CANONICAL VARIABLES OF THE SECOND KIND AND THE REDUCTION OF THE N-BODY PROBLEM

JOHN G. BRYANT

47, avenue Felix Faure, 75015 Paris, France

Abstract. We introduce a new kind of canonical variables that prove very useful when the order of a Hamiltonian system can be reduced by one, as in the case of isoenergetic reduction, and of what we call homogeneous reduction. The Kepler Problem, Geometrical Optics and McGehee Blow-up are discussed as examples. Finally we carry out the isoenergetic reduction of the general N-Body Problem using the new variables, and briefly discuss its application to the problem of collision.

1. The Definition of Canonical Variables of the Second Kind

When we consider a Hamiltonian function written in the standard form $H(p_i, q_i) = T(p_i) - U(q_i)$ (where the (p_i, q_i) are a set of $2n$ canonical variables), and we want to reduce the corresponding Hamiltonian system thanks to the energy integral $H =$ constant $= h$, the question arises as to which variable we should choose to eliminate. The usual procedure is to eliminate one of the p_i, and the reduced system in the remaining variables is then a non-autonomous Hamiltonian system (with the role of the "time" variable now being played by the corresponding q_i). The drawback is that the individual p_i as a rule do not have any intrinsic physical meaning (since they usually depend on the choice of reference frame). Instead of eliminating one of the p_i, it would clearly be more useful to eliminate the function $T(p_i)$ itself, considered as a single variable, and which generally does have a definite physical meaning that is independent of the choice of reference frame. This is especially true in problems like the N-Body Problem, where $T(p_i)$ becomes infinite every time the system approaches a collision (since then $U(q_i)$ becomes infinite).

In the second part of this paper, we will show how the elimination of $T(p_i)$ in the N-Body Problem can be accomplished. First of all we will consider the somewhat simpler case where $T(p_i)$ is written

$$T(p_i) = \frac{p^2}{2} \text{ where } p = \sqrt{\sum_{i=1}^{n} p_i^2}$$

i.e. we consider the hypothetical motion of a particle with mass 1 in n-dimensional space.

The basic idea is to incorporate p in a new set of variables that, while no longer canonical in the traditional sense, still presents some remarkable properties. More precisely, we want to define a transformation

$$(p_i, q_i) \to (p, u, v, v_\lambda, w, w_\lambda)\ ;\ (\lambda = 1, \cdots, n-2)$$

Celestial Mechanics and Dynamical Astronomy **73**: 269–280, 1999.

characterized by the property that the so-called Liouville 1-form (Libermann and Marie, 1987), $\pi = \sum_{i=1}^{n} p_i dq_i$ is written in the new variables

$$\pi = p(du + vdw + \sum_{\lambda=1}^{n-2} v_\lambda dw_\lambda)$$

where

$$p = \sqrt{\sum_{i=1}^{n} p_i^2} \ ; \ u = \frac{1}{p}\sum_{i=1}^{n} p_i q_i \ ; \ v = \frac{1}{p}\sqrt{\sum_{\substack{i,j=1 \\ i<j}}^{n} (p_j q_i - q_j p_i)^2}$$

Note that we have the relation $u^2 + v^2 = \sum_{i=1}^{n} q_i^2$.

We call the new set of variables Canonical Variables of the Second Kind (CVSK). Since the above transformation is not a canonical transformation, the Hamiltonian system no longer has its usual form in the new variables. As we will see however, the new system can be easily computed, and will still be designated as Hamiltonian.

We start with the simplest case $n = 2$. We want to define a transformation $(p_1, p_2, q_1, q_2) \to (p, u, v, w)$ such that the 1-form $\pi = p_1 dq_1 + p_2 dq_2$ can be written $\pi = p(du - vdw)$. The obvious choice is to define "polar coordinates" for the p_i, i.e. we define w so that

$$p_1 = p\cos w \ ; \ p_2 = p\sin w$$

We can then write

$$\begin{aligned} \pi &= p_1 dq_1 + p_2 dq_2 \\ &= p(\cos w dq_1 + \sin w dq_2) \\ &= p[d(q_1 \cos w + q_2 \sin w) + (q_1 \sin w - q_2 \cos w)dw] \end{aligned}$$

Setting $u = q_1 \cos w + q_2 \sin w$; $v = q_1 \sin w - q_2 \cos w$ the transformation $(p_1, p_2, q_1, q_2) \to (p, u, v, w)$ is defined by the formulas

$$\begin{aligned} p &= \sqrt{p_1^2 + p_2^2} \ ; \ \tan w = \frac{p_2}{p_1} \\ u &= \frac{1}{p}(p_1 q_1 + p_2 q_2) \ ; \ v = \frac{1}{p}(p_2 q_1 - p_1 q_2) \end{aligned}$$

Note that it is always possible to have $v > 0$, if need be by changing the order of the (p_1, q_1) and the (p_2, q_2). To write the Hamiltonian equations in the new variables, we make use of the fact that, although the above transformation is not a canonical transformation, the transformation $(p_1, p_2, q_1, q_2) \to (p, p' = pv, u, w)$ *is* canonical, so that the equations in these variables *do* have standard form. We can thus easily deduce from the Hamiltonian system defined by the Hamiltonian $H' = H'(p, p', u, w)$

$$\frac{dp}{dt} = -\frac{\partial H'}{\partial u} \quad ; \quad \frac{du}{dt} = \frac{\partial H'}{\partial p} \quad ; \quad \frac{dp'}{\partial t} = -\frac{\partial H'}{\partial w} \quad ; \quad \frac{dw}{dt} = \frac{\partial H'}{\partial p'}$$

the corresponding system in the (p, u, v, w):

$$\begin{aligned} \frac{dp}{dt} &= -\frac{\partial H}{\partial u} & ; \quad \frac{du}{dt} &= \frac{\partial H}{\partial p} - \frac{v}{p}\frac{\partial H}{\partial v} \\ \frac{dv}{dt} &= -\frac{1}{p}\left(\frac{\partial H}{\partial w} - v\frac{\partial H}{\partial u}\right) & ; \quad \frac{dw}{dt} &= \frac{1}{p}\frac{\partial H}{\partial v} \end{aligned}$$

where we set $H(p, u, v, w) = H'(p, pv, u, w)$. Although the above system no longer has standard form we still call it Hamiltonian. Note that when $\partial H/\partial w = 0, dp'/dt = d(pv)/dt = 0$ and pv = constant.

The introduction of CVSK in the case $n = 2$ has several immediate applications.

a) The Kepler Problem : In variables (p_1, p_2, q_1, q_2), the Hamiltonian of the Kepler Problem can be written

$$H(p_1, p_2, q_1, q_2) = \frac{p_1^2 + p_2^2}{2} - \frac{\alpha}{\sqrt{q_1^2 + q_2^2}} \qquad (\alpha = \text{ constant})$$

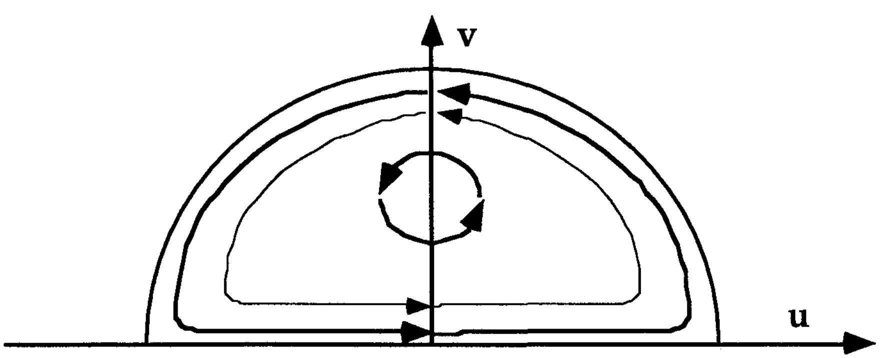
In variables (p, u, v, w), H is written

$$H(p, u, v, w) = \frac{p^2}{2} - \frac{\alpha}{\sqrt{u^2 + v^2}}$$

Since H does not depend on w, we have the first integral pv = constant $= C$. Setting H = constant $= h$, and replacing p by C/v, we have the following relation between u and v

$$\frac{C^2}{2v^2} - \frac{\alpha}{\sqrt{u^2 + v^2}} = h$$

For a given value of h, we have a one-parameter flow in the (u, v)-space (the parameter being C). In the case $h < 0$, this gives the following figure



https://doi.org/10.1017/S0252921100072626 Published online by Cambridge University Press

The flow along the u-axis, corresponding to $C = 0$, represents rectilinear motion. We see that the flow near the origin $(u = 0, v = 0)$, which corresponds to collision, is parallel to the u-axis, i.e., the flow is regular. The introduction of CVSK leads to "instant" regularization.

b) Geometrical Optics : The propagation of a light ray in a medium with refractive index $n(q_1, q_2)$ is governed by Fermat's well-known Principle of Least Time. This is a variational principle and the paths of the ray are obtained from a Lagrangian system. It is not generally known that the paths of the light ray (as well as the motion in time) can also be obtained from the Hamiltonian system associated with the following *homogeneous* Hamiltonian

$$H(p_1, p_2, q_1, q_2) = \frac{c\sqrt{p_1^2 + p_2^2}}{n(q_1, q_2)}$$

where c is the velocity of light in a vacuum. This is shown simply by comparing the two sets of Lagrangian and Hamiltonian equations. Introducing the CVSK and using the formulas $q_1 = u\cos w + v\sin w$; $q_2 = u\sin w - v\cos w$, we set $n(q_1, q_2) = \tilde{n}(u, v, w)$ and H is written

$$\tilde{H}(p, u, v, w) = cp/\tilde{n}(u, v, w)$$

The corresponding Hamiltonian system is written

$$\frac{dp}{dt} = \frac{cp}{\tilde{n}^2}\frac{\partial \tilde{n}}{\partial u} \quad ; \quad \frac{du}{dt} = \frac{c}{\tilde{n}} + \frac{cv}{\tilde{n}^2}\frac{\partial \tilde{n}}{\partial v}$$

$$\frac{dv}{dt} = \frac{c}{\tilde{n}^2}\left(\frac{\partial \tilde{n}}{\partial w} - v\frac{\partial \tilde{n}}{\partial u}\right) \quad ; \quad \frac{dw}{dt} = -\frac{c}{\tilde{n}^2}\frac{\partial \tilde{n}}{\partial v}$$

We see that the last three equations do not contain p, i.e. the system *separates.* The equations in the (u, v, w) determine what is known as a 3^{rd}-order contact system (Arnold, 1978). It has the form

$$\frac{du}{dt} = K - v\frac{\partial K}{\partial v} \; ; \; \frac{dv}{dt} = -\frac{\partial K}{\partial w} + v\frac{\partial K}{\partial u} \; ; \; \frac{dw}{dt} = \frac{\partial K}{\partial v}$$

where $K(u, v, w) = \frac{c}{\tilde{n}(u,v,w)}$ is known as the Contact Hamiltonian, and is equal in this case to the velocity of the light ray. This reduction by an order of one of a homogeneous Hamiltonian system is a general property (Bryant, 1983) that becomes immediately apparent when we use CVSK. Note that this type of reduction, which we call *homogeneous* reduction is distinct from the standard isoenergetic reduction and does not involve any new first integral of the motion.

c) McGehee Blow-up : This is a more elaborate case of homogeneous reduction. The starting point is the Generalized Kepler Problem. In a system of extended polar coordinates $(p_r, p_\varphi, r, \varphi)$ defined by the formulas

$$r = \sqrt{q_1^2 + q_2^2} \; ; \; p_r = \frac{1}{r}(p_1 q_1 + p_2 q_2) \; ; \; p_\varphi = p_2 q_1 - p_1 q_2 \; ; \; \tan\varphi = \frac{q_2}{q_1}$$

we consider the Hamiltonian

$$H(p_r, p_\varphi, r, \varphi) = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) - \frac{\mu(\varphi)}{r}$$

Depending on the form of the function $\mu(\varphi)$, the above Hamiltonian can represent the anisotropic Kepler problem, the rectilinear 3-Body Problem, the Planar Isoceles 3-Body Problem, and, when $\mu =$ constant, the ordinary Kepler Problem. The homogeneous nature of the potential function can be interpreted geometrically thanks to the vector field

$$Y' = -p_r \frac{\partial}{\partial p_r} + 2r\frac{\partial}{\partial r} + p_\varphi \frac{\partial}{\partial p_\varphi}$$

for which we have

$$Y'.H = -p_r \frac{\partial H}{\partial p_r} + 2r\frac{\partial H}{\partial r} + p_\varphi \frac{\partial H}{\partial p_\varphi} = -2H \; .$$

Associated with Y' is the $1-$form

$$\pi' = -p_r dr - 2r dp_r + p_\varphi d\varphi$$

which, like the standard Liouville $1-$form (whose expression in polar coordinates is $\pi = p_r dr + p_\varphi d\varphi$), verifies $d\pi' = \omega$, where $\omega = dp_r \wedge dr + dp_\varphi \wedge d\varphi$ is the symplectic 2-form. The idea is to have π' play the role that π has played up till now. The first step is to choose a set of variables (p_1', p_2', q_1', q_2') such that π' has the form $\pi' = p_1' dq_1' + p_2' dq_2'$. We can choose (Bryant, 1980)

$$p_1' = 2\sqrt{r} \; ; \; p_2' = p_\varphi \; ; \; q_1' = -p_r\sqrt{r} \; ; \; q_2' = \varphi$$

In these variables Y' is written $Y' = p_1' \frac{\partial}{\partial p_1'} + p_2' \frac{\partial}{\partial p_2'}$ and $Y'.H = p_1' \frac{\partial H}{\partial p_1'} + p_2' \frac{\partial H}{\partial p_2'} = -2H$ means that H is homogeneous of degree -2 in the (p_1', p_2'). The second step is to define CVSK for π', and for this we set

$$p' = p_1' = 2\sqrt{r} \; ; \; u' = q_1' = -p_r\sqrt{r} \; ; \; v' = \frac{p_2'}{p_1'} = \frac{p_\varphi}{2\sqrt{r}} \; ; \; w' = q_2' = \varphi$$

so that $\pi' = p'(du' + v'dw')$. Note that in these variables, $Y' = p'\frac{\partial}{\partial p'}$, and $Y'.H = p'\frac{\partial H}{\partial p'} = -2H$ means that H is homogeneous of degree -2 in p'. We now in fact have

$$H(p', u', v', w') = \frac{4}{p'^2}\left(\frac{u'^2}{2} + 2v'^2 - \mu(w')\right)$$

The Hamiltonian equations are written

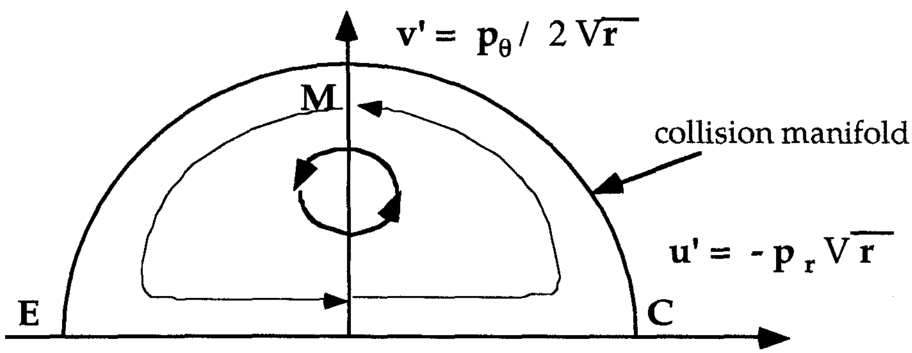
https://doi.org/10.1017/S0252921100072626 Published online by Cambridge University Press

$$\frac{dp'}{dt} = -\frac{\partial H}{\partial u'} = -\frac{4u'}{p'^2}\ ; \ \frac{du'}{dt} = \frac{\partial H}{\partial p'} - \frac{v'}{p'}\frac{\partial H}{\partial v'} = -\frac{4}{p'^3}(u'^2 + 8v'^2 - 2\mu(w'))\ ;$$

$$\frac{dv'}{dt} = -\frac{1}{p'}\left(\frac{\partial H}{\partial w'} - v'\frac{\partial H}{\partial u'}\right) = \frac{4}{p'^3}\left(\frac{d\mu}{dw'} + u'v'\right)\ ;\ \frac{dw'}{dt} = \frac{1}{p'}\frac{\partial H}{\partial v'} = \frac{16v'}{p'^3}$$

After the time change $dt' = 8p'^{-3}dt(= r^{-3/2}dt)$, the equations in u', v', w', which no longer depend on p', are equivalent to McGehee's equations (McGehee, 1974). We see that they result from the homogeneous reduction of the Generalized Kepler Problem, obtained by the introduction of a system of CVSK adapted to the 1-form π'.

In the case of the ordinary Kepler Problem (μ = constant), an interesting comparison can be made with the isoenergetic reduction given beforehand. As before, the problem can be reduced to a 2-dimensional flow, this time in the (u', v') space. Using the first integrals $H = h\ ;\ p'v' = C$ to eliminate p', we obtain a 1-parameter family of curves given by the equation $v'^2(2u'^2 + 8v'^2 - 4\mu) = k(= hC^2)$. When $k < 0$, we have the following figure



A near-collision orbit remains close to the u'-axis until it is near point C when it follows the "collision manifold", i.e. the limiting curve $2u'^2 + 8v'^2 - 4\mu = 0$, and reaches closest approach at point M on the v'-axis. McGehee variables entail a "blow-up" of the origin of physical space and allow a detailed description of near-collision orbits. They do not however supply us with a "natural" extension of actual collision orbits, which correspond to the flow on the u'-axis. Note that the flow near the origin ($u' = 0\ ;\ v' = 0$) now corresponds to the maximum value of r, and no longer has anything to do with collision.

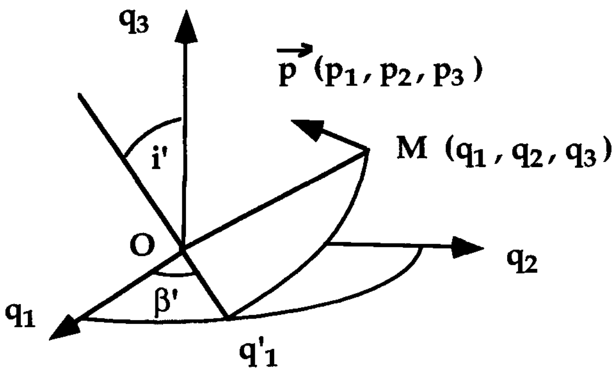
We now consider the case $n = 3$. In order to define a system of CVSK, we make use of an intermediate set of canonical variables. These are obtained thanks to the well-known Andoyer transformation (Boigey, 1981). This transformation goes from the variables $(p_1, p_2, p_3, q_1, q_2, q_3)$ to the Andoyer variables

https://doi.org/10.1017/S0252921100072626 Published online by Cambridge University Press

$(p'_1, p'_2, p'_\beta, q'_1, q'_2, \beta')$ and is defined by the formulas

$$\begin{array}{ll} p_1 = p'_1 \cos\beta' - p'_2 \sin\beta' \cos i' & q_1 = q'_1 \cos\beta' - q'_2 \sin\beta' \cos i' \\ p_2 = p'_1 \sin\beta' + p'_2 \cos\beta' \cos i' & q_2 = q'_1 \sin\beta' + q'_2 \cos\beta' \cos 1' \\ p_3 = p'_2 \sin i' & q_3 = q'_2 \sin i' \end{array}$$

where by definition $\cos i' = p'_\beta/(p'_2 q'_1 - p'_1 q'_2)$. It is easily checked that the transformation verifies $\pi = p_1 dq_1 + p_2 dq_2 + p_3 dq_3 = p'_1 dq'_1 + p'_2 dq'_2 + p'_\beta d\beta'$, i.e. it is a canonical transformation. We have the following geometrical interpretation



The Oq'_1-axis lies on the intersection of the horizontal Oq_1q_2-plane with the so-called plane of instantaneous motion determined by $\alpha OM(q_1, q_2, q_3)$ and $\mathbf{p}(p_1, p_2, p_3)$. The Oq'_2-axis is perpendicular to the Oq'_1-axis in the plane of instantaneous motion. We immediately verify that

$$\|\mathbf{p}\|^2 = p^2 = \sum_{i=1}^{3} p_i^2 = p_1'^2 + p_2'^2 \; ; \; \mathbf{OM.p} = \sum_{i=1}^{3} p_i q_i = p'_1 q'_1 + p'_2 q'_2$$

$$\|\mathbf{OM}\|^2 = \sum_{i=1}^{3} q_i^2 = q_1'^2 + q_2'^2 \; ; \; \|\mathbf{OM} \times \mathbf{p}\| = \sqrt{\sum_{\substack{i,j=1 \\ i<j}}^{3} (p_j q_i - q_j p_i)^2} = p'_2 q'_1 - p'_1 q'_2$$

All we now have to do is apply to the (p'_1, p'_2, q'_1, q'_2) the transformation defined in the case $n = 2$, i.e.

$$p = \sqrt{p_1'^2 + p_2'^2} \; ; \; \tan w = \frac{p'_2}{p'_1} \; ; \; u = \frac{1}{p}(p'_1 q'_1 + p'_2 q'_2) \; ; \; v = \frac{1}{p}(p'_2 q'_1 - p'_1 q'_2)$$

The (p, u, v, w) along with $v_1 = \frac{p'_\beta}{p}$ and $w_1 = \beta'$ verify

$$\pi = p'_1 dq'_1 + p'_2 dq'_2 + p'_\beta d\beta' = p(du + v dw + v_1 dw_1)$$

as well as the conditions on p, u and v given at the beginning of this paper, and therefore determine a system of CVSK.

For each successive value of n, we must first of all define an intermediate set of variables which are in fact generalized Andoyer variables. These are obtained by iteration of the standard Andoyer transformation. In the case $n = 4$ for example, we define a transformation $(p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4) \to (p_1'', p_2'', p_\beta', p_\beta'', q_1'', q_2'', \beta', \beta'')$ by two iterations of the Andoyer transformation. The first one, which leaves p_4 and q_4 invariant and goes from the $(p_1, p_2, p_3, q_1, q_2, q_3)$ to the $(p_1', p_2', p_\beta', q_1', q_2', \beta')$, is the transformation given previously. The second Andoyer transformation, which leaves p_β' and β' invariant, goes from the $(p_1', p_2', p_4, q_1', q_2', q_4)$ to the $(p_1'', p_2'', p_\beta'', q_1'', q_2'', \beta'')$, and is defined by the formulas

$$\begin{array}{ll} p_1' = p_1'' \cos\beta'' - p_2'' \sin\beta'' \cos i'' & q_1' = q_1'' \cos\beta'' - q_2'' \sin\beta'' \cos i'' \\ p_2' = p_1'' \sin\beta'' + p_2'' \cos\beta'' \cos i'' & q_2' = q_1'' \sin\beta'' + q_2'' \cos\beta'' \cos i'' \\ p_4 = p_2'' \sin i'' & q_4 = q_2'' \sin i'' \end{array}$$

where by definition $\cos i'' = p_\beta''/(p_2'' q_1'' - p_1'' q_2'')$. It is easily checked that the total transformation is canonical, since

$$\begin{aligned} \pi &= (p_1 dq_1 + p_2 dq_2 + p_3 dq_3) + p_4 dq_4 = (p_1' dq_1' + p_2' dq_2' + p_\beta' d\beta') + p_4 dq_4 \\ &= (p_1' dq_1' + p_2' dq_2' + p_4 dq_4) + p_\beta' d\beta' = (p_1'' dq_1'' + p_2'' dq_2'' + p_\beta'' d\beta'') + p_\beta' d\beta' \end{aligned}$$

and that the following relations hold

$$\begin{aligned} p^2 &= \sum_{i=1}^{4} p_i^2 = p_1'^2 + p_2'^2 + p_4^2 = p_1''^2 + p_2''^2 , \\ \sum_{i=1}^{4} q_i^2 &= q_1'^2 + q_2'^2 + q_4^2 = q_1''^2 + q_2''^2 \\ \sum_{i=1}^{4} p_i q_i &= p_1' q_1' + p_2' q_2' + p_4 q_4 = p_1'' q_1'' + p_2'' q_2'' \\ \sum_{\substack{i,j=1 \\ i<j}}^{4} (p_j q_i - q_j p_i)^2 &= (p_2' q_1' - p_1' q_2')^2 + (p_4 q_2' - p_2' q_4)^2 + (p_4 q_1' - p_1' q_4)^2 \\ &= (p_2'' q_1'' - p_1'' q_2'')^2 \end{aligned}$$

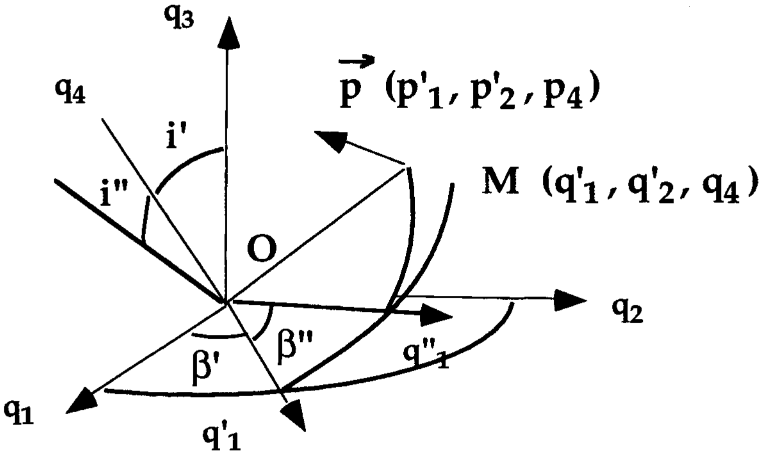
To determine a set of CVSK, we simply apply to the $(p_1'', p_2'', q_1'', q_2'')$ the same transformation as in the case $n = 2$, i.e.

$$p = \sqrt{p_1''^2 + p_2''^2} \; ; \; \tan w = \frac{p_2''}{p_1''} \; ; \; u = \frac{1}{p}(p_1'' q_1'' + p_2'' q_2'') \; ; \; v = \frac{1}{p}(p_2'' q_1'' - p_1'' q_2'')$$

and add the variables $v_1 = p'_\beta/p$; $v_2 = p''_\beta/p$; $w_1 = \beta'$; $w_2 = \beta''$, so that $\pi = p(du + vdw + v_1 dw_1 + v_2 dw_2)$. The Hamiltonian system can be easily computed, and is written in the new variables

$$\frac{dp}{dt} = -\frac{\partial H}{\partial u} ; \frac{du}{dt} = \frac{\partial H}{\partial p} - \frac{v}{p}\frac{\partial H}{\partial v} - \frac{v_1}{p}\frac{\partial H}{\partial v_1} - \frac{v_2}{p}\frac{\partial H}{\partial v_2} ;$$
$$\frac{dv}{dt} = -\frac{1}{p}\left(\frac{\partial H}{\partial w} - v\frac{\partial H}{\partial u}\right) ; \frac{dv_1}{dt} = -\frac{1}{p}\left(\frac{\partial H}{\partial w_1} - v_1\frac{\partial H}{\partial u}\right)$$
$$\frac{dv_2}{dt} = -\frac{1}{p}\left(\frac{\partial H}{\partial w_2} - v_2\frac{\partial H}{\partial u}\right)$$
$$\frac{dw}{dt} = \frac{1}{p}\frac{\partial H}{\partial v} ; \frac{dw_1}{dt} = \frac{1}{p}\frac{\partial H}{\partial v_1} ; \frac{dw_2}{dt} = \frac{1}{p}\frac{\partial H}{\partial v_2}$$

Although the transformation from the $(p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4)$ to the generalized Andoyer variables $(p''_1, p''_2, p'_\beta, p''_\beta, q''_1, q''_2, \beta', \beta'')$ is not easy to write explicitly, it does receive a simple geometrical interpretation, thanks to the following figure



The $0q_4$-axis is perpendicular to the $0q'_1q'_2$ plane. The $0q''_1$-axis lies on the intersection of the $0q'_1q'_2$-plane with the "pseudo plane of instantaneous motion" determined by $\alpha OM(q'_1, q'_2, q_4)$ and $\mathbf{p}(p'_1, p'_2, p_4)$. The $0q''_2$-axis is perpendicular to the $0q''_1$-axis in the pseudo plane of instantaneous motion.

The generalization of the above transformation to arbitrary n is accomplished by the iteration of the Andoyer transformation the number of times necessary, and then applying the transformation given at the beginning of this paper to the generalized Andoyer variables in order to obtain our system of CVSK.

2. The Reduction of the N-Body Problem

We take as our starting point the N-Body Problem formulated in Jacobi variables, for which the elimination of the center of mass motion is immediate, and allows us to interpret the reduced problem as the motion of $N-1$ fictitious bodies with Jacobi reduced masses in 3-dimensional space (Whittaker, 1927). Designating the Jacobi coordinates of each body and their conjugate momenta by $q_{\alpha i}, p_{\alpha i}(\alpha = 1, \cdots, N-1\ ;\ i = 1,2,3)$ and the corresponding reduced mass by m_α, the Hamiltonian function has the form

$$H(p_{\alpha i}, q_{\alpha i}) = T(p_{\alpha i}) - U(q_{\alpha i}) \text{ where } T(p_{\alpha i}) = \sum_{\alpha=1}^{N-1} \frac{p_\alpha^2}{2m_\alpha} \text{ and } p_\alpha^2 = \sum_{i=1}^{3} p_{\alpha i}^2$$

The Liouville 1-form is written $\pi = \sum_{\alpha=1}^{N-1} \pi_\alpha$ with $\pi_\alpha = \sum_{i=1}^{3} p_{\alpha i} dq_{\alpha i}$. We can eliminate the masses from the problem by "normalizing" the $(p_{\alpha i}, q_{\alpha i})$, i.e. by replacing them with $\tilde{p}_{\alpha i} = (1/\sqrt{m_\alpha}) p_{\alpha i}\ ;\ \tilde{q}_{\alpha i} = \sqrt{m_\alpha} q_{\alpha i}$. This simplifies the expression of $T \left(= \frac{1}{2} \sum_\alpha \tilde{p}_\alpha^2\right)$ and preserves the form of the π_α $(= \sum_i \tilde{p}_{\alpha i} d\tilde{q}_{\alpha i})$. (So as to not overburden the notation, we will keep on using $(p_{\alpha i}, q_{\alpha i})$ and assume normalization has already been carried out).

The definition of a system of CVSK is a two-step process. This is so we can take account of the fact that the variables come in $N-1$ groups, and that the problem is invariant by rotation (i.e. there exists an angular momentum integral). The first step is to define CVSK for *each one* of the $N-1$ sets of variables $(p_{\alpha i}, q_{\alpha i})$. This means replacing them by $N-1$ sets $(p_\alpha, u_\alpha, v_\alpha, v_{1\alpha}, w_\alpha, w_{1\alpha})$ according to the transformation corresponding to the case $n=3$. We therefore have

$$\pi_\alpha = \sum_{i=1}^{3} p_{\alpha i} dq_{\alpha i} = p_\alpha (du_\alpha + v_\alpha dw_\alpha + v_{1\alpha} dw_{1\alpha})$$

along with the usual relations

$$p_\alpha^2 = \textstyle\sum_{i=1}^{3} p_{\alpha i}^2 \quad ;\ u_\alpha = \frac{1}{p_\alpha} \sum_{i=1}^{3} p_{\alpha i} q_{\alpha i}\ ,$$

$$v_\alpha = \frac{1}{p_\alpha} \sqrt{\textstyle\sum_{\substack{i,j=1 \\ i<j}}^{3} (p_{\alpha j} q_{\alpha i} - q_{\alpha j} p_{\alpha i})^2} \ ;\ u_\alpha^2 + v_\alpha^2 = \sum_{i=1}^{3} q_{\alpha i}^2$$

We can write

$$\pi = \sum_{\alpha=1}^{N-1} \pi_\alpha = \sum_{\alpha=1}^{N-1} p_\alpha du_\alpha + \sum_{\alpha=1}^{N-1} (p_\alpha v_\alpha) dw_\alpha + \sum_{\alpha=1}^{N-1} (p_\alpha v_{1\alpha}) dw_{1\alpha}$$

The second step is to define a system of CVSK *for the* (p_α, u_α), i.e. we must define a system of $2(N-1)$ variables $(p, \overline{u}, \overline{v}, \overline{v}_\lambda, \overline{w}, \overline{w}_\lambda)$, $(\lambda = 1, \cdots, N-3)$, such that

https://doi.org/10.1017/S0252921100072626 Published online by Cambridge University Press

$$\overline{\pi} = \sum_{\alpha=1}^{N-1} p_\alpha du_\alpha = p(d\overline{u} + \overline{v}d\overline{w} + \sum_{\lambda=1}^{N-3} \overline{v}_\lambda d\overline{w}_\lambda)$$

$$p^2 = \sum_{\alpha=1}^{N-1} p_\alpha^2 \; ; \; \overline{u} = \frac{1}{p} \sum_{\alpha=1}^{N-1} p_\alpha u_\alpha$$

$$\overline{v} = \frac{1}{p} \sqrt{\sum_{\substack{\alpha,\beta=1 \\ \alpha<\beta}}^{N-1} (p_\beta u_\alpha - u_\beta p_\alpha)^2} \; ; \; \overline{u}^2 + \overline{v}^2 = \sum_{\alpha=1}^{N-1} u_\alpha^2$$

To accomplish this, we simply make use of the general transformation with the u_α in the place of the q_i, and where $n = N - 1$. Having thus defined the $(p, \overline{u}, \overline{v}, \overline{v}_\lambda, \overline{w}, \overline{w}_\lambda)$, we add the variables $\tilde{v}_\alpha = p_\alpha v_\alpha / p$; $\tilde{v}_{1\alpha} = p_\alpha v_{1\alpha}/p$, and the Liouville 1-form π takes on the desired form

$$\pi = p \left(d\overline{u} + \overline{v}d\overline{w} + \sum_{\lambda=1}^{N-3} \overline{v}_\lambda d\overline{w}_\lambda + \sum_{\alpha=1}^{N-1} \tilde{v}_\alpha dw_\alpha + \sum_{\alpha=1}^{N-1} \tilde{v}_{1\alpha} dw_{1\alpha} \right)$$

Note that it is possible to give a simple expression to the length C of the angular momentum vector. For this we must assume that this vector lies along the vertical axis. We then have the classic result that C is the sum of the projections of the angular momentum of each of the $N - 1$ bodies on the vertical axis, i.e. $C = \sum_\alpha p_\alpha v_{1\alpha} = p \sum_\alpha \tilde{v}_{1\alpha}$.

The $(q_{\alpha i})$, as well as the potential function $U(q_{\alpha i})$, when expressed in terms of the new variables, do not depend explicitly on p. To see this, we consider the so-called homothetical vector field associated with the Liouville 1-form, which has the following expression in terms of the $(p_{\alpha i}, q_{\alpha i})$

$$Y = \sum_{\alpha=1}^{N-1} \sum_{i=1}^{3} p_{\alpha i} \frac{\partial}{\partial p_{\alpha i}}$$

We therefore have $Y.q_{\alpha i} = 0$. Expressed in the new variables, Y is written simply $Y = p\frac{\partial}{\partial p}$, and the relation $Y.q_{\alpha i} = p\frac{\partial q_{\alpha i}}{\partial p} = 0$ means that $q_{\alpha i}$ and hence $U(q_{\alpha i})$ do not depend on p. It follows that since $T(p_{\alpha i}) = \frac{1}{2}p^2$, the energy integral $H = T - U = \text{constant} = h$ can be easily solved for p, i.e.

$$p = \sqrt{2(U + h)}$$

Replacing p by the above expression in the Hamiltonian equations effectively carries out the isoenergetic reduction of the system.

When the motion approaches collision, $U(q_{\alpha i})$ becomes infinite, and, because of the above relation, so does p. A remarkable feature of the CVSK defined for the problem is that all of the other variables remain finite. This is a direct consequence of the method we used to construct the system of CVSK. Therefore, once p has been eliminated, collision and near-collision orbits in the reduced problem belong to a finite domain in the vicinity of collision.

The above property leads to a "practical" method for extending total collision orbits. A classical property of total collision orbits is that the length C of the angular momentum is zero. The idea for extending such orbits through collision is to incorporate them in a one-parameter continuous family of orbits, the parameter being C. The only total collision orbit in the family is therefore the one where $C = 0$, which appears as a limiting case since we assume $C \geq 0$. If none of the orbits of the family exhibits a lower-order collision, all the orbits except the total collision one are well-defined and finite. A limiting orbit for the family therefore always exists when $C \to 0$, and this is the orbit we adopt as the extension of the collision orbit. Note that this method only extends the geometrical orbits through collision, but not the actual Hamiltonian system (i.e. it has not been regularized).

In a subsequent paper we will describe the above method for extending total collision orbits in more detail, and give several concrete examples. We will also show how to handle lower-order collisions.

References

Arnold, V.I.:1978, *Mathematical methods of classical mechanics*, Graduate texts in mathematics, 60, Springer Verlag.

Boigey, F.: 1981, Transformations canoniques a variables imposées,. Applications a la mecanique celeste, These, Universite P. et M. Curie (Paris), .

Bryant, J.: 1983, Le formalisme de contact en mecanique classique et relativiste, *Ann. Inst. Henri Poincare, Physique Theorique*, **38**, 121–152

Bryant, J.: 1980, Sur une reduction du probleme des N-corps mettant en evidence une structure de contact, *C.R. Acad. Sc. Paris*, **291 A**, 205–208

Libermann, P. and Marie, C-M.: 1987, *Symplectic geometry and analytical mechanics*, Kluwer Academic, .

McGehee, R.: 1974, Triple collision in the collinear three-body problem, *Inv. Math*, **27**, 191–227

Whittaker, E.T.: 1927, *A treatise on the analytical dynamics of particles and rigid bodies*, Cambridge University Press