \ldots \sigma_n})\}.$$

For each connected component $V \subset i_p^{-1}(U)$, let $\mathcal{U}_V$ be the cover of V by connected components $W \subset i_p^*(U_\sigma)$, for all $\sigma \in \Sigma$. By Lemma 5.3, $\pi_! i_p^*(N_\bullet(\mathcal{U}))$ is the disjoint sum of the Čech nerves of these covers $\mathcal{U}_V$, and these nerves are acyclic by Proposition 3.5. Thus $\pi_! i_p^*(N_\bullet(\mathcal{U}))$ is acyclic also, and the lemma is proved. □

Proof of Theorem 5.1. By general homological algebra, it suffices to show that for any injective Abelian group I in $\mathcal{T}$ the sheaf cohomology groups $\mathrm{H}^n(X, \varphi^*(I))$ vanish for $n > 0$. By Lemma 5.4, the sheaf $\varphi^*(I) \restriction U$ is 'Čech-acyclic' for each basic open set $U \subset X$. The result follows by applying Cartan's criterion [1], Proposition V.4.3, [3], Théorème 5.9.2. □

As stated in Section 1, the argument actually proves the somewhat stronger assertion that the higher right derived functors of $\varphi_* \colon \mathrm{Sh}(X) \to \mathcal{T}$ vanish. Before stating this as Corollary 5.6 below, we observe the following corollary.

COROLLARY 5.5. *Let E be any sheaf (of sets) in $\mathcal{T}$. Then in the pullback of topoi*

$$\begin{array}{ccc}
\mathrm{Sh}(\varphi^* E) & \xrightarrow{\ \pi\ } & \mathrm{Sh}(X) \\
\varphi_E \downarrow & & \downarrow \varphi \\
\mathcal{T}/E & \longrightarrow & \mathcal{T}
\end{array}$$

the map φ_E induces isomorphisms $\mathrm{H}^n(\mathcal{T}/E, A) \xrightarrow{\cong} \mathrm{H}^n(\varphi^*(E), \varphi_E^*(A))$, for any Abelian sheaf A in $\mathcal{T}/E$.

Here $\mathcal{T}/E$ denotes the 'induced topos' ([1], Exposé IV.5) of $\mathcal{T}$-objects over E, and $\mathcal{T}/E \to \mathcal{T}$ is the canonical morphism (loc. cit. (5.2.1)).

Proof. We claim that the map φ_E is again of the form $\varphi \colon \mathrm{Sh}(X_\mathcal{T}) \to \mathcal{T}$ so that Corollary 5.5 is actually a special case of Theorem 5.1. More precisely, $\varphi_E \colon \mathrm{Sh}(\varphi^* E) \to \mathcal{T}/E$ is precisely the map $\mathrm{Sh}(X_{(\mathcal{T}/E)}) \to \mathcal{T}/E$, for a suitable choice of the various parameters. Indeed, suppose $X_\mathcal{T}$ is defined using the set of small points $\mathcal{P}$, the object G so that subsheaves of G^n generate $\mathcal{T}$, and the index set I. Then $H = (G \times E \to E)$ is an object of $\mathcal{T}/E$ so that subsheaves of H^n generate $\mathcal{T}/E$. Moreover, the points of $\mathcal{T}/E$ are in bijective correspondence with

pairs (p, e), where p is a point of $\mathcal{T}$ and $e \in E_p$. For such a pair (p, e), the stalk of an object $(f: F \to E)$ at (p, e) is given by $(f: F \to E)_{(p,e)} = f_p^{-1}(e) \subset E_p$. In particular, $H_{(p,e)} = G_p$ for each e. Now for the set of small points of $\mathcal{T}/E$ we can take all these pairs (p, e) where $p \in \mathcal{P}$, and we can then take the same index set I.

The space $X_{(\mathcal{T}/E)}$ defined from these choices then is the space of triples (p, e, α), where p is a small point of $\mathcal{T}$, $e \in E_p$, and $\alpha \in \mathrm{En}(H_{(p,e)}) = \mathrm{En}(G_p)$. But this is exactly the space $\varphi^*(E)$ defined in Section 2. Further details are straightforward. □

COROLLARY 5.6. *For any Abelian sheaf A in $\mathcal{T}$, and any $n > 0$, $(\mathrm{R}^n \varphi_*) \times (\varphi^* A) = 0$.*

Proof. As before, it suffices to prove this for A injective. For an arbitrary sheaf B on X, $\mathrm{R}^n \varphi_*(B)$ is the associated sheaf of the presheaf $E \mapsto \mathrm{H}^n(\varphi^*(E), \pi^*(B))$ (where $\pi: \varphi^*(E) \to X$ is the sheaf projection); see [1], Proposition V.5.1 and [7], Lemma 8.18. For $B = \varphi^*(I)$ where I is injective, the result thus follows from Corollary 5.5. □

6. Étale Cohomology

By way of example, we will give an explicit description of the space $X_{\mathcal{T}}$ in the case where $\mathcal{T}$ is the étale topos over a scheme. The main reference for this section is Grothendieck's Exposé VIII in [1]. For basic properties of strictly Henselian local rings and strict Henselization, see [11].

Fix a ground field k, and a scheme Y (over k). Let $Y_{\text{ét}}$ be the étale site over Y, and let $\widetilde{Y}_{\text{ét}}$ be the associated étale topos. For a point $y \in Y$, $k(y)$ denotes the residue field of the local ring $\mathcal{O}_{Y,y}$, and $\overline{k(y)}$ its separable closure. The Galois group $\mathrm{Gal}(\overline{k(y)}/k(y))$ is denoted by π_y.

The functor A on $Y_{\text{ét}}$ which associates to each object $f: Z \to Y$ of the étale site the ring $\Gamma(Z, f^*(\mathcal{O}_Y))$ is a sheaf, and defines a local ring A in the topos $\widetilde{Y}_{\text{ét}}$. The functor A^{hs} on $Y_{\text{ét}}$ which associates to $f: Z \to Y$ the ring $\Gamma(Z, \mathcal{O}_Z)$ is again a sheaf, and a strictly Henselian local ring in $\widetilde{Y}_{\text{ét}}$ [5]. The extension $A \to A^{\text{hs}}$ is a universal strict Henselization of $\mathcal{O}_Y$ in the topos $\widetilde{Y}_{\text{ét}}$. The sheaf A^{hs} will play the role of the object G.

The étale topos has enough points. We recall from [1], Exposé VIII, that each point $y \in Y$ defines first a geometric point $\overline{y}: \mathrm{Spec}(\overline{k(y)}) \to Y$ of the scheme Y, and then a point of the topos $\widetilde{Y}_{\text{ét}}$, whose inverse image functor is the composition $\Gamma \circ \overline{y}^*: \widetilde{Y}_{\text{ét}} \to \widetilde{\mathrm{Spec}(\overline{k(y)})}_{\text{ét}} \to \mathcal{S}$, and denoted $F \mapsto F_{\overline{y}}$. By loc. cit., Corollaire VIII.3.6, the set of all these points is jointly conservative. So we can take this set of points $y \in Y$ for the set $\mathcal{P}$.

Consider again the extension $A \to A^{\text{hs}}$ in the topos $\widetilde{Y}_{\text{ét}}$. As explained in [1], Exposé VIII.4, for any $y \in Y$ the stalk map $A_{\overline{y}} \to A_{\overline{y}}^{\text{hs}}$ is a (the) strict Henselization

of $\mathcal{O}_{Y,y} = A_{\overline{y}}$, relative to the separable closure $k(y) \hookrightarrow \overline{k(y)}$. Thus we will write $\mathcal{O}_{Y,y}^{\mathrm{hs}}$ for $A_{\overline{y}}^{\mathrm{hs}}$, and we will identify $\mathcal{O}_{Y,y}$ with a subset of $\mathcal{O}_{Y,y}^{\mathrm{hs}}$.

By the universal property of the strict Henselization [4], Section 18, [11], Section VIII.2, the group π_y acts on $\mathcal{O}_{Y,y}^{\mathrm{hs}}$, say from the left. The local ring $\mathcal{O}_{Y,y} \subset \mathcal{O}_{Y,y}^{\mathrm{hs}}$ is fixed under this action.

Let I be a set whose cardinality is at least as big as that of all these strict Henselizations $\mathcal{O}_{Y,y}^{\mathrm{hs}}$.

We can now describe the space $X = X_{\mathcal{T}}$ of Theorem 5.1 in this special case where $\mathcal{T} = \widetilde{Y_{\mathrm{\acute{e}t}}}$. Let $y \in Y$, and consider all functions ('enumerations') $\alpha \colon \mathrm{dom}(\alpha) \to \mathcal{O}_{Y,y}^{\mathrm{hs}}$ defined on a subset $\mathrm{dom}(\alpha) \subset I$; and with the property that $\alpha^{-1}(b)$ is infinite for each $b \in \mathcal{O}_{Y,y}^{\mathrm{hs}}$. Call two such enumerations α and β equivalent, $\alpha \sim \beta$, if $\mathrm{dom}(\alpha) = \mathrm{dom}(\beta)$, and if there is a $g \in \pi_y$ so that $g \cdot \alpha(i) = \beta(i)$ for each $i \in \mathrm{dom}(\alpha)$. The points of the space X are defined to be equivalence classes of pairs (y, α), with (y, α) equivalent to (z, β) iff $y = z$ and $\alpha \sim \beta$.

In this particular case, the topology of the space X, defined in general in Section 2, can be described more explicitly by using standard étale extensions. Fix for this an affine open $U = \mathrm{Spec}(R)$ of Y and (for some n) polynomials $p_1, \ldots, p_n$ in $R[T_1, \ldots, T_n]$ such that the determinant $\det(J)$ of the Jacobian $J = (\partial p_j / \partial T_k)_{j,k}$ is invertible in $R[T_1, \ldots, T_n]/(p_1, \ldots, p_n)$. Moreover, we fix a finite sequence of indices $i = (i_1, \ldots, i_n)$. Together these data define the open set

$$V = \{(y, \alpha) \mid y \in U,\ i_1, \ldots, i_n \in \mathrm{dom}(\alpha),$$
$$\text{and } p_k(\alpha(i_1), \ldots, \alpha(i_n)) = 0 \text{ for } k = 1, \ldots, n \}.$$

Note that this makes sense, since each p_k has coefficients in R, and R maps to the localization $R_y = \mathcal{O}_{Y,y}$ and then to $\mathcal{O}_{Y,y}^{\mathrm{hs}}$. Thus p_k can be evaluated at the tuple $(\alpha(i_1), \ldots, \alpha(i_n))$. These open sets of the form V generate the topology on X.

The construction of Section 2 gives for each étale sheaf $E \in \widetilde{Y_{\mathrm{\acute{e}t}}}$ a sheaf $\varphi^*(E)$ on this topological space X, with stalks $\varphi^*(E)_{(y,\alpha)} = E_{\overline{y}}$. Our theorem asserts that there is a natural isomorphism $\mathrm{H}^n(Y_{\mathrm{\acute{e}t}}, A) \cong \mathrm{H}^n(X, \varphi^* A)$, for any Abelian sheaf A on $Y_{\mathrm{\acute{e}t}}$ and any $n \geqslant 0$.

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