Forum of Mathematics, Pi (2022), **Vol. 10**:e19 1–53
doi:10.1017/fmp.2022.13

CAMBRIDGE UNIVERSITY PRESS

RESEARCH ARTICLE

On genus one mirror symmetry in higher dimensions and the BCOV conjectures

Dennis Eriksson[1], Gerard Freixas i Montplet[2] and Christophe Mourougane[3]

[1]Chalmers University of Technology and University of Gothenburg, Department of Mathematics, Sweden;
E-mail: dener@chalmers.se.
[2]CNRS, Institut de Mathématiques de Jussieu - Paris Rive Gauche, France; E-mail: gerard.freixas@imj-prg.fr.
[3]Université de Rennes 1, CNRS, IRMAR - UMR 6625, F-35000 Rennes, France;
E-mail: christophe.mourougane@univ-rennes1.fr.

Received: 16 June 2020; **Revised:** 11 May 2022; **Accepted:** 18 June 2022

2020 Mathematics Subject Classification: *Primary* – 14J32, 14J33, 58J52; *Secondary* – 32G20

Abstract

The mathematical physicists Bershadsky–Cecotti–Ooguri–Vafa (BCOV) proposed, in a seminal article from 1994, a conjecture extending genus zero mirror symmetry to higher genera. With a view towards a refined formulation of the Grothendieck–Riemann–Roch theorem, we offer a mathematical description of the BCOV conjecture at genus one. As an application of the arithmetic Riemann–Roch theorem of Gillet–Soulé and our previous results on the BCOV invariant, we establish this conjecture for Calabi–Yau hypersurfaces in projective spaces. Our contribution takes place on the B-side, and together with the work of Zinger on the A-side, it provides the first complete examples of the mirror symmetry program in higher dimensions. The case of quintic threefolds was studied by Fang–Lu–Yoshikawa. Our approach also lends itself to arithmetic considerations of the BCOV invariant, and we study a Chowla–Selberg type theorem expressing it in terms of special Γ-values for certain Calabi–Yau manifolds with complex multiplication.

To Jean-Pierre Demailly, in memoriam.

Contents

1. Introduction

The purpose of this article is to establish higher-dimensional cases of genus one mirror symmetry, as envisioned by mathematical physicists Bershadsky–Cecotti–Ooguri–Vafa (henceforth abbreviated BCOV) in their influential paper [BCOV94]. We relate the generating series of genus one Gromov–Witten invariants on Calabi–Yau hypersurfaces to an invariant of a mirror family built out of holomorphic analytic torsions. The invariant, whose existence was conjectured in [BCOV94], was mathematically defined and studied in our previous paper [EFiMM21]. We refer to it as the BCOV invariant τ_{BCOV}. In dimension 3, the construction of the BCOV invariant and its relation to mirror symmetry were established by Fang–Lu–Yoshikawa [FLY08], relying on previous results by [Zin08, Zin09].

Our approach parallels the Kodaira–Spencer formulation of the Yukawa coupling in genus zero and can be recast as a refined version of the Grothendieck–Riemann–Roch theorem *à la* Deligne [Del87]. We hope this point of view will also be inspiring to study higher genus Gromov–Witten invariants and the B-side of mirror symmetry in dimension 3. In this setting, the A-side has received a lot of attention recently.

1.1. The classical BCOV conjecture at genus one

Let X be a Calabi–Yau manifold of dimension n. In this article, this will mean a complex projective connected manifold with trivial canonical sheaf. We now briefly recall the BCOV program at genus one.

On the one hand, on what is referred to as the A-side, we consider enumerative invariants associated to X. For this, recall first that for every curve class β in $H_2(X,\mathbb{Z})$, there is a proper Deligne–Mumford stack of stable maps from genus g curves to X, whose fundamental class is β:

$$\overline{\mathcal{M}}_g(X,\beta) = \{f : C \to X \mid g(C) = g,\ f \text{ stable and } f_*[C] = \beta\}\,.$$

The virtual dimension of this stack can be computed to be (see [Beh97], in particular the introduction)

$$\int_\beta c_1(X) + (\dim(X) - 3)(1-g) = (\dim(X) - 3)(1-g).$$

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

Whenever $\dim(X) = 3$ or $g = 1$, this is of virtual dimension 0, and one can consider the Gromov–Witten invariants

$$\mathrm{GW}_g(X, \beta) = \deg\, [\overline{\mathcal{M}}_g(X, \beta)]^{\mathrm{vir}} \in \mathbb{Q}.$$

Since the main focus of our paper is higher dimensions, we henceforth impose $g = 1$. One then defines the formal power series

$$F_1^A(\tau) = \frac{-1}{24} \int_X \mathrm{c}_{n-1}(X) \cap 2\pi i\tau + \sum_{\beta > 0} \mathrm{GW}_1(X, \beta) e^{2\pi i \langle \tau, \beta \rangle}, \tag{1.1}$$

where τ belongs to the complexified Kähler cone[1] $\mathcal{H}_X$, and β runs over the nonzero effective curve classes.

On the other hand, on what is referred to as the B-side, BCOV introduced a spectral quantity $\mathcal{F}_1^B$ built out of holomorphic Ray–Singer analytic torsions of a mirror Calabi–Yau manifold $X^\vee$. It depends on an auxiliary choice of a Kähler structure ω on $X^\vee$ and can be recast as

$$\mathcal{F}_1^B(X^\vee, \omega) = \prod_{0 \le p,q \le n} (\det \Delta_{\overline{\partial}}^{p,q})^{(-1)^{p+q} pq},$$

where $\det \Delta_{\overline{\partial}}^{p,q}$ is the ζ-regularised determinant of the Dolbeault Laplacian acting on $A^{p,q}(X^\vee)$. In our previous work [EFiMM21], we normalised this quantity to make it independent of the choice of ω:

$$\tau_{\mathrm{BCOV}}(X^\vee) = C(X^\vee, \omega) \cdot \mathcal{F}_1^B(X^\vee, \omega),$$

for some explicit constant $C(X^\vee, \omega)$. Thus $\tau_{\mathrm{BCOV}}(X^\vee)$ only depends on the complex structure of the Calabi–Yau manifold, in accordance with the philosophy that the B-model only depends on variations of the complex structure on $X^\vee$.

Mirror symmetry predicts that given X, there is a mirror family of Calabi–Yau manifolds over a punctured multi-disc around the origin $\varphi\colon \mathcal{X}^\vee \to \mathbf{D}^\times = (\mathbb{D}^\times)^d$, with maximally unipotent monodromies and $d = h^{1,1}(X) = h^1(T_{X^\vee})$.[2] Here we denoted by $X^\vee$ any member of the mirror family. The A-side and B-side should be related by a distinguished biholomorphism onto its image $\mathbf{D}^\times \to \mathcal{H}_X$, which is referred to as the mirror map and is denoted $q \mapsto \tau(q)$. The mirror map sends the origin of the multi-disc to infinity. Fixing a basis of ample classes on X, we can think of it as a change of coordinates on $\mathbf{D}^\times$. In the special case of $d = 1$, one such a map is constructed as a quotient of carefully selected periods in [Mor93].

BCOV conjecture at genus one. Let X be a Calabi–Yau manifold and $\varphi\colon \mathcal{X}^\vee \to \mathbf{D}^\times$ a mirror family as above:

1. There is a procedure called *passing to the holomorphic limit* to extract from $\tau_{\mathrm{BCOV}}(\mathcal{X}_q^\vee)$ as $q \to 0$ a holomorphic function $F_1^B(q)$.
2. The functions F_1^A and F_1^B are related via the mirror map by

$$F_1^B(q) = F_1^A(\tau(q)).$$

Passing to the holomorphic limit is often interpreted as considering a Taylor expansion of $\tau_{\mathrm{BCOV}}(\mathcal{X}_q^\vee)$ in $\tau(q)$ and $\overline{\tau(q)}$ and keeping the holomorphic part. In this article, we will instead use a procedure based on degenerations of Hodge structures.

[1] If $\mathcal{K}_X$ denotes the Kähler cone of X, we define $\mathcal{H}_X$ as $H^{1,1}_{\mathbb{R}}(X)/H^{1,1}_{\mathbb{Z}}(X) + i\mathcal{K}_X$.
[2] Such families are also called *large complex structure limits* of Calabi–Yau manifolds.

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

1.2. Grothendieck–Riemann–Roch formulation of the BCOV conjecture at genus one

The purpose of this subsection is to formulate a version of the BCOV conjecture producing the holomorphic function F_1^B without any reference to spectral theory, holomorphic anomaly equations or holomorphic limits. Our formulation parallels the Hodge theoretic approach to the Yukawa coupling in 3-dimensional genus zero mirror symmetry: the key ingredients going into its construction are the Kodaira–Spencer mappings between Hodge bundles and canonical trivialisations of those.

To state a simplified form of our conjecture, we need to introduce the BCOV line bundle $\lambda_{\mathrm{BCOV}}(\mathcal{X}^\vee/\mathbf{D}^\times)$ of the mirror family $\varphi\colon \mathcal{X}^\vee \to \mathbf{D}^\times$. The BCOV line of a Calabi–Yau manifold $X^\vee$ is defined to be

$$\lambda_{\mathrm{BCOV}}(X^\vee) = \bigotimes_{0\leq p,q\leq n} \det H^q(X^\vee, \Omega^p_{X^\vee})^{(-1)^{p+q}p}.$$

For a family of Calabi–Yau manifolds, it glues together into a holomorphic line bundle on the base. Also, we denote by χ the Euler characteristic of any fibre of φ and by $K_{\mathcal{X}^\vee/\mathbf{D}^\times}$ the relative canonical bundle.

Refined BCOV conjecture at genus one. Let X be a Calabi–Yau manifold and $\varphi\colon \mathcal{X}^\vee \to \mathbf{D}^\times$ a mirror family as in §1.1:

1. There exists a natural isomorphism of line bundles,

$$\mathbf{GRR}\colon \lambda_{\mathrm{BCOV}}(\mathcal{X}^\vee/\mathbf{D}^\times)^{\otimes 12\kappa} \xrightarrow{\sim} \varphi_*(K_{\mathcal{X}^\vee/\mathbf{D}^\times})^{\otimes \chi\kappa}, \tag{1.2}$$

   together with natural trivialising sections of both sides. Here κ is a nonzero integer that only depends on the relative dimension of φ.
2. In the natural trivialisations, the isomorphism **GRR** can be expressed as a holomorphic function, which when written as $\exp\left((-1)^n F_1^B(q)\right)^{24\kappa}$ satisfies

$$F_1^B(q) = F_1^A(\tau(q)).$$

The existence of some isomorphism as in (1.2) is provided by the Grothendieck–Riemann–Roch theorem in Chow theory, the key point of the conjecture being the naturality requirement. In fact, an influential program by Deligne [Del87] suggests that the codimension one part of the usual Grothendieck–Riemann–Roch equality can be lifted to a base change invariant isometry of line bundles when equipped with natural metrics. An intermediate version of this exists via the arithmetic Riemann–Roch theorem of Gillet–Soulé [GS92], which provides an equality of isometry classes of hermitian line bundles. Properly interpreted, this establishes a link between the BCOV invariant and a metric evaluation of (1.2).

A more detailed treatment of the formulation of the conjecture is given in Section 6, and examples related to the existing literature are also discussed. We focus on mirror families with a strong degeneration property formalised by Deligne in [Del] and expressed as a Hodge–Tate condition on the limiting Hodge structures of all the cohomology groups. In this case, from general principles in the theory of degenerations of Hodge structures, we can indeed construct natural trivialisations of the line bundles in (1.2).

1.3. Main results

In this subsection, we discuss the framework and statements of our results. For Calabi–Yau hypersurfaces in projective space, our main theorem settles the BCOV conjecture and its refinement.

Let X be a Calabi–Yau hypersurface in $\mathbb{P}^n_{\mathbb{C}}$, with $n \geq 4$. Its complexified Kähler cone is one-dimensional, induced by restriction from that of the ambient projective space. The mirror family

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

$f\colon \mathcal{Z} \to U$ can be realised using a crepant resolution of the quotient of the Dwork pencil

$$x_0^{n+1} + \ldots + x_n^{n+1} - (n+1)\psi x_0 \ldots x_n = 0, \quad \psi \in U = \mathbb{C} \setminus \mu_{n+1} \tag{1.3}$$

by the subgroup of $\mathsf{GL}_{n+1}(\mathbb{C})$ given by $G = \left\{g \cdot (x_0, \ldots, x_n) = (\xi_0 x_0, \ldots, \xi_n x_n), \xi_i^{n+1} = 1, \prod \xi_i = 1\right\}$. Moreover, $f\colon \mathcal{Z} \to U$ can be naturally extended across μ_{n+1} to a degeneration with ordinary double point singularities, sometimes referred to as a conifold degeneration.

The monodromy around $\psi = \infty$ is maximally unipotent, and the properties of the limiting Hodge structure can be used to define a natural flag of homology cycles. Using this, we can produce natural holomorphic trivialisations $\widetilde{\eta}_k$, in a neighbourhood of $\psi = \infty$, of the determinants of the primitive Hodge bundles $\det(R^k f_* \Omega^{n-1-k}_{\mathcal{Z}/U})_{\mathrm{prim}}$.[3] These holomorphic trivialisations have unipotent lower triangular period matrices. These sections have natural L^2 norms given by Hodge theory. The product $\otimes_{k=0}^{n-1} \widetilde{\eta}_k^{(n-1-k)(-1)^{n-1}}$ is the essential building block of a natural frame $\widetilde{\eta}_{\mathrm{BCOV}}$ of $\lambda_{\mathrm{BCOV}}(\mathcal{Z}/U)$.

Finally, let $F_1^A(\tau(\psi))$ be the generating series defined as in (1.1) for a general Calabi–Yau hypersurface $X \subset \mathbb{P}^n_{\mathbb{C}}$. Here $\psi \mapsto \tau(\psi)$ is the mirror map. Then our main result (Theorem 5.9 and Theorem 6.13) can be stated as follows:[4]

Main Theorem. Let $n \geq 4$. Consider a Calabi–Yau hypersurface $X \subset \mathbb{P}^n_{\mathbb{C}}$ and the mirror family $f\colon \mathcal{Z} \to U$ above.

1. In a neighbourhood of infinity, the BCOV invariant of Z_ψ factors as

$$\tau_{\mathrm{BCOV}}(Z_\psi) = C \left|\exp\left((-1)^{n-1} F_1^B(\psi)\right)\right|^4 \left(\frac{\|\widetilde{\eta}_0\|_{L^2}^{\chi(Z_\psi)/12}}{\|\widetilde{\eta}_{\mathrm{BCOV}}\|_{L^2}}\right)^2,$$

   where $F_1^B(\psi)$ is a multivalued holomorphic function with $F_1^B(\psi) = F_1^A(\tau(\psi))$ as formal series in ψ, and C is a positive constant;
2. Up to a constant, the refined BCOV conjecture at genus one is true for X and its mirror family, with the choices of trivialising sections $\widetilde{\eta}_{\mathrm{BCOV}}$ and $\widetilde{\eta}_0$.

Actually, the theorem also holds in the case of cubic curves (as follows from §1.5) and quartic surfaces. We also show, more generally, in Proposition 6.14 that the refined BCOV conjecture holds, up to a constant, for $K3$ surfaces.

The first part of the theorem extends to arbitrary dimensions previous work of Fang–Lu–Yoshikawa [FLY08, Thm. 1.3] in dimension 3. In their approach, all the Hodge bundles have geometric meaning in terms of Weil–Petersson geometry and Kuranishi families. The lack thereof is an additional complication in our setting.

To our knowledge, our theorem is the first complete example of higher-dimensional mirror symmetry, of BCOV type at genus one, established in the mathematics literature. It confirms various instances that had informally been utilised for computational purposes, such as [KP08, Sec. 6] in dimension 4.

We remark that there is an alternative approach to the BCOV theory, in arbitrary genera, provided by Costello and Li, described in their preprint [CL12]. It would be interesting to compare the results in this article with their program.

1.4. Overview of proof of the main theorem

Arithmetic Riemann–Roch

In the algebro-geometric setting, the arithmetic Riemann–Roch theorem from Arakelov theory allows us to compute the BCOV invariant of a family of Calabi–Yau varieties in terms of L^2 norms of auxiliary

[3] The primitive Hodge bundle $(R^k f_* \Omega^{n-1-k}_{\mathcal{Z}/U})_{\mathrm{prim}}$ is actually of rank one if $2k \neq n-1$.

[4] To facilitate the comparison with the BCOV conjecture, notice that X now has dimension $n-1$ instead of n.

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

sections of Hodge bundles. This bypasses some arguments in former approaches, such as [FLY08], based on the holomorphic anomaly equation (see [EFiMM21, Prop. 5.9]). It determines the BCOV invariant up to a meromorphic function – in fact, a rational function.[5] The divisor of this rational function is encapsulated in the asymptotics of the L^2 norms and the BCOV invariant. In the special case when the base is a Zariski open set of $\mathbb{P}^1_{\mathbb{C}}$, as for the Dwork pencil (1.3) and the mirror family, this divisor is determined by all but one point. Hence so is the function itself, up to constant. The arithmetic Riemann–Roch theorem simultaneously allows us to establish the existence of an isomorphism **GRR** as in (1.2).

Hodge bundles of the mirror family
The construction of the auxiliary sections is first of all based on a comparison of the Hodge bundles of the mirror family with the G-invariant part of the Hodge bundles on the Dwork pencil in (1.3), explained in Section 3. Using the residue method of Griffiths, we construct algebraic sections of the latter. These are then transported into sections η_k of the Hodge bundles of the crepant resolution: that is, the mirror family. This leads us to a systematic geometric study of these sections in connection with Deligne extensions and limiting Hodge structures at various key points, notably at μ_{n+1}, where ordinary double point singularities arise. We rely heavily on knowledge of the Yukawa coupling and our previous work in [EFiMM21, Sec. 2] on logarithmic Hodge bundles and semi-stable reduction. The arguments are elaborated in Section 4.

Asymptotics of L^2 norms and the BCOV invariant
The above arithmetic Riemann–Roch reduction leads us to study the norm of the auxiliary sections outside of the maximally unipotent monodromy point, enabling us to focus on ordinary double points. Applying our previous result [EFiMM21, Thm. 4.4] to the auxiliary sections, we find that the behaviour of their L^2 norms is expressed in terms of monodromy eigenvalues and the possible zeros or poles as determined by the geometric considerations of the preceding paragraph. The monodromy is characterised by the Picard–Lefschetz theorem. As for the asymptotics for the BCOV invariant, they were already accomplished in [EFiMM21, Thm. B]. This endeavor results in Theorem 5.1, which is a description of the rational function occurring in the arithmetic Riemann–Roch theorem.

Connection to enumerative geometry
The BCOV conjecture suggests that we need to study the BCOV invariant close to $\psi = \infty$. However, the formula in Theorem 5.1 is not adapted to the mirror symmetry setting: for example, the sections η_k do not make any reference to $H^{n-1}_{\lim}$. We proceed to normalise the η_k by dividing by holomorphic periods, for a fixed basis of the weight filtration on the homology $(H_{n-1})_{\lim}$, to obtain the sections $\widetilde{\eta}_k$ of the main theorem. Rephrasing Theorem 5.1 with these sections, we thus arrive at an expression for the F_1^B in the theorem. Combined with results of Zinger [Zin08, Zin09], this yields the relation to the generating series of Gromov–Witten invariants in the mirror coordinate. Lastly, the refined BCOV conjecture is deduced in this case through a reinterpretation of the BCOV invariant and the arithmetic Riemann–Roch theorem.

1.5. Applications to Kronecker limit formulas

Classical first Kronecker limit formula
The simplest Calabi–Yau varieties are elliptic curves, which can conveniently be presented as $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, for τ in the Poincaré upper half-plane. The generating series in (1.1) of Gromov–Witten invariants is then given by $-\frac{1}{24}\log\Delta(\tau)$, where $\Delta(\tau) = q\prod(1-q^n)^{24}$ and $q = e^{2\pi i\tau}$. The corresponding function

[5] This rational function compares to the so-called *holomorphic ambiguity* in the physics literature.

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

$\mathcal{F}_1^B$ is computed as $\exp(\zeta_\tau'(0))$, where

$$\zeta_\tau(s) = (2\pi)^{-2s} \sum_{(m,n)\neq(0,0)} \frac{(\operatorname{Im}\tau)^s}{|m+n\tau|^{2s}}.$$

The BCOV conjecture at genus one is deduced from the equality

$$\exp(-\zeta_\tau'(0)) = \frac{1}{(2\pi)^2} \operatorname{Im}(\tau)|\Delta(\tau)|^{1/6}. \tag{1.4}$$

This is a formulation of the first Kronecker limit formula; see, for example, [Yos99, Intro.]. In the mirror symmetry interpretation, the correspondence $\tau \mapsto q$ is the (inverse) mirror map. Equation (1.4) can be recovered from a standard application of the arithmetic Riemann–Roch theorem. In this vein, we will interpret all results of this shape as generalisations of the Kronecker limit formula. This includes the Theorem 5.1 cited above, as well as a Theorem 2.6 for Calabi–Yau hypersurfaces in Fano manifolds.

Chowla–Selberg formula

While being applicable to algebraic varieties over $\mathbb{C}$, the Riemann–Roch theorem in Arakelov geometry has the further advantage of providing arithmetic information when the varieties are defined over $\mathbb{Q}$. The arithmetic Riemann–Roch theorem is suited to evaluating the BCOV invariant of certain arithmetically defined Calabi–Yau varieties with additional automorphisms. As an example, for the special fibre Z_0 of our mirror family in (1.3), Theorem 7.2 computes the BCOV invariant as a product of special values of the Γ-function. This is reminiscent of the Chowla–Selberg theorem [SC67], which derives from (1.4) an expression of the periods of a CM elliptic curve as a product of special Γ-values. Assuming deep conjectures of Gross–Deligne [Gro78], we would be able to write any BCOV invariant of a CM Calabi–Yau manifold in such terms.

2. The BCOV invariant and the arithmetic Riemann–Roch theorem

In this section, we describe a general method to express the BCOV invariant of a family of Calabi–Yau varieties in terms of L^2 norms of rational sections of determinants of Hodge bundles. The approach is based on the arithmetic Riemann–Roch theorem. As an application, we consider the case of the universal family of Calabi–Yau hypersurfaces in the projective space.

2.1. Kähler manifolds and L^2 norms

Let X be a compact complex manifold. In this article, a hermitian metric on X means a smooth hermitian metric on the holomorphic vector bundle T_X. Let h be a hermitian metric on X. The Arakelov theoretic Kähler form attached to h is given in local holomorphic coordinates by

$$\omega = \frac{i}{2\pi} \sum_{j,k} h\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right) dz_j \wedge d\overline{z}_k. \tag{2.1}$$

We assume that the complex hermitian manifold (X, h) is Kähler: that is, the differential form ω is closed. The hermitian metric h induces hermitian metrics on the $\mathcal{C}^\infty$ vector bundles of differential forms of type (p, q), which we still denote h. Then the spaces $A^{p,q}(X)$ of global sections inherit a L^2 hermitian inner product

$$h_{L^2}(\alpha, \beta) = \int_X h(\alpha, \beta) \frac{\omega^n}{n!}. \tag{2.2}$$

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

The coherent cohomology groups $H^q(X, \Omega_X^p)$ can be computed as Dolbeault cohomology, which in turn can be computed in $A^{p,q}(X)$ by taking $\overline{\partial}$-harmonic representatives. Via this identification, $H^q(X, \Omega_X^p)$ inherits a L^2 inner product. Similarly, the hermitian metric h also induces hermitian metrics on the vector bundles and spaces of complex differential forms of degree k. The complex de Rham cohomology $H^k(X, \mathbb{C})$ has an induced L^2 inner product by taking d-harmonic representatives. The canonical Hodge decomposition

$$H^k(X, \mathbb{C}) \simeq \overset{\perp}{\bigoplus_{p,q}} H^q(X, \Omega_X^p)$$

is an isometry for the L^2 metrics.

2.2. The BCOV invariant

We briefly recall the construction of the BCOV invariant [EFiMM21, Sec. 5]. Let X be a Calabi–Yau manifold of dimension n. Fix a Kähler metric h on X, with Kähler form ω as in (2.1). Let $T(\Omega_X^p, \omega)$ be the holomorphic analytic torsion of the vector bundle Ω_X^p of holomorphic differential p-forms endowed with the metric induced by h and with respect to the Kähler form ω on X. The *BCOV torsion* of (X, ω) is

$$T(X, \omega) = \prod_{0 \leq p \leq n} T(\Omega_X^p, \omega)^{(-1)^p p}.$$

Let $\Delta_{\overline{\partial}}^{p,q}$ be the Dolbeault Laplacian acting on $A^{p,q}(X)$ and $\det \Delta_{\overline{\partial}}^{p,q}$ its ζ-regularised determinant (excluding the zero eigenvalue). Unraveling the definition of holomorphic analytic torsion, we find for the BCOV torsion

$$T(X, \omega) = \prod_{0 \leq p,q \leq n} (\det \Delta_{\overline{\partial}}^{p,q})^{(-1)^{p+q} pq}.$$

It depends on the choice of the Kähler metric. A suitable normalisation makes it independent of choices. For this purpose, we introduce two real-valued quantities. For the first one, let η be a basis of $H^0(X, K_X)$, and define as in [FLY08, Sec. 4]:

$$A(X, \omega) = \exp\left(-\frac{1}{12} \int_X (\log \varphi) \mathrm{c}_n(T_X, h)\right), \quad \text{with} \quad \varphi = \frac{i^{n^2} \eta \wedge \overline{\eta}}{\|\eta\|_{\mathrm{L}^2}^2} \frac{n!}{(2\pi\omega)^n}. \tag{2.3}$$

For the second one, we consider the largest torsion-free quotient of the cohomology groups $H^k(X, \mathbb{Z})$, denoted by $H^k(X, \mathbb{Z})_{\mathrm{nt}}$. These are lattices in the real cohomology groups $H^k(X, \mathbb{R})$. The latter have euclidean structures induced from the L^2 inner products on the $H^k(X, \mathbb{C})$. We define $\mathrm{vol}_{\mathrm{L}^2}(H^k(X, \mathbb{Z}), \omega)$ to be the square of the covolume of the lattice $H^k(X, \mathbb{Z})_{\mathrm{nt}}$ with respect to this euclidean structure, and we put

$$B(X, \omega) = \prod_{0 \leq k \leq 2n} \mathrm{vol}_{\mathrm{L}^2}(H^k(X, \mathbb{Z}), \omega)^{(-1)^{k+1} k/2}. \tag{2.4}$$

The *BCOV invariant* of X is then defined to be

$$\tau_{\mathrm{BCOV}}(X) = \frac{A(X, \omega)}{B(X, \omega)} T(X, \omega) \in \mathbb{R}_{>0}. \tag{2.5}$$

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

The BCOV invariant depends only on the complex structure of X [EFiMM21, Prop. 5.8]. The definition in (2.5) differs from that of [EFiMM21, Def. 5.7] by a factor $(2\pi)^{n^2\chi(X)/2}$ due to the different choice of normalisation of the L^2 metric:

$$\langle \alpha, \beta \rangle = \int_X h(\alpha, \beta) \frac{(2\pi\omega)^n}{n!}.$$

2.3. The arithmetic Riemann–Roch theorem

In this subsection, we work over an arithmetic ring. This means an excellent regular domain A together with a finite set Σ of embeddings $\sigma\colon A \hookrightarrow \mathbb{C}$, closed under complex conjugation. For example, A could be a number field with the set of all its complex embeddings, or the complex field $\mathbb{C}$. Denote by K the field of fractions of A.

Let X be an arithmetic variety: that is, a regular, integral, flat and quasi-projective scheme over A. For every embedding $\sigma\colon A \hookrightarrow \mathbb{C}$, the base change $X_\sigma = X \times_{A,\sigma} \mathbb{C}$ is a quasi-projective and smooth complex variety whose associated analytic space X_σ^{an} is therefore a quasi-projective complex manifold. It is convenient to define X^{an} as the disjoint union of the X_σ^{an}, indexed by σ. For instance, when A is a number field, then X^{an} is the complex analytic space associated to X as an arithmetic variety over $\mathbb{Q}$. Differential geometric objects on X^{an} such as line bundles, differential forms, metrics and so on may equivalently be seen as collections of corresponding objects on the X_σ^{an}. The complex conjugation induces an anti-holomorphic involution on X^{an}, and it is customary in Arakelov geometry to impose some compatibility of the analytic data with this action. Let us now recall the definitions of the arithmetic Picard and first Chow groups of X.

Definition 2.1. A smooth hermitian line bundle on X consists in a pair (L, h), where

- L is a line bundle on X.
- h is a smooth hermitian metric on the holomorphic line bundle L^{an} on X^{an} deduced from L, invariant under the action of the complex conjugation. Hence, h is a conjugation invariant collection $\{h_\sigma\}_{\sigma\colon A\to\mathbb{C}}$, where h_σ is a smooth hermitian metric on the holomorphic line bundle L_σ^{an} on X_σ^{an} deduced from L by base change and analytification.

The set of isomorphism classes of hermitian line bundles (L, h), with the natural tensor product operation, is a commutative group denoted by $\widehat{\mathrm{Pic}}(X)$ and called the arithmetic Picard group of X.

Definition 2.2. The first arithmetic Chow group $\widehat{\mathrm{CH}}^1(X)$ of X is the commutative group

- generated by arithmetic divisors – that is, couples (D, g_D) – where D is a Weil divisor on X and g_D is a Green current for the divisor D^{an}, compatible with complex conjugation. Hence, by definition, g_D is a degree 0 current on X^{an} that is a dd^c-potential for the current of integration $\delta_{D^{\mathrm{an}}}$

$$dd^c\, g_D + \delta_{D^{\mathrm{an}}} = [\omega_D],$$

  up to some smooth differential $(1,1)$ form ω_D on X^{an}.
- with relations $\big(\mathrm{div}(\phi), [-\log|\phi|^2]\big)$ for nonzero rational functions ϕ on X.

The arithmetic Picard and first Chow groups are related via the first arithmetic Chern class

$$\widehat{\mathrm{c}}_1 : \widehat{\mathrm{Pic}}(X) \to \widehat{\mathrm{CH}}^1(X),$$

which maps a hermitian line bundle (L, h) to the class of the arithmetic divisor $\big(\mathrm{div}(\ell), [-\log\|\ell\|_h^2]\big)$, where ℓ is any nonzero rational section of L. This is in fact an isomorphism. We refer the reader to [GS90b, Sec. 2] for a complete discussion.

More generally, Gillet–Soulé developed a theory of arithmetic cycles and Chow rings [GS90a], an arithmetic K-theory and characteristic classes [GS90b, GS90c] and an arithmetic Riemann–Roch

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

theorem [GS92]. While for the comprehension of the theorem below only $\widehat{\mathrm{CH}}^1$, $\widehat{\mathrm{Pic}}$ and $\widehat{\mathrm{c}}_1$ are needed, the proof uses all this background, for which we refer to the above references.

Let now $f\colon \mathcal{X} \to S$ be a smooth projective morphism of arithmetic varieties of relative dimension n, with generic fibre X_∞. To simplify the exposition, we assume that $S \to \operatorname{Spec} A$ is surjective and has geometrically connected fibres. In particular, we assume that S_σ^{an} is connected for every embedding σ. More importantly, we suppose that the fibres X_s are Calabi–Yau, and hence they satisfy $K_{X_s} = \mathcal{O}_{X_s}$. We define the BCOV line bundle on S as the determinant of cohomology of the virtual vector bundle $\sum_p (-1)^p p\Omega^p_{\mathcal{X}/S}$: that is, in additive notation for the Picard group of S

$$\lambda_{\mathrm{BCOV}}(\mathcal{X}/S) = \sum_{p=0}^{n} (-1)^p p\lambda(\Omega^p_{\mathcal{X}/S}) = \sum_{p,q} (-1)^{p+q} p \det R^q f_*\Omega^p_{\mathcal{X}/S}. \tag{2.6}$$

If there is no possible ambiguity, we will sometimes write λ_{BCOV} instead of $\lambda_{\mathrm{BCOV}}(\mathcal{X}/S)$.

For the following statement, we fix an auxiliary conjugation invariant Kähler metric h on $T_{\mathcal{X}^{\mathrm{an}}}$. We denote by ω the associated Kähler form, normalised according to the conventions in Arakelov theory as in (2.1). We assume that the restriction of ω to fibres (still denoted by ω) has a rational cohomology class. All the L^2 metrics below are computed with respect to ω as in (2.2). Depending on the Kähler metric, the line bundle λ_{BCOV} carries a Quillen metric h_{Q}

$$h_{\mathrm{Q},s} = T(X_s, \omega) \cdot h_{\mathrm{L}^2,s}.$$

Following [EFiMM18, Def. 4.1] and [EFiMM21, Def. 5.2], the Quillen-BCOV metric on λ_{BCOV} is defined by multiplying h_{Q} by the correcting factor A in (2.3): for every $s \in S^{\mathrm{an}}$, we put

$$h_{\mathrm{Q,BCOV},s} = A(X_s, \omega) \cdot h_{\mathrm{Q},s}.$$

It is shown in [EFiMM18, Sec. 4.1] and [EFiMM21, Sec. 5.2] that the Quillen-BCOV metric is actually a smooth hermitian metric, independent of the choice of ω. Besides, according to [EFiMM21, Def. 5.4], one defines the L^2-BCOV metric on λ_{BCOV} by

$$h_{\mathrm{L}^2,\mathrm{BCOV},s} = B(X_s, \omega) \cdot h_{\mathrm{L}^2,s}, \tag{2.7}$$

where h_{L^2} stands for the combination of L^2-metrics on the Hodge bundles and B was introduced in (2.4). In [EFiMM21, Prop. 4.2], we showed that the function $s \mapsto B(X_s, \omega)$ is actually locally constant and that $h_{\mathrm{L}^2,\mathrm{BCOV}}$ is a smooth hermitian metric, independent of the choice of ω. Notice that the BCOV invariant defined in (2.5) can then be written as the quotient of the Quillen-BCOV and L^2-BCOV metrics:

$$\tau_{\mathrm{BCOV}}(X_s) = \frac{h_{\mathrm{Q,BCOV},s}}{h_{\mathrm{L}^2,\mathrm{BCOV},s}}. \tag{2.8}$$

Theorem 2.3. *Under the above assumptions, there is an equality in* $\widehat{\mathrm{CH}}^1(S)_{\mathbb{Q}} = \widehat{\mathrm{CH}}^1(S) \otimes \mathbb{Q}$

$$\widehat{\mathrm{c}}_1(\lambda_{\mathrm{BCOV}}, h_{\mathrm{Q,BCOV}}) = \frac{\chi(X_\infty)}{12} \widehat{\mathrm{c}}_1(f_*K_{\mathcal{X}/S}, h_{\mathrm{L}^2}). \tag{2.9}$$

Hence, for any complex embedding σ, any rational section η of $f_*K_{\mathcal{X}/S}$, any rational section $\eta_{p,q}$ of $\det R^q f_*\Omega^p_{\mathcal{X}/S}$, *we have an equality of functions on* S_σ^{an}

$$\log \tau_{\mathrm{BCOV},\sigma} = \log |\Delta|^2_\sigma + \frac{\chi(X_\infty)}{12} \log \|\eta\|^2_{\mathrm{L}^2,\sigma} - \sum_{0 \le p,q \le n} (-1)^{p+q} p \log \|\eta_{p,q}\|^2_{\mathrm{L}^2,\sigma} + \log C_\sigma, \tag{2.10}$$

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

where

- $\Delta \in K(S)^\times \otimes_{\mathbb{Z}} \mathbb{Q}$.
- $C_\sigma \in \pi^r \mathbb{Q}_{>0}$, *where* $r = \frac{1}{2}\sum(-1)^{k+1}k^2 b_k$ *and* b_k *is the* k*th Betti number of* X_∞.

Proof. The proof is a routine application of the arithmetic Riemann–Roch theorem of Gillet–Soulé [GS92, Thm. 7]. We give the details for the convenience of the reader. Consider the virtual vector bundle $\sum(-1)^p p\Omega^p_{\mathcal{X}/S}$, with virtual hermitian structure deduced from the metric h and denoted $h^\bullet$. Its determinant of cohomology λ_{BCOV} carries the Quillen metric h_{Q}. The theorem of Gillet–Soulé provides an equality in $\widehat{\mathrm{CH}}^1(S)_{\mathbb{Q}}$

$$\begin{aligned}\widehat{\mathrm{c}}_1(\lambda_{\mathrm{BCOV}}, h_{\mathrm{Q}}) &= f_*\left(\widehat{\mathrm{ch}}(\sum(-1)^p p\Omega^p_{\mathcal{X}/S}), h^\bullet)\widehat{\mathrm{Td}}(T_{\mathcal{X}/S}, h)\right)^{(1)} \\ &\quad - a\left(\mathrm{ch}(\sum(-1)^p p\Omega^p_{\mathcal{X}^{\mathrm{an}}/S^{\mathrm{an}}})\,\mathrm{Td}(T_{\mathcal{X}^{\mathrm{an}}/S^{\mathrm{an}}})R(T_{\mathcal{X}^{\mathrm{an}}/S^{\mathrm{an}}})\right)^{(1)} \\ &= \frac{1}{12} f_*\left(\widehat{\mathrm{c}}_1(K_{\mathcal{X}/S}, h^*)\widehat{\mathrm{c}}_n(T_{\mathcal{X}/S}, h)\right), \end{aligned} \tag{2.11}$$

where $h^* = (\det h)^{-1}$ is the hermitian metric on $K_{\mathcal{X}/S}$ induced from h. Notice that the topological factor containing the R-genus in (2.11) vanishes in our situation since

$$\mathrm{ch}\left(\sum(-1)^p p\Omega^p_{\mathcal{X}^{\mathrm{an}}/S^{\mathrm{an}}}\right)\mathrm{Td}(T_{\mathcal{X}^{\mathrm{an}}/S^{\mathrm{an}}}) = -\mathrm{c}_{n-1} + \frac{n}{2}\mathrm{c}_n - \frac{1}{12}\mathrm{c}_1\mathrm{c}_n + \text{higher-degree terms},$$

and R has only odd degree terms and $\mathrm{c}_1(T_{\mathcal{X}^{\mathrm{an}}/S^{\mathrm{an}}}) = 0$. Now the evaluation map $f^*f_*K_{\mathcal{X}/S} \to K_{\mathcal{X}/S}$ is an isomorphism, but it is in general not an isometry if we equip $f_*K_{\mathcal{X}/S}$ with the L^2 metric and $K_{\mathcal{X}/S}$ with the metric h^*. Comparing both metrics yields a relation in $\widehat{\mathrm{CH}}^1(\mathcal{X})$

$$\widehat{\mathrm{c}}_1(K_{\mathcal{X}/S}, h^*) = f^*\widehat{\mathrm{c}}_1(f_*K_{\mathcal{X}/S}, h_{\mathrm{L}^2}) + [(0, -\log\varphi)]. \tag{2.12}$$

Here φ is the smooth function on X^{an} given by

$$\varphi = \frac{i^{n^2}\eta\wedge\overline{\eta}}{\|\eta\|^2_{\mathrm{L}^2}}\frac{n!}{(2\pi\omega)^n},$$

where η denotes a local trivialisation of $f_*K_{\mathcal{X}^{\mathrm{an}}/S^{\mathrm{an}}}$, thought of as a section of $K_{\mathcal{X}^{\mathrm{an}}/S^{\mathrm{an}}}$ via the evaluation map. Multiplying (2.12) by $\widehat{\mathrm{c}}_n(T_{\mathcal{X}/S}, h)$ and applying f_* and the projection formula for arithmetic Chow groups, we find

$$\begin{aligned} f_*\left(\widehat{\mathrm{c}}_1(K_{\mathcal{X}/S}, h^*)\widehat{\mathrm{c}}_n(T_{\mathcal{X}/S}, h)\right) &= f_*\left(f^*\widehat{\mathrm{c}}_1(f_*K_{\mathcal{X}/S}, h_{\mathrm{L}^2})\widehat{\mathrm{c}}_n(T_{\mathcal{X}/S}, h)\right) + f_*\left([(0, -\log\varphi)]\widehat{\mathrm{c}}_n(T_{\mathcal{X}/S}, h)\right) \\ &= \chi(X_\infty)\widehat{\mathrm{c}}_1(f_*K_{\mathcal{X}/S}, h_{\mathrm{L}^2}) + \left[\left(0, -\int_{\mathcal{X}^{\mathrm{an}}/S^{\mathrm{an}}}(\log\varphi)\mathrm{c}_n(T_{\mathcal{X}^{\mathrm{an}}/S^{\mathrm{an}}}, h)\right)\right], \end{aligned}$$

where $\mathrm{c}_n(T_{\mathcal{X}/S}, h)$ is the nth Chern–Weil differential form of $(T_{\mathcal{X}^{\mathrm{an}}/S^{\mathrm{an}}}, h)$. Together with (2.11), this shows that the metric

$$h_{\mathrm{Q,BCOV}} = h_{\mathrm{Q}} \cdot \exp\left(-\frac{1}{12}\int_{\mathcal{X}^{\mathrm{an}}/S^{\mathrm{an}}}(\log\varphi)\mathrm{c}_n(T_{\mathcal{X}^{\mathrm{an}}/S^{\mathrm{an}}}, h)\right)$$

indeed satisfies (2.9).

The outcome of (2.10) is a translation of the meaning of the equality in (2.9) in $\widehat{\mathrm{CH}}^1(S)_{\mathbb{Q}}$ in terms of the constructions in (2.8) and (2.7).

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

By [EFiMM21, Prop. 4.2], the normalising factor B is constant on each connected manifold S_σ^{an} and would be rational if the L^2 inner products on cohomology groups were computed with $h/2\pi$.

With this understood, we find

$$\mathrm{vol}_{\mathrm{L}^2}(H^k(X_s,\mathbb{Z}),\omega) \in (2\pi)^{-kb_k}\mathbb{Q}^\times_{>0} \tag{2.13}$$

for any $s \in S_\sigma^{\mathrm{an}}$. Together with the definition of B in (2.4), this is responsible for the constants C_σ. □

Remark 2.4.

1. The use of the arithmetic Riemann–Roch theorem requires an algebraic setting but directly yields the existence of the rational function Δ. By contrast, previous techniques (see, e.g., [FLY08, Sections 7 & 10]) rely on subtle integrability estimates of the functions in (2.10) to ensure that the *a priori* pluriharmonic function $\log|\Delta|^2_\sigma$ is indeed the logarithm of a rational function. The arithmetic Riemann–Roch theorem further provides the field of definition of Δ and the constants C_σ.
2. In the case of a Calabi–Yau 3-fold defined over a number field, similar computations were done by Maillot–Rössler [MR12, Sec. 2].

2.4. Kronecker limit formulas for families of Calabi–Yau hypersurfaces

In this section, we give an example of the use of Theorem 2.3, and we determine the BCOV invariant for families of Calabi–Yau hypersurfaces in Fano manifolds. The argument provides a simplified model for the later computation of the BCOV invariant of the mirror family of Calabi–Yau hypersurfaces.

Let V be a complex Fano manifold with very ample anti-canonical bundle $-K_V$. We consider the anti-canonical embedding of V into $|-K_V| = \mathbb{P}(H^0(V,-K_V)) \simeq \mathbb{P}^N$, whose smooth hyperplane sections are Calabi–Yau manifolds. The dual projective space $\check{\mathbb{P}} = \mathbb{P}(H^0(V,-K_V)^\vee) \simeq \check{\mathbb{P}}^N$ parametrises hyperplane sections and contains an irreducible subvariety $\Delta \subseteq \check{\mathbb{P}}$ that corresponds to singular such sections [GKZ08, Chap. 1, Prop. 1.3]. We assume that Δ is a hypersurface in $\check{\mathbb{P}}$. This is in general not true, and a necessary condition is proven in [GKZ08, Chap. 1, Cor. 1.2]. Denote by U the quasi-projective complement $U := \check{\mathbb{P}} \setminus \Delta$. Denote by $f\colon \mathcal{X} \to \check{\mathbb{P}}$ the universal family of hyperplane sections. Therefore f is smooth on U, and the corresponding BCOV line bundle λ_{BCOV} is thus defined on U.

Lemma 2.5. *For some positive integer m, the line bundles $(f_*K_{\mathcal{X}/U})^{\otimes m}$ and $\lambda_{\mathrm{BCOV}}^{\otimes m}$ have trivialising sections. These are unique up to constants.*

Proof. A standard computation shows that $\mathrm{Pic}(U) = \mathbb{Z}/\deg\Delta$, providing the first claim of the lemma. For the second assertion, for any of the line bundles under consideration, let θ and θ' be two trivialisations on U. Therefore, $\theta = h\theta'$ for some invertible function h on U. The previous description of $\mathrm{Pic}(U)$ shows that the divisor of h, as a rational function on $\check{\mathbb{P}}$, is supported on Δ. As Δ is irreducible, in the projective space $\check{\mathbb{P}}$ this is only possible if the divisor vanishes. We conclude that h is necessarily constant. □

For the following statement, we need a choice of auxiliary Kähler metric on $\mathcal{X}$ (restricted to U), whose Arakelov theoretic Kähler form has fibrewise rational cohomology class. We compute L^2 norms on Hodge bundles and on λ_{BCOV} with respect to this choice.

Theorem 2.6. *For some integer $m > 0$ as in the lemma, let β be a trivialisation of $\lambda_{\mathrm{BCOV}}^{\otimes m}$ and η a trivialisation of $(f_*K_{\mathcal{X}/U})^{\otimes m}$. Then there is a global constant C such that, for any Calabi–Yau hyperplane section $X_H = V \cap H$, we have*

$$\tau_{\mathrm{BCOV}}(X_H) = C\|\eta\|_{\mathrm{L}^2}^{\chi/6m}\|\beta\|_{\mathrm{L}^2}^{-2/m}.$$

Proof. We apply Theorem 2.3 to $f\colon \mathcal{X} \to U$ (over $\mathbb{C}$), which in terms of β and η becomes

$$m\log\tau_{\mathrm{BCOV}}(X_H) = \log|g|^2 + \frac{\chi}{12}\log\|\eta\|^2_{\mathrm{L}^2} - \log\|\beta\|^2_{\mathrm{L}^2} + \log C$$

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

for some regular invertible function g on U and some constant C. By construction, as a rational function on $\check{\mathbb{P}}$, g must have its zeros or poles along Δ. Since Δ is irreducible, this forces g to be constant. □

Remark 2.7.

1. When V is a toric variety with very ample anti-canonical class, all of the constructions can be done over the rational numbers. The sections β and η can be taken to be defined over $\mathbb{Q}$ and unique up to a rational number. With this choice, the constant C takes the form stated in Theorem 2.3.
2. In the case when the discriminant Δ has higher codimension, we have $\mathrm{Pic}(U) \simeq \mathrm{Pic}(\check{\mathbb{P}})$. In particular, λ_{BCOV} uniquely extends to a line bundle $\check{\mathbb{P}}$. The existence of the canonical (up to constant) trivialisations β and η is no longer true. However, one can propose a variant of the theorem where β and η are trivialisations outside a chosen ample divisor in $\check{\mathbb{P}}$.

3. The Dwork and mirror families, and their Hodge bundles

The main object of interest in this section is the mirror family of Calabi–Yau hypersurfaces. It is obtained from the Dwork pencil of Calabi–Yau varieties by first modding out by a group of generic symmetries and then performing a crepant resolution. We study the structure of the Hodge bundles of the mirror family. In the even dimension, we show that the primitive Hodge bundles in the middle degree can be decomposed into two direct factors. One will be seen to be constant in Section 4, and the other one is called the minimal part. For the latter, we construct explicit trivialisations via Griffiths' residue method.

Throughout, our arguments combine analytic and algebraic aspects of the same geometric objects. Except when there is a risk of confusion, we won't make any distinction in the notations between an algebro-geometric object and its analytification. Likewise, we won't specify the field of definition of various algebraic varieties and schemes. However, we will precisely indicate the category where the statements take place.

3.1. The geometry of the Dwork family

We review general facts on the Dwork pencil of Calabi–Yau hypersurfaces and the construction of an equivariant normal crossings model. Initially, we work with algebraic varieties over the field of complex numbers. Rationality refinements will be made along the way.

Let $n \geq 4$ be an integer. The Dwork pencil $\mathcal{X} \to \mathbb{P}^1$ is defined by the hypersurface of $\mathbb{P}^n \times \mathbb{P}^1$ of the equation

$$F_\psi(x_0, \ldots, x_n) := \sum_{j=0}^{n} x_j^{n+1} - (n+1)\psi x_0 \ldots x_n = 0, \quad [x_0 : x_1 : \ldots : x_n] \in \mathbb{P}^n, \quad \psi \in \mathbb{P}^1.$$

The smooth fibres of this family are Calabi–Yau manifolds of dimension $n-1$. The singular fibres are

- Fibre at $\psi = \infty$, given by the divisor with normal crossings $x_0 \cdot \ldots \cdot x_n = 0$.
- The fibres where $\psi^{n+1} = 1$. These fibres have ordinary double point singularities. The singular points have projective coordinates $(x_0, \ldots, x_n)$ with $x_0 = 1$ and $x_j^{n+1} = 1$ for all $j \geq 1$, and $\prod_j x_j = \psi^{-1}$.

Denote by μ_{n+1} the group of the $(n+1)$th roots of unity. Let K be the kernel of the multiplication map $\mu_{n+1}^{n+1} \to \mu_{n+1}$. Let also Δ be the diagonal embedding of μ_{n+1} in K and $G := K/\Delta$. The group G acts naturally on the fibres X_ψ of $\mathcal{X} \to \mathbb{P}^1$ by multiplication of the projective coordinates.

The above constructions can be realised as schemes over $\mathbb{Q}$. Indeed, F_ψ is already defined over $\mathbb{Q}$, and the groups K, Δ are finite algebraic groups over $\mathbb{Q}$, and hence so does the quotient G. The action

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

of G on F_ψ is defined over $\mathbb{Q}$ as well, as one can see by examining the compatibility with the action of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ on the $\mathbb{C}$ points of $\mathcal{X}$ or alternatively by writing the coaction at the level of algebras.

The following argument was provided to us by Nicholas Shepherd-Barron, whom we warmly thank for letting us include it in our article.

Proposition 3.1. *The family $\mathcal{X} \to \mathbb{P}^1$ admits a G-equivariant projective normal crossings model $\mathcal{X}' \to \mathbb{P}^1$, with $\mathcal{X}'$ nonsingular, which is semi-stable at $\psi = \infty$ and defined over $\mathbb{Q}$.*

Proof. Outside of the singular points, there is nothing to modify. The points corresponding to ordinary double point singularities are provided by the affine equations $x_0 = 1$ and $x_j^{n+1} = 1$ for $j \geq 1$, and blowing up along the corresponding locus of $\mathcal{X}$ provides a normal crossings model. The locus of ordinary double points is defined over $\mathbb{Q}$ and is G-equivariant and thus so is also the blowup.

We next consider our family at the point at infinity. Introduce the divisor D_0 in $\mathbb{P}^n$ given by $\sum_j x_j^{n+1} = 0$ and the divisor $D_\infty = \sum_j H_j$, where H_j is the hyperplane cut out by $x_j = 0$. The axis of the pencil $\mathcal{X} \to \mathbb{P}^1$ is $D_0 \cap D_\infty$ and hence $\mathcal{X} = \mathrm{Bl}_{D_0 \cap D_\infty}(\mathbb{P}^n)$. We construct another model by blowing up $\mathbb{P}^n$ in $D_0 \cap H_0$ to get $\mathcal{X}_1$. Continue to blow up the strict transform of D_0 in $\mathcal{X}_1$ intersected by the strict transform of H_1, and so on. Each such blowup is a blowup in a smooth centre that is G-equivariant. The final result is a G-equivariant $\widetilde{\mathcal{X}}$ projective manifold with an equivariant morphism $\nu : \widetilde{\mathcal{X}} \to \mathbb{P}^n$. Denote by $\widetilde{D_0}$ (respectively, $\widetilde{D_\infty}$) the strict transforms of D_0 (respectively, D_∞). By construction they are disjoint, and computation shows that $\nu^* D_0 \sim \widetilde{D_0} + \sum E_i$ and $\nu^*(D_\infty) \sim \widetilde{D_\infty} + \sum E_i$, where the E_i denote the strict transforms of the exceptional divisors. Since $D_0 - D_\infty$ is the divisor of a rational function, hence linearly equivalent to zero, and $\widetilde{D_\infty}$ is disjoint from $\widetilde{D_0}$, we find a morphism $p : \widetilde{\mathcal{X}} \to \mathbb{P}^1$ such that $p^{-1}(\infty) = \widetilde{D_\infty}$ and $p^{-1}(0) = \widetilde{D_0}$. This is the searched for semi-stable model at infinity. From the local description, we also see that $\nu^{-1}(D_0 \cap D_\infty) = \sum E_i$ that is principal, so that $\widetilde{\mathcal{X}} \to \mathbb{P}^1$ factors over $\mathcal{X} \to \mathbb{P}^1$.

All of the above constructions can be defined over $\mathbb{Q}$, and taking them together with the previous considerations with the ordinary double points provides a model $\mathcal{X}' \to \mathbb{P}^1$ as in the statement of the proposition. □

3.2. The mirror family

The first step towards the construction of a mirror family is the formation of the quotient $\mathcal{Y} = \mathcal{X}/G$. As the action of G on $\mathcal{X}$ is defined over $\mathbb{Q}$, the space $\mathcal{Y}$ and projection map $\mathcal{Y} \to \mathbb{P}^1$ are also. The following lemma shows that except for the fibre at infinity, this is a family of singular Calabi–Yau varieties with mild singularities.

Lemma 3.2. *The total space of the restricted family $\mathcal{Y} \to \mathbb{A}^1$ has rational Gorenstein singularities. It has a relative canonical line bundle $K_{\mathcal{Y}/\mathbb{A}^1}$ obtained by descent from $K_{\mathcal{X}/\mathbb{A}^1}$.*

Proof. To lighten notations, let us write in this proof $\mathcal{X}$ and $\mathcal{Y}$ for the corresponding restrictions to $\mathbb{A}^1$. The total space $\mathcal{X}$ is nonsingular, and $\mathcal{Y}$ is a quotient of it by the action of a finite group. Therefore, $\mathcal{Y}$ has rational singularities. In particular, it is normal and Cohen–Macaulay. Consequently, if $\mathcal{Y}^{ns}$ is the nonsingular locus of $\mathcal{Y}$ and $j : \mathcal{Y}^{ns} \hookrightarrow \mathcal{Y}$ the open immersion, then we have a relation between relative dualising sheaves $j_* \omega_{\mathcal{Y}^{ns}/\mathbb{A}^1} = \omega_{\mathcal{Y}/\mathbb{A}^1}$. We will use this below.

Now for the Gorenstein property and the descent claim. Notice that since $\mathbb{A}^1$ is nonsingular, $\mathcal{Y}$ is Gorenstein if and only if the fibres of $\mathcal{Y} \to \mathbb{A}^1$ are Gorenstein. We will implicitly confound both the absolute and relative points of view. We introduce $\mathcal{X}^\circ$ the complement of the fixed locus of G and $\mathcal{X}^*$ the smooth locus of $\mathcal{X} \to \mathbb{A}^1$. These are G-invariant open subschemes of $\mathcal{X}$ and constitute an open cover because the ordinary double points in the fibres of $\mathcal{X} \to \mathbb{A}^1$ are disjoint from the fixed point locus of G. Then $\mathcal{Y}^\circ = \mathcal{X}^\circ/G$ and $\mathcal{Y}^* = \mathcal{X}^*/G$ form an open cover of $\mathcal{Y}$, and it is enough to proceed for each one separately.

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

Since G acts freely on $\mathcal{X}^\circ$, the quotient $\mathcal{Y}^\circ$ is nonsingular and is therefore Gorenstein. The morphism $\mathcal{X}^\circ \to \mathcal{Y}^\circ$ is étale, and hence $K_{\mathcal{X}^\circ/\mathbb{A}^1}$ descends to $K_{\mathcal{Y}^\circ/\mathbb{A}^1}$.

For $\mathcal{Y}^*$, we observe that G preserves a relative holomorphic volume form on $\mathcal{X}^*$. Indeed, in affine coordinates $z_k = \frac{x_k}{x_j}$ on the open set $x_j \neq 0$, and where $\partial F_\psi/\partial z_i \neq 0$, the expression

$$\theta_0 = \frac{(-1)^{i-1} dz_0 \wedge \ldots \widehat{dz_i} \wedge \ldots \wedge \widehat{dz_j} \wedge \ldots \wedge dz_n}{\partial F_\psi/\partial z_i}\Big|_{F_\psi=0} \tag{3.1}$$

provides such an invariant relative volume form. This entails that $K_{\mathcal{X}^*/\mathbb{A}^1}$ descends to an invertible sheaf $\mathcal{K}$ on $\mathcal{Y}^*$. Now the singular locus of $\mathcal{Y}^*$ is contained in the image of the fixed point set of G on $\mathcal{X}^*$. We infer that $\mathcal{K}$ is an invertible extension of the relative canonical bundle of $(\mathcal{Y}^*)^{ns} \to \mathbb{A}^1$. But $\mathcal{Y}^*$ is normal so that $\mathcal{K} \simeq j_* j^* \mathcal{K}$. Then as mentioned at the beginning of the proof, $j_* \omega_{\mathcal{Y}^{ns}/\mathbb{A}^1} = \omega_{\mathcal{Y}/\mathbb{A}^1}$, and we conclude, since $\mathcal{K}$ is also an extension of $\omega_{\mathcal{Y}^{ns}/\mathbb{A}^1}$. □

Because the BCOV invariant has not been fully developed for Calabi–Yau orbifolds (see nevertheless [Yos17] for some three-dimensional cases), we need crepant resolutions of the varieties Y_ψ. This needs to be done in families so that the results of §2.3 apply. The family of crepant resolutions $\mathcal{Z} \to \mathbb{P}^1$ that we exhibit will be called the *mirror family*, although it is not unique. We also have to address the rationality of the construction.

Lemma 3.3. *There is a projective birational morphism $\mathcal{Z} \to \mathcal{Y}$ of algebraic varieties over $\mathbb{Q}$, such that*

1. *$\mathcal{Z}$ is smooth.*
2. *If $\psi^{n+1} = 1$, the fibre Z_ψ has a single ordinary double point singularity.*
3. *If $\psi = \infty$, Z_∞ is a simple normal crossings divisor in $\mathcal{Z}$.*
4. *Otherwise, $Z_\psi \to Y_\psi$ is a crepant resolution of singularities. In particular, Z_ψ is a smooth Calabi–Yau variety.*
5. *The smooth complex fibres Z_ψ are mirror to the X_ψ in that their Hodge numbers satisfy $h^{p,q}(Z_\psi) = h^{n-1-p,q}(X_\psi)$. In particular, the smooth Z_ψ are Calabi–Yau with $\chi(Z_\psi) = (-1)^{n-1}\chi(X_\psi)$.*

Proof. The proof of (1)–(4) is based on [DHZ98, Sec. 8 (v)], [DHZ06] and [BG14, Prop. 3.1], together with Hironaka's resolution of singularities. We recall the strategy to justify the existence of a model over $\mathbb{Q}$.

Introduce $W = \mathbb{P}^n/G$. We claim this is a split toric variety over $\mathbb{Q}$. First of all, it can be realised as the hypersurface in $\mathbb{P}^{n+1}_{\mathbb{Q}}$ of equation

$$W\colon\ y_0^{n+1} = \prod_{j=1}^{n+1} y_j.$$

Second, the associated torus is split over $\mathbb{Q}$. It is actually given by $\mathbb{G}_{\mathrm{m}\,\mathbb{Q}} \times \mathbb{T}$, where $\mathbb{T}$ is the kernel of the multiplication map $\mathbb{G}^{n+1}_{\mathrm{m}\,\mathbb{Q}} \to \mathbb{G}_{\mathrm{m}\,\mathbb{Q}}$. Finally, the action of the torus on W is defined over $\mathbb{Q}$:

$$((t_0, t_1, \ldots, t_{n+1}), (y_0, y_1, \ldots, y_{n+1})) \mapsto (t_0 y_0, t_0 t_1 y_1, \ldots, t_0 t_{n+1} y_{n+1}).$$

Once we know that W is a split toric variety over $\mathbb{Q}$ with the same equation as in [DHZ06, Application 5.5], the toric and crepant projective resolution exhibited in [DHZ06] automatically works over $\mathbb{Q}$ as well. We write $\widetilde{W}$ for this resolution of W.

We now consider $\mathcal{Y}$ as a closed integral $\mathbb{Q}$-subscheme of $W \times \mathbb{P}^1$. Let $\widetilde{\mathcal{Y}}$ be the strict transform of $\mathcal{Y}$ in $\widetilde{W} \times \mathbb{P}^1$. By [DHZ98, Sec. 8 (v)], the fibres of $\widetilde{\mathcal{Y}}$ at $\psi \in \mathbb{C} \setminus \mu_{n+1}$ are projective crepant resolutions of the fibres Y_ψ. In particular, $\widetilde{\mathcal{Y}}$ is smooth over $\mathbb{C} \setminus \mu_{n+1}$, and in turn this implies smoothness over

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

the complement U of the closed subscheme $V(\psi^{n+1}-1)$ of $\mathbb{A}^1_{\mathbb{Q}}$. Necessarily, the fibres of $\widetilde{\mathcal{Y}}$ over U have trivial canonical bundle as well. For the fibres at $\psi^{n+1}=1$, the claim of the lemma requires two observations:

- The ordinary double points of X_ψ are permuted freely and transitively by G and are identified to a single point in the quotient Y_ψ. This entails that the total space $\mathcal{Y}$ is nonsingular in a neighbourhood of these points and that they remain ordinary double points of $\mathcal{Y}\to\mathbb{P}^1$.
- The centre of the toric resolution is disjoint from the ordinary double points since it is contained in the locus of $\mathbb{P}^n/G$ where two or more projective coordinates vanish. Therefore, the morphism $\widetilde{\mathcal{Y}}\to\mathcal{Y}$ is an isomorphism in a neighbourhood of these points. Finally, on the complement, $\widetilde{\mathcal{Y}}_\psi$ is a resolution of singularities of Y_ψ. Indeed, this is a local question in a neighbourhood of the fixed points of G so that the above references [DHZ98, DHZ06] still apply.

Finally, $\widetilde{\mathcal{Y}}$ is by construction smooth on the complement of the fibre $\psi=\infty$. After a resolution of singularities given by blowups with smooth centres in $\widetilde{Y}_\infty$ (defined over $\mathbb{Q}$), we obtain a smooth algebraic variety $\mathcal{Z}$ over $\mathbb{Q}$ such that Z_∞ is a simple normal crossings divisor in $\mathcal{Z}$. This sets (1–4).

For (5), we refer for instance to [BD96, Thm. 6.9, Conj. 7.5 & Ex. 8.7]. This is specific to the Dwork pencil. More generally, we can cite work of Yasuda, who proves an invariance property of orbifold Hodge structures (and hence orbifold Hodge numbers) under crepant resolutions, for quotient Gorenstein singularities [Yas04, Thm. 1.5]. Orbifold Hodge numbers coincide with stringy Hodge numbers of global (finite) quotient orbifolds, whose underlying group respects a holomorphic volume form [BD96, Thm. 6.14]. Finally, by [BB96, Thm. 4.15], stringy Hodge numbers satisfy the expected mirror symmetry property for the mirror pairs constructed by Batyrev [Bat94]. □

From the proof of Lemma 3.3, we keep the notation $U\subset\mathbb{P}^1$ for the smooth locus of the mirror family $f\colon\mathcal{Z}\to\mathbb{P}^1$. For later use, we record the following lemma.

Lemma 3.4. *Let $h^{p,q}$ be the rank of the Hodge bundle $R^qf_*\Omega^p_{\mathcal{Z}/U}$. Then*

- $h^{p,q}=1$ *if* $p+q=n-1$ *and* $p\neq q$.
- $h^{p,p}=\sum_{j=0}^{p}(-1)^j\binom{n+1}{j}\binom{(p+1-j)n+p}{n}+\delta_{2p,n-1}$.
- $h^{p,q}=0$ *otherwise.*

In particular,

$$\chi(Z_\psi)=(-1)^{n-1}\chi(X_\psi)=(-1)^{n-1}\left(\frac{(-n)^{n+1}-1}{n+1}+n+1\right).$$

Proof. The items are a consequence of the mirror symmetry property for the Hodge numbers in Lemma 3.3 and the computation of the cohomology of a hypersurface in projective space (see [BD96, Ex. 8.7]). □

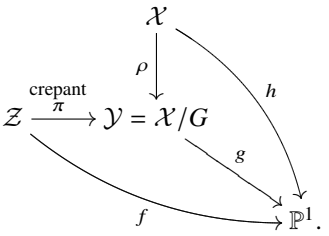
Definition 3.5. The point $\infty\in\mathbb{P}^1$ is called the MUM point of the family $f\colon\mathcal{Z}\to\mathbb{P}^1$. The points $\xi\in\mathbb{P}^1$ with $\xi^{n+1}=1$ are called the ODP points.

The terminology MUM stands for *Maximally Unipotent Monodromy*, and it will be justified later in Lemma 4.3. The terminology ODP stands for *Ordinary Double Point*.

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

3.3. Generalities on Hodge bundles

We gather general facts on the Hodge bundles of our families of Calabi–Yau varieties, summarised in the following diagram:



(3.2)

Recall the notation U for the Zariski open subset of $\mathbb{P}^1$ where f (respectively, h) is smooth. When it is clear from the context, we will still write $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ for the total spaces of the fibrations restricted to U. Otherwise, we add an index U to mean the restriction to U. We let $\mathcal{Y}^\circ$ be the nonsingular locus of $\mathcal{Y}_U$. It is the étale quotient of $\mathcal{X}^\circ$, the complement in $\mathcal{X}_U$ of the fixed point set of G. They are both open subsets whose complements have codimension ≥ 2.

In this subsection, most of the arguments take place in the complex analytic category.

Hodge bundles in arbitrary degree

Our discussion is based on a minor adaptation of [Ste77, Sec. 1] to the relative setting. First of all, we observe that the higher direct images $R^k g_* \mathbb{C}$ are locally constant sheaves, and actually $R^k g_* \mathbb{C} \simeq (R^k h_* \mathbb{C})^G$. Indeed, we have the equality $\mathbb{C}_{\mathcal{Y}} = (\rho_* \mathbb{C}_{\mathcal{X}})^G$. Moreover, since G is finite, so is ρ, and taking G-invariants is an exact functor in the category of sheaves of $\mathbb{C}[G]$-modules. A spectral sequence argument allows us to conclude. Similarly, one has $R^k g_* \mathbb{Q} \simeq (R^k h_* \mathbb{Q})^G$.

Let now $\widetilde{\Omega}^\bullet_{\mathcal{Y}/U}$ be the relative holomorphic de Rham complex of $\mathcal{Y} \to U$, in the orbifold sense. It is constructed as follows. If $j \colon \mathcal{Y}^\circ \hookrightarrow \mathcal{Y}_U$ is the open immersion, then we let $\widetilde{\Omega}^\bullet_{\mathcal{Y}_U} := j_* \Omega^\bullet_{\mathcal{Y}^\circ}$, and we derive the relative version $\widetilde{\Omega}^\bullet_{\mathcal{Y}/U}$ out of it in the usual manner. An equivalent presentation is

$$\widetilde{\Omega}^\bullet_{\mathcal{Y}/U} = (\rho_* \Omega^\bullet_{\mathcal{X}/U})^G.$$

The complex $\widetilde{\Omega}^\bullet_{\mathcal{Y}/U}$ is a resolution of $g^{-1}\mathcal{O}_U$. Hence its kth relative hypercohomology computes $(R^k g_* \mathbb{C}) \otimes \mathcal{O}_U$ and satisfies

$$R^k g_* \widetilde{\Omega}^\bullet_{\mathcal{Y}/U} \simeq (R^k h_* \Omega^\bullet_{\mathcal{X}/U})^G \tag{3.3}$$

compatibly with $R^k g_* \mathbb{C} \simeq (R^k h_* \mathbb{C})^G$. It has a Hodge filtration and a Gauss–Manin connection defined in the usual way, satisfying a relationship analogous to (3.3). Equipped with this extra structure, $R^k g_* \mathbb{Q}$ defines a variation of pure rational Hodge structures of weight k.

In [Ste77, Lemma 1.11], a canonical identification $\widetilde{\Omega}^\bullet_{\mathcal{Y}_U} = \pi_* \Omega^\bullet_{\mathcal{Z}_U}$ is established. It induces a natural morphism

$$\widetilde{\Omega}^\bullet_{\mathcal{Y}/U} \longrightarrow \pi_* (\Omega^\bullet_{\mathcal{Z}/U}). \tag{3.4}$$

The restriction of (3.4) to $\mathcal{Y}^\circ$ is given by pulling back differential forms. We derive a natural map

$$(R^k h_* \Omega^\bullet_{\mathcal{X}/U})^G \simeq R^k g_* \widetilde{\Omega}^\bullet_{\mathcal{Y}/U} \longrightarrow R^k f_* \Omega^\bullet_{\mathcal{Z}/U}, \tag{3.5}$$

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

which is an injective morphism of variations of pure Hodge structures of weight k (see [Ste77, Cor. 1.5]). It is in particular compatible with restricting to the fibres and remains injective on those. It can be checked to be compatible with the topological $\mathbb{Q}$-structures, and hence we have an injective morphism of variations of rational Hodge structures over U

$$(R^k h_* \mathbb{Q})^G \hookrightarrow R^k f_* \mathbb{Q}. \tag{3.6}$$

Notice that at this stage, the compatibility of (3.5) with the algebraic geometric $\mathbb{Q}$-structure has not been addressed. This will be studied in later subsections.

Hodge bundles in the middle degree

In the case $k = n-1$, considering the isotypical components of the action of G on $R^{n-1}h_*\mathbb{C}$, we have a direct sum decomposition,

$$R^{n-1}h_*\mathbb{C} = (R^{n-1}h_*\mathbb{C})^G \oplus \mathbb{E}_{\mathbb{C}}, \quad \text{where} \quad \mathbb{E}_{\mathbb{C}} = \bigoplus_{\substack{\chi\colon G\to\mathbb{C}^\times \\ \chi\neq 1}} (R^{n-1}h_*\mathbb{C})_\chi. \tag{3.7}$$

This decomposition is easily seen to be orthogonal for the intersection form on $R^{n-1}h_*\mathbb{C}$. In particular, the restriction of the intersection form to $(R^{n-1}h_*\mathbb{C})^G$ is nondegenerate, and Poincaré duality holds for $R^{n-1}g_*\mathbb{C} \simeq (R^{n-1}h_*\mathbb{C})^G$. Notice that the orthogonal of $(R^{n-1}h_*\mathbb{Q})^G$ in $R^{n-1}h_*\mathbb{Q}$ defines a rational structure on $\mathbb{E}_{\mathbb{C}}$, and hence (3.7) can be refined rationally.

We next relate the intersection forms of $(R^{n-1}h_*\mathbb{Q})^G$ and $R^{n-1}f_*\mathbb{Q}$. Before the first statement in this direction, we recall from Lemma 3.2 that $\mathcal{Y}_U$ is Gorenstein and $K_{\mathcal{X}/U}$ descends to the relative canonical bundle $K_{\mathcal{Y}/U}$.

Lemma 3.6.

1. *$\widetilde{\Omega}^{n-1}_{\mathcal{Y}/U}$ is the relative canonical bundle $K_{\mathcal{Y}/U}$.*
2. *The natural morphism $R^{n-1}g_*\widetilde{\Omega}^\bullet_{\mathcal{Y}/U} \longrightarrow R^{n-1}f_*\Omega^\bullet_{\mathcal{Z}/U}$ induces a commutative diagram*

$$\begin{array}{ccc}
R^q g_*\widetilde{\Omega}^p_{\mathcal{Y}/U} \otimes R^{n-1-q}g_*\widetilde{\Omega}^{n-1-p}_{\mathcal{Y}/U} & \longrightarrow & R^{n-1}g_*K_{\mathcal{Y}/U} \\
\downarrow & & \| \\
R^q f_*\Omega^p_{\mathcal{Z}/U} \otimes R^{n-1-q}f_*\Omega^{n-1-p}_{\mathcal{Z}/U} & \longrightarrow & R^{n-1}f_*K_{\mathcal{Z}/U}
\end{array}
\qquad
\begin{array}{l}
\xrightarrow{\text{tr}} \mathcal{O}_U \\
\xrightarrow{\text{tr}} \mathcal{O}_U
\end{array}$$

3. *The natural isomorphism $R^{n-1}g_*\widetilde{\Omega}^\bullet_{\mathcal{Y}/U} \simeq (R^{n-1}h_*\Omega^\bullet_{\mathcal{X}/U})^G$ induces a commutative diagram*

$$\begin{array}{ccccc}
R^q g_*\widetilde{\Omega}^p_{\mathcal{Y}/U} \otimes R^{n-1-q}g_*\widetilde{\Omega}^{n-1-p}_{\mathcal{Y}/U} & \longrightarrow & R^{n-1}g_*K_{\mathcal{Y}/U} & \xrightarrow{\text{tr}} & \mathcal{O}_U \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle |G|\cdot} \\
R^q h_*\Omega^p_{\mathcal{X}/U} \otimes R^{n-1-q}h_*\Omega^{n-1-p}_{\mathcal{X}/U} & \longrightarrow & R^{n-1}h_*K_{\mathcal{X}/U} & \xrightarrow{\text{tr}} & \mathcal{O}_U
\end{array}$$

Proof. For the first property, we notice that $\rho^* K_{\mathcal{Y}/U} = K_{\mathcal{X}/U}$, since both coincide outside a codimension ≥ 2 closed subset and $\mathcal{X}_U$ is smooth. Then we have the string of equalities

$$\widetilde{\Omega}^{n-1}_{\mathcal{Y}/U} = (\rho_* K_{\mathcal{X}/U})^G = (K_{\mathcal{Y}/U} \otimes \rho_* \mathcal{O}_{\mathcal{X}_U})^G = K_{\mathcal{Y}/U} \otimes (\rho_* \mathcal{O}_{\mathcal{X}_U})^G = K_{\mathcal{Y}/U}.$$

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

For the first diagram, only the commutativity of the triangle requires a justification. For this, we rely on general facts in duality theory. Our references are stated in the algebraic category. Corresponding complex analytic properties are obtained by analytification. With this understood, the commutativity of the triangle is a consequence of the three following facts: i) the transitivity of trace maps with respect to the composition of morphisms [Har66, Thm. 10.5 (TRA1)], ii) the crepant resolution property $\pi^* K_{\mathcal{Y}/U} = K_{\mathcal{Z}/U}$ and iii) $\mathcal{Y}_U$ has rational singularities so that $R\pi_* \mathcal{O}_{\mathcal{Z}_U} = \mathcal{O}_{\mathcal{Y}_U}$. The argument is similar for the second diagram. Briefly, one combines i) the transitivity of trace maps, ii) the duality $\rho_* K_{\mathcal{X}_U/\mathcal{Y}_U} = \mathrm{Hom}_{\mathcal{O}_{\mathcal{Y}_U}}(\rho_* \mathcal{O}_{\mathcal{X}_U}, \mathcal{O}_{\mathcal{Y}_U})$ and iii) the trace $\mathrm{tr}\colon \rho_* K_{\mathcal{X}_U/\mathcal{Y}_U} \to \mathcal{O}_{\mathcal{Y}_U}$ is given by $\varphi \mapsto \varphi(1)$ [Har66, proof of Prop. 6.5], and the composite map

$$K_{\mathcal{Y}/U} \longrightarrow \rho_* K_{\mathcal{X}/U} = K_{\mathcal{Y}/U} \otimes \rho_* K_{\mathcal{X}_U/\mathcal{Y}_U} \xrightarrow{\mathrm{id} \otimes \mathrm{tr}} K_{\mathcal{Y}/U}$$

is the multiplication by $|G|$. This is clear over $\mathcal{Y}^\circ$ since it is the étale quotient of $\mathcal{X}^\circ$ by G. It is then necessarily true everywhere. □

Proposition 3.7. *Let Q be the intersection form on $R^{n-1} f_* \mathbb{Q}$ and Q' the intersection form on $R^{n-1} h_* \mathbb{Q}$. Then via the injection in (3.6), we have $Q = \frac{1}{|G|} Q'$ on $(R^{n-1} h_* \mathbb{Q})^G$.*

Proof. It is enough to check the relationship after extending the scalars to $\mathbb{C}$, in which case we can use the Hodge decomposition. The proposition then follows from Lemma 3.6 and the fact that in the middle degree, the intersection form is induced by the cohomological cup product and the trace map. We notice that in dimension $n-1$, the topological and complex geometric trace maps differ by a factor $(2\pi i)^{n-1}$, but this is inconsequential for the problem at hand. □

Remark 3.8.

1. In the case of direct images of relative canonical sheaves, the discussion in the proof of Lemma 3.6 reduces to the chain of isomorphisms of line bundles

$$(h_* K_{\mathcal{X}/U})^G \xrightarrow{\sim} g_* K_{\mathcal{Y}/U} \xrightarrow{\sim} f_* K_{\mathcal{Z}/U}. \tag{3.8}$$

   We leave it to the reader to check that these are the natural morphisms already defined in the algebraic category over $\mathbb{Q}$.
2. Because of Proposition 3.7, and for the purposes of this article, it is natural to scale the intersection form on $(R^{n-1} h_* \mathbb{Q})^G$ as $\frac{1}{|G|} Q'$. This will be of minor importance below.

3.4. The Kodaira–Spencer maps and the Yukawa coupling

Recall that for a general variation of Hodge structures $(\mathcal{H}, \mathcal{F}^\bullet)$ on a complex manifold X, Griffiths transversality entails that the Gauss–Manin connection factors as an $\mathcal{O}_X$-linear morphism $\mathcal{F}^p / \mathcal{F}^{p+1} \to \left(\mathcal{F}^{p-1} / \mathcal{F}^p\right) \otimes \Omega^1_X$. This is the Kodaira–Spencer map, and in the setting of $R^{n-1} f_* \Omega^\bullet_{\mathcal{Z}/U}$, we also write it in the form

$$\mathrm{KS}^{(q)}\colon T_U \longrightarrow \mathrm{Hom}_{\mathcal{O}_U}(R^q f_* \Omega^{n-1-q}_{\mathcal{Z}/U}, R^{q+1} f_* \Omega^{n-2-q}_{\mathcal{Z}/U}). \tag{3.9}$$

A repeated application of the Kodaira–Spencer maps gives a morphism

$$Y\colon \mathrm{Sym}^{n-1} T_U \longrightarrow \mathrm{Hom}_{\mathcal{O}_U}(f_* K_{\mathcal{Z}/U}, R^{n-1} f_* \mathcal{O}_{\mathcal{Z}}) \simeq (f_* K_{\mathcal{Z}/U})^{\otimes -2}. \tag{3.10}$$

We can explicitly evaluate the morphism Y in terms of the sections $\psi d/d\psi$ of T_U and the section θ_0 of $(h_* K_{\mathcal{X}/U})^G \simeq f_* K_{\mathcal{Z}/U}$ (see (3.8)) constructed in (3.1). Then the morphism Y identifies with a rational function on U, denoted $Y(\psi)$. This is the definition of the so-called (unnormalised) *Yukawa coupling*.

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

Working with $(R^{n-1}h_*\Omega^\bullet_{\mathcal{X}/U})^G$ instead, one similarly defines a function $\widetilde{Y}(\psi)$. Via the morphism in (3.5), the functions $\widetilde{Y}(\psi)$ and $Y(\psi)$ can be compared. The only subtle point to bear in mind is the use of Serre duality in the definition of the Yukawa coupling. For Hodge bundles of complementary bi-degree, Serre duality is induced by the cup-product and the trace morphism. Hence, an application of Lemma 3.6 shows that $Y(\psi)$ and $\widetilde{Y}(\psi)$ are equal up to the order of G. With this understood, we can invoke the computation of the Yukawa coupling in [BvS95, Cor. 4.5.6 & Ex. 4.5.7], which summarises to

$$\tilde{Y}(\psi) = \int_{X_\psi} \left(\theta_0 \wedge \nabla^{n-1}_{\psi d/d\psi}\theta_0\right) = c\frac{\psi^{n-1}}{1-\psi^{n+1}}, \tag{3.11}$$

for some irrelevant constant $c \neq 0$. To ease the comparison with the expression in [BvS95], we make the following observations. First, their factor λz is $1/\psi^{n+1}$. Secondly, their evaluation of Y amounts to working with the section $\psi\theta_0$ instead of θ_0.

3.5. The middle degree Hodge bundles

We now further compare the middle degree Hodge bundles of the Dwork pencil $h\colon \mathcal{X} \to U$ and that of the mirror $f\colon \mathcal{Z} \to U$ by drawing on specific features of these families. We introduce primitivity notions for the relative Hodge bundles induced by any projective factorisation of f and the natural projective embedding of h. Observe that the latter is G-equivariant and defined over $\mathbb{Q}$. We also require the polarisation for $\mathcal{Z} \to U$ to be defined over $\mathbb{Q}$. Then the primitive Hodge bundles are defined in the algebraic category over $\mathbb{Q}$.

Construction of sections

We begin by constructing explicit sections of the middle degree Hodge bundles of $h\colon \mathcal{X} \to U$ via Griffiths' residue method [Gri69].

Our reasoning starts in the complex analytic category. Denote by $H = x_0 \cdot x_1 \cdot \ldots \cdot x_n$ and $\Omega = \sum(-1)^i x_i dx_0 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n \in H^0(\mathbb{P}^n, \Omega^n_{\mathbb{P}^n}(n+1))$. For $\psi \in U$, the residue along X_ψ

$$\theta_k = \mathrm{res}_{X_\psi}\left(\frac{k!H^k\Omega}{F_\psi^{k+1}}\right)$$

defines a G-invariant element of $H^{n-1}(X_\psi)$, still denoted θ_k. For $k = 0$, this indeed agrees with the holomorphic volume form in (3.1). Varying ψ gives us sections of $R^{n-1}h_*\Omega^\bullet_{\mathcal{X}/U}$, also denoted by θ_k. The constructed sections are primitive by [Gri69, Thm. 8.3].

From the definition of the sections θ_k, one can check the following recurrence:

$$\nabla_{d/d\psi}\,\theta_k = \mathrm{res}_{X_\psi}\left(\frac{\partial}{\partial\psi}\left(\frac{k!H^k\Omega}{F_\psi^{k+1}}\right)\right) = (n+1)\theta_{k+1}. \tag{3.12}$$

Lemma 3.9.

1. *For $k = 0, \ldots, n-1$, we have*

$$\theta_k \in F^{n-1-k}H^{n-1}(X_\psi)^G_{\mathrm{prim}}.$$

   Moreover, the spaces $H^{n-1-k,k}(X_\psi)^G_{\mathrm{prim}}$ are all one-dimensional, and the image of θ_k in $H^{n-1-k,k}(X_\psi)^G_{\mathrm{prim}}$ is a basis for $\psi \in U$. In particular, the local system $(R^{n-1}h_*\mathbb{Q})^G_{\mathrm{prim}}$ is of rank n.
2. *The sections θ_k trivialise $(R^{n-1}h_*\Omega^\bullet_{\mathcal{X}/U})^G_{\mathrm{prim}}$ outside of 0 and are algebraic and defined over $\mathbb{Q}$.*

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

Proof. For the first item, the spaces $H^{n-1-k,k}(X_\psi)^G_{\mathrm{prim}}$ are necessarily one-dimensional, which follows from a computation in the case of Fermat hypersurfaces; see [DMOS82, p. 82, Rmk. 7.5]. For the rest of (1), we use Griffiths' description of the Hodge filtration of a hypersurface in terms of residues of rational forms, reviewed in [Voi07, Chap. 6].

By [Voi07, Thm. 6.10], we indeed have for $k = 0, \dots, n-1$, $\theta_k \in F^{n-1-k}H^{n-1}(X_\psi)^G_{\mathrm{prim}}$. We need to verify that the projections of the sections θ_k onto $H^{n-1-k,k}(X_\psi)^G_{\mathrm{prim}}$ are everywhere nonzero on U. The following argument was suggested by the anonymous referee, whom we thank for allowing us to include it. A detailed study of Griffiths' residue map – see, for example, [Voi07, Cor. 6.12] – provides an isomorphism $[\mathbb{C}[x_0, \dots, x_n]/J]_{(n+1)\cdot k} \to H^{n-1-k,k}(X_\psi)_{\mathrm{prim}}$, where J denotes the Jacobian ideal of X_ψ in $\mathbb{P}^n$ and the index $(n+1)k$ refers to the homogeneous part of the corresponding degree. Recall the notation $H = x_0 \cdot \ldots \cdot x_n$. Since $\mathbb{C}[x_0, \dots, x_n]^G = \mathbb{C}[x_0^{n+1}, \dots, x_n^{n+1}, H]$, we find that

$$\left[\mathbb{C}[x_0^{n+1}, \dots, x_n^{n+1}, H]/J^G\right]_{(n+1)\cdot k} \simeq H^{n-1-k,k}(X_\psi)^G_{\mathrm{prim}},$$

where $J^G = J \cap \mathbb{C}[x_0, \dots, x_n]^G$. A straightforward computation shows that $x_i^{n+1} \equiv \frac{\psi}{n+1} H$ modulo J^G, so that in fact

$$\mathbb{C}[x_0^{n+1}, \dots, x_n^{n+1}, H]/J^G \simeq \mathbb{C}[H]/J^G.$$

Now the image of θ_k in $H^{n-1-k,k}(X_\psi)^G_{\mathrm{prim}}$ corresponds to the image of $k!H^k$ in $[\mathbb{C}[H]/J^G]_{(n+1)k}$ through the above isomorphisms, and the latter is a generator of $[\mathbb{C}[H]/J^G]_{(n+1)k}$, hence nonzero. Thus the projection of θ_k gives a basis of $H^{n-1-k,k}(X_\psi)^G_{\mathrm{prim}}$.

For the second item, we just need to address the second half of the statement. We observe that the section θ_0 of $(h_* K_{\mathcal{X}/U})^G$ is algebraic and defined over $\mathbb{Q}$. By the algebraic theory of the Gauss–Manin connection [KO68], we know that the latter preserves the algebraic de Rham cohomology $(R^{n-1}h_*\Omega^\bullet_{\mathcal{X}/U})^G_{\mathrm{prim}}$ and is defined over $\mathbb{Q}$. Because the vector field $d/d\psi$ is algebraic and defined over $\mathbb{Q}$, the claim follows from the recurrence in (3.12). □

Remark 3.10. An alternative approach to the nonvanishing of the projection of the sections θ_k onto $H^{n-1-k,k}(X_\psi)^G_{\mathrm{prim}}$ is based on the explicit expression of the Yukawa coupling in (3.11) and the realisation of the sections θ_k as iterated Gauss–Manin derivatives via (3.12). If either of θ_k have zero projection for some ψ, applying the Kodaira–Spencer map in (3.9) and the recurrence in (3.12), we see that all the projections of $\theta_{k'}$ with $k' \geq k$ are also zero at ψ. This implies that the Yukawa coupling, divided by ψ^{n-1} to work with the tangent vector $d/d\psi$ instead of $\psi d/d\psi$, also has a zero at ψ. But the expression in (3.11) divided by ψ^{n-1} has no zeros on U, from which we conclude.

The minimal component of the cohomology

Below, we show that the image of the primitive middle cohomology of the Dwork family under (3.6) is a direct factor of the cohomology of the mirror. Later, in Lemma 4.2, we will see that the complement is irrelevant for most considerations.

Lemma 3.11.

1. *The natural morphism in (3.6) induces an injective morphism of variations of polarised Hodge structures over* U^{an}

$$(R^{n-1}h_*\mathbb{Q})^G_{\mathrm{prim}} \hookrightarrow (R^{n-1}f_*\mathbb{Q})_{\mathrm{prim}}. \tag{3.13}$$

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

2. *The natural morphism*

$$(R^{n-1}h_*\Omega^\bullet_{\mathcal{X}/U})^G_{\mathrm{prim}} \hookrightarrow (R^{n-1}f_*\Omega^\bullet_{\mathcal{Z}/U})_{\mathrm{prim}} \tag{3.14}$$

deduced from (3.13) $\otimes\, \mathcal{O}_{U^{\mathrm{an}}}$ exists in the algebraic category over $\mathbb{Q}$.

Proof. For the proof of (1), it is enough to show that (3.5) restricts to a map between the primitive cohomologies. It will automatically be compatible with the polarisations, by Proposition 3.7. See Remark 3.8 (2) regarding the scaling of the intersection forms. By Lemma 3.9, it suffices to check that the sections θ_k of $(R^{n-1}h_*\Omega^\bullet_{\mathcal{X}/U})^G_{\mathrm{prim}}$ map into primitive classes. Let θ'_k be the image of θ_k under (3.5). As (3.5) is compatible with Gauss-Manin connections, the θ'_k satisfy the analogous recurrence to (3.12). Because $f_*K_{\mathcal{Z}/U}$ is primitive and the Gauss–Manin connection preserves primitive cohomology, we see that the θ'_k land in the primitive cohomology.

The claim in (2) is addressed in a similar manner. By Lemma 3.9, we already know that the sections θ_k constitute an algebraic trivialisation of $(R^{n-1}h_*\Omega^\bullet_{\mathcal{X}/U})^G_{\mathrm{prim}}$ defined over $\mathbb{Q}$. We need to prove that their images θ'_k in $(R^{n-1}f_*\Omega^\bullet_{\mathcal{Z}/U})_{\mathrm{prim}}$ are algebraic and defined over $\mathbb{Q}$ as well. This is the case of θ'_0, because the natural isomorphism $(h_*K_{\mathcal{X}/U})^G \simeq f_*K_{\mathcal{Z}/U}$ (see (3.8)) is algebraic and defined over $\mathbb{Q}$. In this respect, see Remark 3.8 (1). Because the θ'_k satisfy the analogous recurrence to (3.12), and the Gauss–Manin connection and the vector field $d/d\psi$ are algebraic and defined over $\mathbb{Q}$, we conclude. □

Notice that the image of $(R^{n-1}h_*\mathbb{Q})^G_{\mathrm{prim}}$ under (3.13) is the smallest subvariation of Hodge structures of $R^{n-1}f_*\mathbb{Q}$ whose Hodge filtration contains $f_*K_{\mathcal{Z}/U}$ (see (3.8)). This motivates the following definition:

Definition 3.12. The image of $(R^{n-1}h_*\mathbb{Q})^G_{\mathrm{prim}}$ in $(R^{n-1}f_*\mathbb{Q})_{\mathrm{prim}}$ under the morphism in (3.13) is denoted by $(R^{n-1}f_*\mathbb{Q})_{\mathrm{min}}$ and called the minimal component or minimal part. Likewise, we decorate algebraic variants (see (3.14)) and associated objects by min. For example, this applies to Hodge bundles and homology constructions.

The next step consists of isolating the complement of the minimal component. In preparation for the statement, we recall that the topological intersection form on $R^{n-1}f_*\mathbb{C}$ has a counterpart on the de Rham cohomology $R^{n-1}f_*\Omega^\bullet_{\mathcal{Z}/U}$, which is already defined in the algebraic category over $\mathbb{Q}$. Indeed, the construction of the latter involves the cohomological cup-product, the graded product structure on the complex $\Omega^\bullet_{\mathcal{Z}/U}$ and the algebraic geometric trace map

$$R^{n-1}f_*\Omega^\bullet_{\mathcal{Z}/U} \otimes R^{n-1}f_*\Omega^\bullet_{\mathcal{Z}/U} \xrightarrow{\cup} R^{2(n-1)}f_*\Omega^\bullet_{\mathcal{Z}/U} = R^{n-1}f_*K_{\mathcal{Z}/U} \xrightarrow{\mathrm{tr}} \mathcal{O}_U.$$

After forming $(R^{n-1}f_*\mathbb{C}) \otimes \mathcal{O}_{U^{\mathrm{an}}}$, the topological and algebraic intersection pairings agree up to a factor $(2\pi i)^{n-1}$, which accounts for the comparison of the trace maps. We are now ready for the next result.

Proposition 3.13 (Minimal decomposition).

1. *Let $\mathbb{V}$ be the orthogonal of $(R^{n-1}f_*\mathbb{Q})_{\mathrm{min}}$ in $(R^{n-1}f_*\mathbb{Q})_{\mathrm{prim}}$ for the topological intersection form. Then there is an orthogonal decomposition of variations of polarised rational Hodge structures over U^{an}*

$$(R^{n-1}f_*\mathbb{Q})_{\mathrm{prim}} = (R^{n-1}f_*\mathbb{Q})_{\mathrm{min}} \oplus \mathbb{V}. \tag{3.15}$$

Furthermore, we have

$$\mathbb{V} = \begin{cases} 0 & \text{if } n-1 \text{ is odd,} \\ \text{of pure type } \left(\frac{n-1}{2}, \frac{n-1}{2}\right) & \text{if } n-1 \text{ is even.} \end{cases}$$

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

2. *Let $\mathcal{V}$ be the orthogonal of $(R^{n-1}f_*\Omega^\bullet_{\mathcal{Z}/U})_{\min}$ in $(R^{n-1}f_*\Omega^\bullet_{\mathcal{Z}/U})_{\mathrm{prim}}$ for the algebraic geometric intersection form. Then there is a direct sum decomposition of locally free coherent sheaves with connection over U, in the algebraic category over $\mathbb{Q}$,*

$$(R^{n-1}f_*\Omega^\bullet_{\mathcal{Z}/U})_{\mathrm{prim}} = (R^{n-1}f_*\Omega^\bullet_{\mathcal{Z}/U})_{\min} \oplus \mathcal{V}. \tag{3.16}$$

Furthermore, the analytification of (3.16) is naturally identified with (3.15) $\otimes\, \mathcal{O}_{U^{\mathrm{an}}}$.

Proof. We first deal with (1). In the case of $n-1$ being odd, $(R^{n-1}f_*\mathbb{Q})_{\mathrm{prim}} = (R^{n-1}f_*\mathbb{Q})_{\min}$, by the very definition of the minimal component and by Lemma 3.4 and Lemma 3.9. In the case $n-1$ is even, we first notice that the intersection pairing is flat for the Gauss–Manin connection and that the orthogonal complement of a subvariation of rational Hodge structures in a variation of polarised rational Hodge structures is also a variation of polarised rational Hodge structures. Thus $\mathbb{V}$ is a variation of polarised rational Hodge structures. To obtain the decomposition of (3.15) with the required properties, we can reduce to the following general fact. Let (H, Q) be a polarised Hodge structure over $\mathbb{Q}$, of weight $2d$, and (E, Q) a sub-Hodge structure, such that $E^{(p,q)} = H^{(p,q)}$ for $p \neq q$. Let $V = E^\perp$ be the orthogonal of E for the intersection form Q. Then $H = E \oplus V$ and V is a Hodge structure over $\mathbb{Q}$, of pure type (d, d). To prove this fact, by linear algebra and the nondegeneracy of the intersection form, it is enough to verify that $E \cap V$ is trivial. Take any element x in the intersection, and decompose it in $H_{\mathbb{C}}$ according to the bidegree as $x = \sum x^{p,q}$. Then $\overline{x^{p,q}} \in E^{q,p} \subset E_{\mathbb{C}}$. On the other hand, $i^{p-q}Q(x, \overline{x^{p,q}}) = i^{p-q}Q(x^{p,q}, \overline{x^{p,q}}) \geq 0$, with equality only if $x^{p,q} = 0$. But this is the case since $x \in E^\perp$, proving the decomposition. It follows from the assumption $E^{(p,q)} = H^{(p,q)}$ for $p \neq q$ that the complement is of pure type (d, d).

For item (2), we first notice that since $(R^{n-1}f_*\Omega^\bullet_{\mathcal{Z}/U})_{\min}$ and $(R^{n-1}f_*\Omega^\bullet_{\mathcal{Z}/U})_{\mathrm{prim}}$ are locally free coherent sheaves, so is $\mathcal{V}$. Besides, the algebraic Gauss–Manin connection preserves $\mathcal{V}$ since it preserves the minimal component and the algebraic intersection form is flat. By the compatibility of the topological and algebraic intersection forms, the analytification of $\mathcal{V}$ is canonically identified with $\mathbb{V} \otimes \mathcal{O}_{U^{\mathrm{an}}}$. For the validity of the direct sum decomposition, we can reduce to the analytic setting, in which case it follows from (3.15) $\otimes\, \mathcal{O}_{U^{\mathrm{an}}}$. □

Remark 3.14. After Proposition 3.13, and with the conventions adopted in Definition 3.12, for the homology local systems, we have

$$(R^{n-1}f_*\mathbb{Q})^\vee_{\mathrm{prim}} = (R^{n-1}f_*\mathbb{Q})^\vee_{\min} \oplus \mathbb{V}^\vee. \tag{3.17}$$

We can thus consider $(R^{n-1}f_*\mathbb{Q})^\vee_{\min}$ as a subsystem of $(R^{n-1}f_*\mathbb{Q})^\vee_{\mathrm{prim}}$, which in turn can be seen as a subsystem of the homology local system $(R^{n-1}f_*\mathbb{Q})^\vee$. This allows us to interpret $(R^{n-1}f_*\mathbb{Q})^\vee_{\min}$ in terms of homology classes and Poincaré duals of these in terms of integration.

In the application of the arithmetic Riemann–Roch theorem to the BCOV conjecture, we will need sections of the Hodge bundles rather than the Hodge filtration (see Theorem 2.3). This is the reason behind the following definition.

Definition 3.15. We define η_k° as the trivialising section of $(R^k f_*\Omega^{n-1-k}_{\mathcal{Z}/U})_{\min}$, deduced from θ_k via the morphism in (3.13) and by projecting to the Hodge bundle. We also define $\eta_k = -(n+1)^{k+1}\psi^{k+1}\eta_k^\circ$.

Remark 3.16.

1. By construction, the section η_k vanishes at order $k+1$ at $\psi = 0$.
2. The sections η_k are algebraic and defined over $\mathbb{Q}$ by Lemma 3.9 and Lemma 3.11.

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

Lemma 3.17. *The sections η_k° satisfy the recurrence*

$$\mathrm{KS}^{(k)}\left(\frac{d}{d\psi}\right)\eta_k^\circ = (n+1)\eta_{k+1}^\circ. \tag{3.18}$$

Consequently,

$$\mathrm{KS}^{(k)}\left(\psi\frac{d}{d\psi}\right)\eta_k = \eta_{k+1}. \tag{3.19}$$

Proof. The first recurrence follows from (3.12), Lemma 3.11, the link between the Gauss–Manin connection ∇ and the Kodaira–Spencer maps $\mathrm{KS}^{(q)}$ and the definition of η_k°. The second recurrence follows from the first by the very definition of the sections η_k and the $\mathcal{O}_U$-linearity of the Kodaira–Spencer maps. □

4. The degeneration of the Hodge bundles of the mirror family

In the previous section, we exhibited explicit trivialising sections of the minimal part of the middle degree Hodge bundles of the mirror family $\mathcal{Z} \to U$. The next goal is to extend these sections to the whole compactification $\mathbb{P}^1$. We also address the trivialisation of the Hodge bundles other than the minimal part and in any degree. For these goals, we exploit the approach to degenerating Hodge structures via relative logarithmic de Rham cohomology.

4.1. Generalities on geometric degenerations of Hodge structures

We recall some background from Steenbrink [Ste76, Ste77] and our previous work [EFiMM21, Sec. 2 & Sec. 4]. We also refer to Illusie's survey [Ill94, Sec. 2.2 & Sec. 2.3]. Let $f\colon \mathcal{X} \to \mathbb{D}$ be a projective morphism of reduced analytic spaces over the unit disc $\mathbb{D}$. We suppose that the fibres X_t with $t \neq 0$ are smooth and connected. We consider the variation of Hodge structures associated to $R^k f_*\mathbb{Q}$ over the punctured disc $\mathbb{D}^\times$. Let T be its monodromy operator and ∇ the Gauss–Manin connection on the holomorphic vector bundle $(R^k f_*\mathbb{Q}) \otimes \mathcal{O}_{\mathbb{D}^\times} = R^k f_*\Omega^\bullet_{\mathcal{X}/\mathbb{D}^\times}$. Recall that T is a quasi-unipotent transformation of the cohomology of the general fibre. The flat vector bundle $(R^k f_*\Omega^\bullet_{\mathcal{X}/\mathbb{D}^\times}, \nabla)$ has a unique extension to a vector bundle on $\mathbb{D}$, such that ∇ extends to a regular singular connection whose residue $\mathrm{Res}_0\,\nabla$ is an endomorphism with eigenvalues in $[0,1) \cap \mathbb{Q}$. This is the Deligne (lower) canonical extension, denoted by ${}^\ell R^k f_*\Omega^\bullet_{\mathcal{X}/\mathbb{D}^\times}$. Occasionally, we may simply refer to it as the Deligne extension of $R^k f_*\mathbb{C}$. It can be realised as the hypercohomology $R^k f'\Omega^\bullet_{\mathcal{X}'/\mathbb{D}}(\log)$ of the logarithmic de Rham complex of a normal crossing model $f'\colon \mathcal{X}' \to \mathbb{D}$. The Hodge filtration $\mathcal{F}^\bullet$ on $R^k f_*\Omega^\bullet_{\mathcal{X}/\mathbb{D}^\times}$ extends to a filtration by vector sub-bundles, still denoted by $\mathcal{F}^\bullet$. Its locally free graded quotients are of the form $R^{k-p} f'\Omega^p_{\mathcal{X}'/\mathbb{D}}(\log)$. If the monodromy operator is unipotent, then the fibre of $R^k f'\Omega^\bullet_{\mathcal{X}'/\mathbb{D}}(\log)$ at 0, together with the restricted Hodge filtration, can be identified with the cohomology of the generic fibre $H^k_{\lim}$ with the limiting Hodge filtration $F^\bullet_\infty$. The identification depends on the choice of a holomorphic coordinate on $\mathbb{D}$. There is also the monodromy weight filtration $W_\bullet$ on $H^k_{\lim}$ attached to the nilpotent operator $N = -2\pi i\,\mathrm{Res}_0\,\nabla$. The triple $(H^k_{\lim}, F^\bullet_\infty, W_\bullet)$ is called the limiting mixed Hodge structure. It is isomorphic to Schmid's limiting mixed Hodge structure [Sch73] on the cohomology of the general fibre. In particular, $W_\bullet$ admits a rational structure. This structure is not needed in the current section, but it will be used later in Section 6 in the greater generality of higher-dimensional parameter spaces. In the general quasi-unipotent case, one first performs a semi-stable reduction and then constructs the limiting mixed Hodge structure.

More generally, for a subvariation of Hodge structures $\mathbb{E}$ of $R^k f_*\mathbb{Q}$, which is a direct summand, the previous constructions can also be carried out and relate to those of $R^k f_*\mathbb{Q}$ as follows. For concreteness,

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

let us comment on the case of $f\colon \mathcal{X} \to \mathbb{D}$ as above, with normal crossings model $f'\colon \mathcal{X}' \to \mathbb{D}$. Denote by $j\colon \mathbb{D}^\times \hookrightarrow \mathbb{D}$ the open immersion. Then the Deligne extension of $\mathcal{E} = \mathbb{E} \otimes \mathcal{O}_{\mathbb{D}^\times}$ equals $j_*\mathcal{E} \cap {}^\ell R^k f_* \Omega^\bullet_{\mathcal{X}/\mathbb{D}^\times}$, or equivalently $j_*\mathcal{E} \cap R^k f'_* \Omega^\bullet_{\mathcal{X}'/\mathbb{D}}(\log)$, where the intersection is taken in $j_* R^k f_* \Omega^\bullet_{\mathcal{X}/\mathbb{D}^\times}$. Let us denote it by ${}^\ell\mathcal{E}$. To construct the limiting mixed Hodge structure of $\mathbb{E}$, we may first perform a ramified base change and suppose that f' is semi-stable. Secondly, we intersect the limiting mixed Hodge structure $(H^k_{\lim}, F^\bullet_\infty, W_\bullet)$ of $R^k f_* \mathbb{Q}$ with ${}^\ell\mathcal{E}(0)$, the fibre at 0 of ${}^\ell\mathcal{E}$. In our work, we will encounter this setting for the standard case of the primitive cohomology but also for the decompositions in (3.7) (G-invariants) and the minimal decomposition of (3.15). Accordingly, the resulting objects will be decorated with the symbol prim, G or min. For example, we will have notations such as $R^{n-1} f'_* \Omega^\bullet_{\mathcal{X}'/\mathbb{D}}(\log)_{\min}$.

Analogously, for a projective normal crossings degeneration $f\colon \mathcal{X} \to S$ between complex algebraic manifolds, with one-dimensional S, there are algebraic counterparts of all the above: logarithmic de Rham cohomology, Gauss–Manin connection, Hodge filtration and so on. This is compatible with the analytic theory after localising to a holomorphic coordinate neighbourhood of a given point $p \in S$. We will in particular speak of the limiting mixed Hodge structure at p and simply write $H^k_{\lim}$ if there is no danger of confusion.

Finally, we will also need the limiting mixed Hodge structure $(H_k)_{\lim}$ on the homology, and in particular the dual weight filtration $W'_\bullet$ defined as $W'_{-r} = (H^k_{\lim}/W_{r-1})^\vee$. See [Del71, (4.2.2)] or [EZT14, (3.1.3.1) and (3.2.2.7)] for more information about dual filtrations.

4.2. Triviality of some variations of Hodge structures

We return to the geometric setting of Section 3 and maintain the notations therein. For the mirror family, we prove that outside of $(R^{n-1} f_* \mathbb{Q})_{\min}$, all the variations of Hodge structures appearing in our work correspond to trivial local systems. In particular, the local systems outside of the middle degree and the local system $\mathbb{V}$ from Proposition 3.13 are all trivial. We also derive consequences for the associated Hodge bundles in the algebraic category.

We fix the normal crossings model $f'\colon \mathcal{Z}' \to \mathbb{P}^1$ obtained by blowing up the locus of the ordinary double points of $f\colon \mathcal{Z} \to \mathbb{P}^1$, which is defined over $\mathbb{Q}$. We also introduce a polarisation induced by a projective factorisation of f' defined over $\mathbb{Q}$. The corresponding logarithmic Hodge bundles and their primitive parts are locally free sheaves over $\mathbb{P}^1$, already defined in the algebraic category and over $\mathbb{Q}$.

By Lemma 3.4, we have $R^d f'_* \Omega^\bullet_{\mathcal{Z}'/\mathbb{P}^1}(\log) = 0$ for d odd, not equal to $n-1$, while if $d = 2p \neq n-1$, $R^d f'_* \Omega^\bullet_{\mathcal{Z}'/\mathbb{P}^1}(\log) = R^p f'_* \Omega^p_{\mathcal{Z}'/\mathbb{P}^1}(\log)$. We then have the following result outside of middle degrees:

Lemma 4.1. *For $2p \neq n-1$, the following hold:*

1. *The local system $R^{2p} f_* \mathbb{Q}$ on $U^{\mathrm{an}} = \mathbb{P}^1 \setminus (\mu_{n+1} \cup \{\infty\})$ is trivial.*
2. *The Hodge bundle $R^p f'_* \Omega^p_{\mathcal{Z}'/\mathbb{P}^1}(\log)$ is trivial in the algebraic category over $\mathbb{Q}$.*

Proof. We first prove that the local system $R^{2p} f_* \mathbb{Q}$ is trivial. Take a base point $b \in U^{\mathrm{an}}$, and let $\rho\colon \pi_1(U^{\mathrm{an}}, b) \to \mathsf{GL}(H^{2p}(Z_b, \mathbb{Q}))$ be the monodromy representation determining the local system. The fundamental group $\pi_1(U^{\mathrm{an}}, b)$ is generated by loops γ_ξ circling around $\xi \in \mu_{n+1}$ and a loop γ_∞ circling around ∞, with a relation $\prod_\xi \gamma_\xi = \gamma_\infty$. Because the singularities of $\mathcal{Z} \to \mathbb{P}^1$ at the points ξ are ordinary double points, and $2p \neq n-1$, the local monodromies $\rho(\gamma_\xi)$ are trivial. Therefore $\rho(\gamma_\infty)$ is trivial as well, and so is ρ.

Now the first claim implies the triviality of $R^p f'_* \Omega^p_{\mathcal{Z}'/\mathbb{P}^1}(\log) = R^{2p} f'_* \Omega^\bullet_{\mathcal{Z}'/\mathbb{P}^1}(\log)$ in the analytic category, since the latter realises the Deligne extension of $R^{2p} f_* \Omega^\bullet_{\mathcal{Z}/U}$. By the GAGA principle, $R^p f'_* \Omega^p_{\mathcal{Z}'/\mathbb{P}^1}(\log)$ is algebraically trivial as a complex vector bundle. This already implies the second claim. Indeed, let E be a vector bundle over $\mathbb{P}^1_{\mathbb{Q}}$, which is trivial after base change to $\mathbb{C}$. Then the natural

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

morphism $H^0(\mathbb{P}^1_{\mathbb{Q}}, E) \otimes \mathcal{O}_{\mathbb{P}^1_{\mathbb{Q}}} \to E$ is necessarily an isomorphism, since it is an isomorphism after a flat base change. □

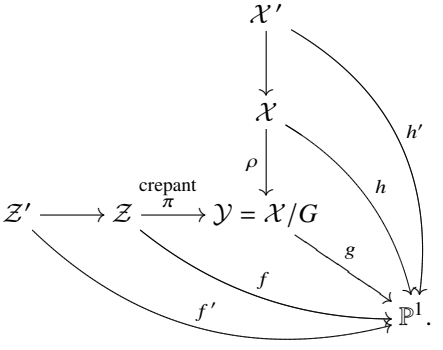
Lemma 4.2. *With the same notations as in Proposition 3.13, we have*

1. *The local system $\mathbb{V}$ on U^{an} is trivial.*
2. *The locally free coherent sheaf with connection $\mathcal{V}$ over U is trivial in the algebraic category over $\mathbb{Q}$.*

Proof. We first show that if the local monodromy of $(R^{n-1}f_*\mathbb{C})_{\mathrm{min}}$ around one ODP point is trivial, then it is so around all the ODP points. Since $(R^{n-1}f_*\mathbb{C})_{\mathrm{min}}$ is isomorphic to $(R^{n-1}h_*\mathbb{C})^G_{\mathrm{prim}}$ as a local system, it is enough to show that the latter descends along the natural projection $(U - \{0\}) \to (U - \{0\})/\mu_{n+1}$, where μ_{n+1} acts by multiplication on $(U - \{0\}) \subset \mathbb{P}^1$. Notice that for any $\zeta \in \mu_{n+1}$, the automorphism $\psi \mapsto \zeta \cdot \psi$ lifts to an automorphism of the family $g\colon \mathcal{Y} \to (U - \{0\})$ via the formula $[x_0 : x_1 : \ldots : x_n] \mapsto [x_0' : x_1' : \ldots : x_n']$, where $x_i' = x_i$ except for one i, for which $x_i' = \zeta^{-1} \cdot x_i$. Since we work in the quotient by the group G, all the choices of i correspond to the same action. We conclude that the local systems $(R^k h_*\mathbb{C})^G \simeq R^k g_*\mathbb{C}$ descend for all k. Observe that $R^2h_*\mathbb{C}$ is actually constant with fibre $H^2(\mathbb{P}^n, \mathbb{C})$, by Lefschetz, with G acting trivially. Therefore, the polarisation necessarily descends. We conclude that $(R^{n-1}h_*\mathbb{C})^G_{\mathrm{prim}} = \ker\left(L\colon (R^{n-1}h_*\mathbb{C})^G \to (R^{n+1}h_*\mathbb{C})^G\right)$ descends too, as was to be shown.

We now show that $\mathbb{V}$ is a trivial local system. It is enough to argue for $\mathbb{V}_{\mathbb{C}}$. In the odd-dimensional case, there is nothing to prove. In the even-dimensional case, we first recall that by the Picard–Lefschetz formula, the local monodromies on $(R^{n-1}f_*\mathbb{C})_{\mathrm{prim}}$ around the ODP points are semi-simple with a single nontrivial eigenvalue -1 of multiplicity one. It follows that around each ODP point, exactly one of the sublocal systems $(R^{n-1}f_*\mathbb{C})_{\mathrm{min}}$ and $\mathbb{V}_{\mathbb{C}}$ has trivial local monodromy. By the argument in the previous lemma, if the monodromies around one and hence all the ODP points on $(R^{n-1}f_*\mathbb{C})_{\mathrm{min}}$ were trivial, it would follow that the monodromy around ∞ would also be trivial. As this is excluded by Lemma 4.3 below, we infer that $\mathbb{V}$ is a trivial local system.

We next address the triviality of $\mathcal{V}$ asserted by the second point. We will now make use of the G-equivariant normal crossings model $h'\colon \mathcal{X}' \to \mathbb{P}^1$ of Proposition 3.1. We summarise the current geometric setting in the following diagram, which builds upon (3.2):



We first argue analytically. By the minimal decomposition (Proposition 3.13), the morphism in (3.13) induces a morphism between Deligne extensions

$$^{\ell}(R^{n-1}h_*\Omega^{\bullet}_{\mathcal{X}/U})^G_{\mathrm{prim}} \hookrightarrow {}^{\ell}(R^{n-1}f_*\Omega^{\bullet}_{\mathcal{Z}/U})_{\mathrm{prim}}. \tag{4.1}$$

We can reformulate (4.1) as a morphism

$$\psi\colon (R^{n-1}h'_*\Omega^{\bullet}_{\mathcal{X}'/\mathbb{P}^1}(\log))^G_{\mathrm{prim}} \hookrightarrow (R^{n-1}f'_*\Omega^{\bullet}_{\mathcal{Z}'/\mathbb{P}^1}(\log))_{\mathrm{prim}},$$

extending (3.14) to $\mathbb{P}^1$. By the GAGA principle, this morphism is algebraic. Notice that the coherent sheaves involved in ψ are locally free and defined over $\mathbb{Q}$. By Lemma 3.11 (2), $\psi_{|U}$ is defined over $\mathbb{Q}$.

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

Therefore, $\psi_{|U}$ is invariant under the action of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. Because U is a nonempty Zariski open subset of $\mathbb{P}^1$, which is an integral scheme, we infer that ψ is invariant under $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. Therefore, ψ is defined over $\mathbb{Q}$, and so is its cokernel. We denote by $\widetilde{\mathcal{V}}$ this cokernel of ψ modulo its torsion part. Then $\widetilde{\mathcal{V}}$ is a vector bundle.

By Proposition 3.13 (2), $\widetilde{\mathcal{V}}_{|U}$ is canonically isomorphic to $\mathcal{V}$, over $\mathbb{Q}$, and in particular inherits a connection from $\mathcal{V}$. Also, by the same proposition, we know that the analytification of (3.16) is canonically identified with the tensor product of (3.15) with $\mathcal{O}_{U^{\mathrm{an}}}$. By taking Deligne extensions, we deduce that $\widetilde{\mathcal{V}}^{\mathrm{an}}$ is a vector bundle with regular singular connection, canonically isomorphic to the Deligne extension of $\mathbb{V} \otimes \mathcal{O}_{U^{\mathrm{an}}}$. By the first part of the lemma, we thus infer that $\widetilde{\mathcal{V}}^{\mathrm{an}}$ is a trivial vector bundle with connection, and in particular any trivialisation over $\mathbb{P}^1$ is flat. As in the proof of Lemma 4.1 (2), we deduce that $\widetilde{\mathcal{V}}$ is a trivial vector bundle over $\mathbb{P}^1$, defined over $\mathbb{Q}$. From all the above, we conclude that the restriction to U of any trivialisation of $\widetilde{\mathcal{V}}$, defined over $\mathbb{Q}$, induces a flat trivialisation of $\mathcal{V} \simeq \widetilde{\mathcal{V}}_{|U}$, defined over $\mathbb{Q}$. This concludes the proof. □

4.3. Behaviour of η_k at the MUM point

For the mirror family $f\colon \mathcal{Z} \to \mathbb{P}^1$, let $\mathbb{D}_\infty$ be a holomorphic disc neighbourhood at infinity, with parameter $t = 1/\psi$. To lighten notations, we still denote by $f\colon \mathcal{Z} \to \mathbb{D}_\infty$ the restricted family. To simplify notation, we write $H^{n-1}_{\lim}$ for the limiting mixed Hodge structure at infinity of $(R^{n-1}f_*\mathbb{Q})_{\min}$.

Lemma 4.3.

1. *The monodromy T of $(R^{n-1}f_*\mathbb{Q})_{\min}$ at ∞ is maximally unipotent. In particular, the nilpotent operator N on $H^{n-1}_{\lim}$ satisfies $N^{n-1} \neq 0$.*
2. *The graded pieces $\mathrm{Gr}^W_k H^{n-1}_{\lim}$ are one-dimensional if k is even and trivial otherwise. For all $1 \leq k \leq n-1$, N induces isomorphisms*

$$\mathrm{Gr}^W_k N\colon \mathrm{Gr}^W_k H^{n-1}_{\lim} \xrightarrow{\sim} \mathrm{Gr}^W_{k-2} H^{n-1}_{\lim}.$$

3. *For all $1 \leq p \leq n-1$, N induces isomorphisms*

$$\mathrm{Gr}^p_{F_\infty} N\colon \mathrm{Gr}^p_{F_\infty} H^{n-1}_{\lim} \xrightarrow{\sim} \mathrm{Gr}^{p-1}_{F_\infty} H^{n-1}_{\lim}.$$

Proof. The maximally unipotent property for $(R^{n-1}f_*\mathbb{Q})_{\min} \simeq (R^{n-1}h_*\mathbb{Q})^G_{\mathrm{prim}}$ is proven in odd relative dimension in [HSBT10, Cor. 1.7]. Exactly the same argument yields the claim in even relative dimension. In particular, $N^{n-1} \neq 0$. This settles the first point. Because moreover N^{n-1} induces an isomorphism $\mathrm{Gr}^W_{2(n-1)} H^{n-1}_{\lim} \xrightarrow{\sim} \mathrm{Gr}^W_0 H^{n-1}_{\lim}$, we deduce that $\mathrm{Gr}^W_0 H^{n-1}_{\lim} \neq 0$. By Lemma 3.9, $H^{n-1}_{\lim}$ is n-dimensional, and the second item follows for dimension reasons. Finally, we use that $\mathrm{Gr}^p_{F_\infty} H^{n-1}_{\lim}$ is one-dimensional again by Lemma 3.9 and then necessarily $\mathrm{Gr}^p_{F_\infty} H^{n-1}_{\lim} = \mathrm{Gr}^p_{F_\infty} \mathrm{Gr}^W_{2p} H^{n-1}_{\lim} = \mathrm{Gr}^W_{2p} H^{n-1}_{\lim}$. Hence the second point implies the third. □

By the maximally unipotent monodromy and for dimension reasons, the T-invariant classes of the minimal cohomology of a general fibre span a rank one trivial subsystem of $(R^{n-1}f_*\mathbb{C})_{\min}$ on $\mathbb{D}^\times_\infty$. We fix a basis γ' of this trivial system. It extends to a nowhere vanishing holomorphic section of the Deligne extension of $(R^{n-1}f_*\mathbb{C})_{\min}$. The fibre at 0 is then a basis for W_0, which identifies with $\ker N$ by the above lemma. We still write γ' for this limit element. Similarly, $(R^{n-1}f_*\mathbb{C})^\vee_{\min}$ has a rank one trivial subsystem spanned by the class of a T-invariant homological cycle γ. We may choose γ to correspond to γ' by Poincaré duality.[6] Hence, for any $\eta \in H^{n-1}(Z_t)$, $t \in \mathbb{D}^\times_\infty$, the period $\langle \gamma, \eta \rangle$ equals the intersection pairing $Q(\gamma', \eta)$. It is possible to explicitly construct an invariant cycle. Although we will need this

[6]Recall from Proposition 3.13 and Remark 3.14 that classes in $(R^{n-1}f_*\mathbb{C})^\vee_{\min}$ can be seen as homological cycles, and Poincaré duality can be used on the minimal component.

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

in a moment, we postpone the discussion to §5.2, where a broader study of homological cycles is delivered.

In preparation for the following lemma, we recall from the preliminaries in §4.1 that $f_*K_{\mathcal{Z}/\mathbb{D}_\infty}(\log)$ is isomorphic to $\mathcal{F}^{n-1}R^{n-1}\Omega^\bullet_{\mathcal{Z}/\mathbb{D}_\infty}(\log)$ and that $R^{n-1}\Omega^\bullet_{\mathcal{Z}/\mathbb{D}_\infty}(\log)$ realises the Deligne extension of $(R^{n-1}f_*\mathbb{C})\otimes\mathcal{O}_{\mathbb{D}^\times_\infty}$ to $\mathbb{D}_\infty$.

Lemma 4.4. *Let η be a holomorphic trivialisation of $f_*K_{\mathcal{Z}/\mathbb{D}_\infty}(\log)$. Then the period $\langle\gamma,\eta\rangle$ defines a holomorphic function on $\mathbb{D}_\infty$, nonvanishing at the origin.*

Proof. The argument is well-known – see, for example, [Mor93, Prop.] and [Voi99, Lemma 3.10] – but we sketch it due to its relevance.

The pairing $\langle\gamma,\eta\rangle = Q(\gamma',\eta)$ is clearly a holomorphic function on $\mathbb{D}^\times_\infty$ since both γ' and η are holomorphic sections of $(R^{n-1}f_*\mathbb{C})\otimes\mathcal{O}_{\mathbb{D}^\times_\infty}$. Moreover, they are both global sections of the Deligne extension. This ensures that $|Q(\gamma',\eta)|$ has at most a logarithmic singularity at 0. It follows that $Q(\gamma',\eta)$ is actually a holomorphic function.

For the nonvanishing property, we make use of the interplay between the intersection pairing seen on $H^{n-1}_{\lim}$ and the monodromy weight filtration [Sch73, Lemma 6.4], together with Lemma 4.3. Let $\eta'\in H^{n-1}_{\lim}$ be the fibre of η at 0. We need to show that $Q(\gamma',\eta')\neq 0$. Suppose the contrary. Since γ' is a basis of $W_0=\ker N=\operatorname{Im}N^{n-1}$, we have $\eta'\in(\operatorname{Im}N^{n-1})^\perp$. The intersection pairing is nondegenerate and satisfies $Q(Nx,y)+Q(x,Ny)=0$. Therefore, we find that $\eta'\in(\operatorname{Im}N^{n-1})^\perp=\ker N^{n-1}=W_{2n-3}$. But η' is a basis of $F^{n-1}H^{n-1}_{\lim}=F^{n-1}\operatorname{Gr}^W_{2n-2}H^{n-1}_{\lim}$, and therefore $\eta'\notin W_{2n-3}$. We thus have reached a contradiction. □

Before the next theorem, we consider the logarithmic extension of the Kodaira–Spencer maps in (3.9): if D is the divisor $[\infty]+\sum_{\xi^{n+1}=1}[\xi]$, then

$$\mathrm{KS}^{(q)}\colon T_{\mathbb{P}^1}(-\log D)\longrightarrow\operatorname{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(R^qf_*\Omega^{n-1-q}_{\mathcal{Z}'/\mathbb{P}^1}(\log),R^{q+1}f_*\Omega^{n-2-q}_{\mathcal{Z}'/\mathbb{P}^1}(\log)).\tag{4.2}$$

They preserve the minimal and primitive components.

Theorem 4.5. *The section η_k is a holomorphic trivialisation of $R^kf_*\Omega^{n-1-k}_{\mathcal{Z}/\mathbb{D}_\infty}(\log)_{\min}$.*

Proof. First of all, we prove that η_0 is a meromorphic section of $f_*K_{\mathcal{Z}/\mathbb{D}_\infty}(\log)$. Indeed, η_0 is an algebraic section of $f_*K_{\mathcal{Z}/U}$ (see Lemma 3.17), hence a rational section of $f_*K_{\mathcal{Z}'/\mathbb{P}^1}(\log)$ and thus a meromorphic section of $f_*K_{\mathcal{Z}/\mathbb{D}_\infty}(\log)$.

Second, we establish the claim of the theorem for η_0. By Lemma 4.4, we need to show that the holomorphic function $\langle\gamma,\eta_0\rangle$ on $\mathbb{D}^\times_\infty$ extends holomorphically to $\mathbb{D}_\infty$ and does not vanish at the origin. This property can be checked by a standard explicit computation reproduced below (5.8).

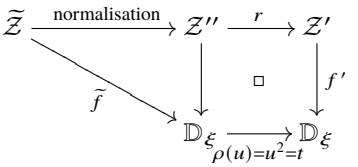
Finally, for the sections η_k, we use the recurrence in (3.19) and the logarithmic extension of the Kodaira–Spencer maps in (4.2). It follows that the sections η_k are global sections of the sheaves $R^kf_*\Omega^{n-1-k}_{\mathcal{Z}/\mathbb{D}_\infty}(\log)_{\min}$. Let us denote by η'_k the fibre at 0 of the sections η_k. Specialising (3.19) at 0, we find $(\operatorname{Gr}^{n-1-k}_{F_\infty}N)\eta'_k=\eta'_{k+1}$. By Lemma 4.3 (3), and because $\eta'_0\neq 0$, we see that $\eta'_k\neq 0$ for all k. This concludes the proof. □

4.4. Behaviour of η_k at the ODP points

Recall the normal crossings model $f'\colon\mathcal{Z}'\to\mathbb{P}^1$. We restrict it to a disc neighbourhood $\mathbb{D}_\xi$ of some $\xi\in\mu_{n+1}$. Concretely, we fix the coordinate $t=\psi-\xi$. We write $f'\colon\mathcal{Z}'\to\mathbb{D}_\xi$ for the restricted family. We now deal with the limiting mixed Hodge structure $H^{n-1}_{\lim}$ at ξ of $(R^{n-1}f_*\mathbb{Q})_{\min}$. Since the monodromy around ξ is not unipotent in general, the construction of $H^{n-1}_{\lim}$ requires a preliminary semi-stable reduction. This can be achieved as

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

follows:



(4.3)

Hence $\widetilde{f} : \widetilde{\mathcal{Z}} \to \mathbb{D}_\xi$ is the normalised base change of f' by ρ. An explicit computation in local coordinates shows it is indeed semi-stable. The special fibre $\widetilde{f}^{-1}(0)$ consists of two components intersecting transversally. One is the strict transform $\widetilde{Z}$ of Z_ξ. We denote by E the other component. Then E is a nonsingular quadric of dimension $n-1$, and $\widetilde{Z} \cap E$ is a nonsingular quadric of dimension $n-2$. In terms of this data, the monodromy weight filtration is computed as follows.

Lemma 4.6. *The graded pieces of the weight filtration on $H^{n-1}_{\lim}$ are given by*

- *if $n-1$ is odd, then*

$$\operatorname{Gr}^W_k H^{n-1}_{\lim} = \begin{cases} \mathbb{Q}\left(-\frac{n-2}{2}\right), & \text{if } k = n-2, \\ \text{a direct factor of } H^{n-1}(\widetilde{Z}), & \text{if } k = n-1, \\ \mathbb{Q}\left(-\frac{n}{2}\right), & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

- *if $n-1$ is even, then*

$$\operatorname{Gr}^W_k H^{n-1}_{\lim} = \begin{cases} \text{a direct factor of } \quad H\left(H^{n-3}(\widetilde{Z}\cap E)(-1) \to H^{n-1}(\widetilde{Z}) \oplus H^{n-1}(E) \to H^{n-1}(\widetilde{Z}\cap E)\right), & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } k = n-1, & \\ 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } k \neq n-1. & \end{cases}$$

Hence, $H^{n-1}_{\lim}$ is a pure Hodge structure of weight $n-1$.

Proof. The proof follows from [Ste77, Ex. 2.15], noticing that $(R^{n-1}\widetilde{f}_*\mathbb{Q})_{\min} = \rho^*(R^{n-1}f_*\mathbb{Q})_{\min}$ is a direct factor of $R^{n-1}\widetilde{f}_*\mathbb{Q}$, whose complement is a trivial variation of Hodge structures by Lemma 4.1 and Lemma 4.2. For the case $n-1$ is even, we moreover recall that $\mathbb{V}$ as in Proposition 3.13 has pure bidegree $((n-1)/2, (n-1)/2)$. □

We will need the comparison of the middle degree minimal Hodge bundles between before and after semi-stable reduction. We follow [EFiMM21, Sec. 2 & Prop. 3.10]. There are natural morphisms

$$\varphi^{p,q} : \rho^* R^q f'_* \Omega^p_{\mathcal{Z}'/\mathbb{D}_\xi}(\log)_{\min} \hookrightarrow R^q \widetilde{f}_* \Omega^p_{\widetilde{\mathcal{Z}}/\mathbb{D}_\xi}(\log)_{\min}. \tag{4.4}$$

Lemma 4.7. *Suppose that $p+q = n-1$. Let $Q^{p,q}$ be the cokernel of $\varphi^{p,q}$ in (4.4).*

- *If $p \neq q$, then $Q^{p,q} = 0$.*
- *If $p = q = \frac{n-1}{2}$, then $Q^{p,p} = \mathcal{O}_{\mathbb{D}_\xi,0}/u\mathcal{O}_{\mathbb{D}_\xi,0}$.*

Proof. The results in [EFiMM21, Sec. 2 & Prop. 3.10] are explicitly stated for the whole Hodge bundles and describe the cokernels in terms of the semi-simple part of the monodromy acting on the limiting Hodge structure. For their minimal components, however, see Remark 2.7 (iii) in [EFiMM21] together with Proposition 3.13 and Lemma 4.2. □

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

We are now fully equipped for the proof of

Theorem 4.8. *The sections η_k extend to meromorphic sections of the logarithmic Hodge bundles $R^k f'_* \Omega^{n-1-k}_{\mathcal{Z}'/\mathbb{A}^1}(\log)_{\min}$. Furthermore, denote by $\operatorname{ord}_\xi \eta_k$ the order of zero or pole of η_k at ξ, as a rational section of $R^k f'_* \Omega^{n-1-k}_{\mathcal{Z}'/\mathbb{A}^1}(\log)_{\min}$.*

- *If $n-1$ is odd, then $\operatorname{ord}_\xi \eta_k = 0$ for $k \leq n/2 - 1$ and $\operatorname{ord}_\xi \eta_k = -1$ otherwise.*
- *If $n-1$ is even, then $\operatorname{ord}_\xi \eta_k = 0$ for $k \leq \frac{n-3}{2}$ and $\operatorname{ord}_\xi \eta_k = -1$ otherwise.*

Proof. Throughout the proof, we write $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ for the respective total spaces over $\mathbb{A}^1$. We begin by showing that η_0 extends to a global section of $f'_* K_{\mathcal{Z}'/\mathbb{A}^1}(\log)$, nonvanishing at ξ. Since the singular fibres of $\mathcal{Z} \to \mathbb{A}^1$ present only ordinary double points, there is an equality

$$f_* K_{\mathcal{Z}/\mathbb{A}^1} = f'_* K_{\mathcal{Z}'/\mathbb{A}^1}(\log).$$

This can be seen as the coincidence of the upper and lower extensions of $f_* K_{\mathcal{Z}/U}$ to $\mathbb{A}^1$ (apply [EFiMM21, Cor. 2.8 & Prop. 2.10] and the Picard–Lefschetz formula for the monodromy). Since $\mathcal{Y}$ has rational singularities (see Lemma 3.2), the natural morphism $g_* K_{\mathcal{Y}/\mathbb{A}^1} \to f_* K_{\mathcal{Z}/\mathbb{A}^1}$ is an isomorphism. Also $g_* K_{\mathcal{Y}/\mathbb{A}^1} = (h_* K_{\mathcal{X}/\mathbb{A}^1})^G$. Indeed, let $\mathcal{X}^\circ$ be the complement of the fixed point locus of G in $\mathcal{X}$ and similarly for $\mathcal{Y}^\circ$, so that $\mathcal{Y} \setminus \mathcal{Y}^\circ$ has codimension ≥ 2. Then because $\mathcal{Y}$ is normal Gorenstein and $\mathcal{Y}^\circ = \mathcal{X}^\circ/G$ is an étale quotient and $\mathcal{X}$ is nonsingular, we find

$$g_* K_{\mathcal{Y}/\mathbb{A}^1} = g_* K_{\mathcal{Y}^\circ/\mathbb{A}^1} = (h_* K_{\mathcal{X}^\circ/\mathbb{A}^1})^G = (h_* K_{\mathcal{X}/\mathbb{A}^1})^G.$$

By construction of η_0 (see Definition 3.15), it is enough to prove that θ_0 defines a trivialisation of $h_* K_{\mathcal{X}/\mathbb{A}^1}$ around ξ. Denote by $\mathcal{X}^*$ the complement in $\mathcal{X}$ of the ordinary double points, so that $\mathcal{X} \setminus \mathcal{X}^*$ has codimension ≥ 2. Because $\mathcal{X}$ is nonsingular, we have $h_* K_{\mathcal{X}/\mathbb{A}^1} = h_* K_{\mathcal{X}^*/\mathbb{A}^1}$. Now the expression in (3.1) for θ_0 defines a relative holomorphic volume form on the whole $\mathcal{X}^*$ and hence a trivialisation of $h_* K_{\mathcal{X}^*/\mathbb{A}^1}$ as desired.

That the sections η_k define meromorphic sections of the sheaves $R^k f'_* \Omega^{n-1-k}_{\mathcal{Z}'/\mathbb{A}^1}(\log)_{\min}$ follows from the corresponding property for η_0 plus the recurrence in (3.19) and the existence of the logarithmic extension of the Kodaira–Spencer maps in (4.2). From the same recurrence, we reduce the computation of $\operatorname{ord}_\xi \eta_k$ to the computation of the orders at ξ of the rational morphisms $\mathrm{KS}^{(j)}(\psi d/d\psi)$, with respect to the logarithmic extension of the Hodge bundles

$$\operatorname{ord}_\xi \eta_k = \operatorname{ord}_\xi \eta_0 + \sum_{j=0}^{k-1} \operatorname{ord}_\xi \mathrm{KS}^{(j)}\left(\psi \frac{d}{d\psi}\right) = \sum_{j=0}^{k-1} \operatorname{ord}_\xi \mathrm{KS}^{(j)}\left(\psi \frac{d}{d\psi}\right).$$

Let us define $M^{(j)} = \operatorname{ord}_\xi \mathrm{KS}^{(j)}\left(\psi \frac{d}{d\psi}\right)$. Because η_0 trivialises $f_* K_{\mathcal{Z}/\mathbb{A}^1}$ at ξ, formula (3.11) shows that

$$\sum_{j=0}^{n-2} M^{(j)} = \operatorname{ord}_\xi Y(\psi) = -1. \tag{4.5}$$

We argue that all but one of the $M^{(j)}$ are zero. For this, we relate $M^{(j)}$ to the action of the nilpotent operator N on the limiting mixed Hodge structure at ξ. Recall that we defined the coordinate $t = \psi - \xi$ on a disc neighbourhood $\mathbb{D}_\xi$ of ξ. The first observation is

$$\operatorname{ord}_{t=0} \mathrm{KS}^{(j)}\left(t \frac{d}{dt}\right) = \operatorname{ord}_\xi \mathrm{KS}^{(j)}\left((\psi - \xi)\frac{d}{d\psi}\right) = M^{(j)} + 1 \geq 0, \tag{4.6}$$

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

since the Kodaira–Spencer maps along logarithmic tangent vectors preserve the logarithmic Hodge bundles (see (4.2)). Hence, we see that $M^{(j)} \geq -1$. We now need to distinguish two cases, depending on the parity of $n-1$.

Odd case: If $n-1$ is odd, then the monodromy is unipotent and the fibre of $\mathrm{KS}^{(p)}(td/dt)$ at $t=0$ is already $\mathrm{Gr}^p_{F_\infty} N\colon \mathrm{Gr}^p_{F_\infty} H^{n-1}_{\lim} \to \mathrm{Gr}^{p-1}_{F_\infty} H^{n-1}_{\lim}$. From Lemma 4.6, we deduce that unless $p = n/2$, $\mathrm{Gr}^p_{F_\infty} N = 0$ so that $\mathrm{ord}_{t=0}\,\mathrm{KS}^{(p)}(td/dt) > 0$ and hence $M^{(p)} \geq 0$. By (4.5), we necessarily have $M^{(n/2)} = -1$ and the other $M^{(j)} = 0$.

Even case: If $n-1$ is even, the nilpotent operator N is trivial, but the monodromy is no longer unipotent. The construction of the limiting mixed Hodge structure thus involves a semi-stable reduction. Choose a square root u of t as in (4.3). Then since $u\frac{d}{du} = 2t\frac{d}{dt}$, we get, comparing the Gauss-Manin connection before and after semi-stable reduction, a commutative diagram of maps of line bundles

$$\begin{array}{ccc} \rho^* R^q f'_* \Omega^p_{\mathcal{Z}'/\mathbb{D}_\xi}(\log)_{\min} & \xrightarrow{\mathrm{KS}^{(q)}\left(u\frac{d}{du}\right)} & \rho^* R^{q+1} f'_* \Omega^{p-1}_{\mathcal{Z}'/\mathbb{D}_\xi}(\log)_{\min} \\ \downarrow{\scriptstyle \varphi^{p,q}} & & \downarrow{\scriptstyle \varphi^{p-1,q+1}} \\ R^q \widetilde{f}_* \Omega^p_{\widetilde{\mathcal{Z}}/\mathbb{D}_\xi}(\log)_{\min} & \xrightarrow{\mathrm{KS}^{(q)}\left(2t\frac{d}{dt}\right)} & R^{q+1} \widetilde{f}_* \Omega^{p-1}_{\widetilde{\mathcal{Z}}/\mathbb{D}_\xi}(\log)_{\min}. \end{array}$$

Together with $\mathrm{ord}_{u=0} = 2\,\mathrm{ord}_{t=0}$, we conclude that

$$\mathrm{ord}_{u=0}\,\varphi^{p,q} + \mathrm{ord}_{u=0}\,\mathrm{KS}^{(q)}\left(u\frac{d}{du}\right) = \mathrm{ord}_{u=0}(\varphi^{p-1,q+1}) + 2\,\mathrm{ord}_{t=0}\,\mathrm{KS}^{(q)}\left(t\frac{d}{dt}\right). \tag{4.7}$$

By Lemma 4.7, $\mathrm{ord}_{u=0}(\varphi^{p,q}) = 0$ except for the case $(p,q) = ((n-1)/2, (n-1)/2)$, where $\mathrm{ord}_{u=0}(\varphi^{p,q}) = 1$. From (4.7), we then conclude that

$$\mathrm{ord}_{u=0}\,\mathrm{KS}^{((n-3)/2)}\left(u\frac{d}{du}\right) = 1 + 2\,\mathrm{ord}_{t=0}\,\mathrm{KS}^{((n-3)/2)}\left(t\frac{d}{dt}\right) \tag{4.8}$$

$$1 + \mathrm{ord}_{u=0}\,\mathrm{KS}^{((n-1)/2)}\left(u\frac{d}{du}\right) = 2\,\mathrm{ord}_{t=0}\,\mathrm{KS}^{((n-1)/2)}\left(t\frac{d}{dt}\right). \tag{4.9}$$

In both (4.8) and (4.9), the order of vanishing of Kodaira–Spencer along the vector field $u\frac{d}{du}$ is strictly positive, since the restriction to 0 is the nilpotent operator $N = 0$. It follows that

$$\mathrm{ord}_{t=0}\,\mathrm{KS}^{((n-3)/2)}\left(t\frac{d}{dt}\right) \geq 0, \quad \text{i.e.,} \quad M^{((n-3)/2)} \geq -1,$$

and

$$\mathrm{ord}_{t=0}\,\mathrm{KS}^{((n-1)/2)}\left(t\frac{d}{dt}\right) \geq 1, \quad \text{i.e.,} \quad M^{((n-1)/2)} \geq 0.$$

Since all other $M^{(j)} \geq 0$ as in the odd case, we conclude from (4.5) that all these inequalities are equalities. □

5. The BCOV invariant of the mirror family

In this section, we prove the first part of the Main Theorem in the introduction to the effect that the BCOV invariant of the mirror family encapsulates the Gromov–Witten invariants of a general

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

Calabi–Yau hypersurface. The proof proceeds by applying the arithmetic Riemann–Roch theorem as in Section 2, by choosing the algebraic trivialisations of the Hodge bundles studied in Section 3. This is then worked out in terms of canonical sections of the Hodge bundles, whose existence is tied to the limiting Hodge structure $H^{n-1}_{\lim}$ at the MUM point. In the process, a transcendental expression built out of periods arises, matching Zinger's formula for the sought generating function of Gromov–Witten invariants.

5.1. The Kronecker limit formula for the mirror family

For the mirror family $f\colon \mathcal{Z} \to U$, we proceed to prove an expression for the BCOV invariant $\tau_{\mathrm{BCOV}}(Z_\psi)$ in terms of the L^2 norms of the sections η_k (see Definition 3.15). The strategy follows the same lines as for families of Calabi–Yau hypersurfaces §2.4.

We fix a polarisation and a projective factorisation of f, defined over $\mathbb{Q}$. We denote by L the corresponding *algebraic* Lefschetz operator, which is the cup-product against the algebraic cycle class of a hyperplane section. We will abusively confound L with the algebraic cycle class of a hyperplane section. With this choice of L, the primitive decomposition of the Hodge bundles $R^p f_*\Omega^q_{\mathcal{Z}/U}$ holds over $\mathbb{Q}$. Let h be a Kähler metric and ω the Kähler form normalised as in (2.1), and assume that the fibrewise cohomology class is in the topological hyperplane class. Hence, under the correspondence between algebraic and topological cycle classes, L is sent to $(2\pi i)[\omega] \in R^2 f_*\mathbb{Q}(1)$.

Below, all the L^2 norms are computed with respect to ω as in (2.2).

Theorem 5.1. *There exists a real positive constant $C \in \pi^c\overline{\mathbb{Q}}^\times$ such that*

$$\tau_{\mathrm{BCOV}}(Z_\psi) = C\left|\frac{(\psi^{n+1})^a}{(1-\psi^{n+1})^b}\right|^2 \frac{\|\eta_0\|_{\mathrm{L}^2}^{\chi/6}}{\left(\prod_{k=0}^{n-1}\|\eta_k\|_{\mathrm{L}^2}^{2(n-1-k)}\right)^{(-1)^{n-1}}},$$

where $\chi = \chi(Z_\psi)$ and

$$\begin{aligned} a &= (-1)^{n-1}\frac{n(n-1)}{6} - \frac{\chi}{12(n+1)},\\ b &= (-1)^{n-1}\frac{n(3n-5)}{24}\\ c &= \frac{1}{2}\sum_k (-1)^{k+1}k^2 b_k. \end{aligned}$$

Proof. We apply the version of the arithmetic Riemann–Roch theorem formulated in Theorem 2.3 to the family $f\colon \mathcal{Z} \to U$ as being defined over $\mathbb{Q}$.

Choices of sections. We need to specify the section η and the sections $\eta_{p,q}$ in (2.10). The section η is chosen to be η_0, as defined in Definition 3.15. We next describe our choices of $\eta_{p,q}$:

- If $p+q \neq n-1$ and $p \neq q$, then the corresponding Hodge bundle vanishes by Lemma 3.4 and thus gives no contribution.
- For $2p \neq n-1$, Lemma 4.1 guarantees that $\det R^p f'_*\Omega^p_{\mathcal{Z}'/\mathbb{P}^1}(\log) = \det R^{2p} f'_*\Omega^\bullet_{\mathcal{Z}'/\mathbb{P}^1}(\log)$ is trivial, in the algebraic category over $\mathbb{Q}$, and any trivialisation is flat for the Gauss–Manin. We choose $\eta_{p,p}$ to be any trivialisation defined over $\mathbb{Q}$ and then restrict it to U. Notice that the L^2 norm $\|\eta_{p,p}\|_{\mathrm{L}^2}$ is then constant.
- For $p+q = n-1$ and $p \neq q$, the (p,q) Hodge bundle is primitive and has rank one. Then we take $\eta_{p,q} = \eta_{\mathrm{Q}}$ in Definition 3.15. By Lemma 3.17, η_{Q} is defined over $\mathbb{Q}$.
- For $p+q = n-1$ and $p = q$, which can only occur when $n-1$ is even, the (p,q) Hodge bundle is no longer primitive of rank one. We employ first the algebraic primitive decomposition and then the

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

minimal decomposition of Proposition 3.13 (2)

$$\begin{aligned}\det R^p f_*\Omega^p_{\mathcal{Z}/U} &= \det(R^p f_*\Omega^p_{\mathcal{Z}/U})_{\mathrm{prim}} \otimes \det LR^{p-1} f_*\Omega^{p-1}_{\mathcal{Z}/U}\\ &\simeq \det(R^p f_*\Omega^p_{\mathcal{Z}/U})_{\mathrm{prim}} \otimes \det R^{p-1} f_*\Omega^{p-1}_{\mathcal{Z}/U}\\ &\simeq \det(R^p f_*\Omega^p_{\mathcal{Z}/U})_{\mathrm{min}} \otimes \det \mathcal{V} \otimes \det R^{p-1} f_*\Omega^{p-1}_{\mathcal{Z}/U}.\end{aligned} \tag{5.1}$$

We define $\eta_{\frac{n-1}{2},\frac{n-1}{2}}$ as the element corresponding to $\eta_{\frac{n-1}{2}} \otimes v \otimes \eta_{\frac{n-3}{2},\frac{n-3}{2}}$ under this isomorphism, for any algebraic flat trivialisation $v \in \det \mathcal{V}$, defined over $\mathbb{Q}$, provided by Lemma 4.2. We claim that

$$\|\eta_{\frac{n-1}{2},\frac{n-1}{2}}\|^2_{\mathrm{L}^2} \sim_{\mathbb{Q}^\times} \|\eta_{\frac{n-1}{2}}\|^2_{\mathrm{L}^2}\|v\|^2_{\mathrm{L}^2}\|\eta_{\frac{n-3}{2},\frac{n-3}{2}}\|^2_{\mathrm{L}^2}, \tag{5.2}$$

where $\sim_{\mathbb{Q}^\times}$ denotes equality up to a rational number. For this, we bring together several facts. The first is that the Lefschetz decomposition is orthogonal for the L^2 metrics regardless of the normalisation of the Kähler forms. The second is that the algebraic cycle class of L corresponds to $(2\pi i)[\omega]$ in analytic de Rham cohomology. The third fact is that the operator $[2\pi\omega] \wedge \cdot$ is an isometry up to a rational constant, since $2\pi\omega$ is the Hodge theoretic Kähler form (see, for instance, [Huy05, Prop. 1.2.31]). The last fact is that the minimal component decomposition of Proposition 3.13 is also orthogonal for the L^2 norm, since it is orthogonal for the intersection form by construction. This settles (5.2). Furthermore, we notice that as for $\eta_{\frac{n-3}{2},\frac{n-3}{2}}$, the L^2 norm of v is constant, since it is flat by construction and it is the wedge product of a collection of sections of pure Hodge bidegree $((n-1)/2,(n-1)/2)$. Therefore, the norm $\|\eta_{\frac{n-1}{2},\frac{n-1}{2}}\|^2_{\mathrm{L}^2}$ equals $\|\eta_{\frac{n-1}{2}}\|^2_{\mathrm{L}^2}$ up to a constant.

Determining the rational function Δ. To establish the theorem, we need to specify the element $\Delta \in \mathbb{Q}(\psi)^\times \otimes \mathbb{Q}$ in (2.10) (formal rational power of a rational function), which satisfies

$$\log \tau_{\mathrm{BCOV}} = \log|\Delta|^2 + \frac{\chi}{12}\log\|\eta\|^2_{\mathrm{L}^2} - \sum_{p,q}(-1)^{p+q} p \log\|\eta_{p,q}\|^2_{\mathrm{L}^2} + \log C_\sigma. \tag{5.3}$$

We will determine Δ up to an algebraic number. To this end, it suffices to know its divisor. Unless $\psi = 0$ or $\psi = \xi$, where $\xi^{n+1} = 1$, Δ has no zeroes or poles by construction, since the sections $\eta_{p,q}$ are holomorphic and nonvanishing, and $\log \tau_{\mathrm{BCOV}}$ is smooth. Hence we are led to consider the logarithmic behaviour of the right-hand side of (5.3) at these points. Since for $2p \neq n-1$ the sections $\eta_{p,p}$ have constant L^2 norm, we only need to examine the functions $\log\|\eta_{p,q}\|_{\mathrm{L}^2}$ with $p+q = n-1$.

Behaviour at $\psi = 0$. This corresponds to a smooth fibre of $f\colon \mathcal{Z} \to U$. Hence $\log \tau_{\mathrm{BCOV}}$ is smooth at $\psi = 0$, as are the L^2 metrics. However, the sections $\eta_{p,q}$ with $p+q = n-1$ admit zeros at $\psi = 0$ (see Remark 3.16), with $\mathrm{ord}_0\,\eta_{p,q} = q+1 = n-p$. This means a in the theorem is given by

$$(n+1)a = (-1)^{n-1}\sum_{p=0}^{n-1} p(n-p) - \frac{\chi}{12} = (-1)^{n-1}\frac{(n-1)n(n+1)}{6} - \frac{\chi}{12}.$$

Behaviour at $\psi = \xi \in \mu_{n+1}$. This corresponds to a singular fibre of $f\colon \mathcal{Z} \to \mathbb{P}^1$, which has a unique ordinary double point. By Theorem 4.8, we control $\mathrm{ord}_\xi\,\eta_k$ according to the parity of $n-1$. Here we encounter the additional problem that the L^2 norms might have contributions from the semi-simple part of the monodromy T_s. More precisely, consider the local parameter $t = \psi - \xi$ around ξ, and write $\eta_{p,q} = t^{b_{p,q}}\sigma_{p,q}$, where $\sigma_{p,q}$ trivialises $\det R^q f_*\Omega^p_{\mathcal{Z}'/\mathbb{P}^1}(\log)$. Then by construction of $\eta_{p,q}$ and by [EFiMM21, Thm. C], we have

$$\log\|\eta_{p,q}\|^2_{\mathrm{L}^2} = (b_{p,q} + \alpha_{p,q})\log|t|^2 + o(\log|t|^2)$$

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

with

$$\alpha_{p,q} = -\frac{1}{2\pi i}\,\mathrm{tr}\left({}^{\ell}\log T_s \mid \mathrm{Gr}^p_{F_\infty} H^{n-1}_{\lim}\right) \in \mathbb{Q}.$$

Here ${}^{\ell}\log$ refers to the lower branch of the logarithm: that is, with argument in $2\pi(-1,0]$. Let us combine all this information:

Odd case: If $n-1$ is odd, according to Theorem 4.8, if $k \leq \frac{n}{2}-1, \mathrm{ord}_\xi\, \eta_k = 0$ and $\mathrm{ord}_\xi\, \eta_k = -1$ otherwise. In this case the monodromy is unipotent, so that $\alpha_{p,q} = 0$ for all $p+q = n-1$. Moreover, by [EFiMM21, Thm. B], we have that $\log \tau_{\mathrm{BCOV}} = \frac{n}{24}\log|t|^2 + o(\log|t|^2)$. Putting all these contributions together, we find that

$$-b = \frac{n}{24} + (-1)^{n-1}\sum_{k=n/2}^{n-1}(n-1-k)\cdot(-1) = \frac{n(3n-5)}{24}.$$

Even case: If $n-1$ is even, according to Theorem 4.8, if $k \leq \frac{n-3}{2}, \mathrm{ord}_\xi\, \eta_k = 0$ and $\mathrm{ord}_\xi\, \eta_k = -1$ otherwise. Also, unless $p = q = (n-1)/2$, $\alpha_{p,q} = 0$. In the remaining case $p = q = (n-1)/2$, by [EFiMM21, Prop. 3.10], we have $\alpha_{p,p} = 1/2$. Finally, from [EFiMM21, Thm. B], we have that $\log \tau_{\mathrm{BCOV}} = \frac{3-n}{24}\log|t|^2 + o(\log|t|^2)$. Putting all these contributions together, we find that

$$-b = \frac{3-n}{24} + (-1)^{n-1}\left((n-1)/2(-1+1/2) + \sum_{k=(n+1)/2}^{n-1}(n-1-k)\cdot(-1)\right) = -\frac{n(3n-5)}{24}.$$

Rationality considerations. To complete the proof of the theorem, we still need to tackle the constant C. Two sources contribute: i) for $2p \neq n-1$, the L^2 norms $\|\eta_{p,p}\|_{\mathrm{L}^2}$ are constant; and ii) if $n-1 = 2p$, after (5.2), there might be extra contributions from $\|\eta_{\frac{n-3}{2},\frac{n-3}{2}}\|_{\mathrm{L}^2}$ and from $\|v\|_{\mathrm{L}^2}$.

First for $2p \neq n-1$. Let $\psi_0 \in \mathbb{Q}$ be in the smooth locus so that we have the period isomorphism

$$H^{2p}(Z_{\psi_0}, \Omega^\bullet_{Z_{\psi_0}/\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H^{2p}(Z_{\psi_0}, \mathbb{Q}) \otimes \mathbb{C}.$$

Taking rational bases on both sides, the determinant can be defined in $\mathbb{C}^\times/\mathbb{Q}^\times$. It equals $(2\pi i)^{pb_{2p}}$. Since $\|\eta_{p,p}\|_{\mathrm{L}^2}$ is constant, it can be evaluated at $\psi = \psi_0$. We find

$$\|\eta_{p,p}\|^2_{\mathrm{L}^2} \sim_{\mathbb{Q}^\times} (2\pi)^{2pb_{2p}}\,\mathrm{vol}_{\mathrm{L}^2}(H^{2p}(Z_{\psi_0},\mathbb{Z}),\omega). \tag{5.4}$$

Now recall from (2.13) that with the Arakelov theoretic normalisation of the Kähler form, and under the integrality assumption on its cohomology class, we have $\mathrm{vol}_{\mathrm{L}^2}(H^{2p}(Z_{\psi_0},\mathbb{Z}),\omega) \sim_{\mathbb{Q}^\times} (2\pi)^{-2pb_{2p}}$. All in all, we arrive at the pleasant

$$\|\eta_{p,p}\|^2_{\mathrm{L}^2} \sim_{\mathbb{Q}^\times} 1. \tag{5.5}$$

If $n-1 = 2p$ is even, we will show that

$$\|\eta_{\frac{n-1}{2},\frac{n-1}{2}}\|^2_{\mathrm{L}^2} \sim_{\mathbb{Q}^\times} \|\eta_{\frac{n-1}{2}}\|^2_{\mathrm{L}^2}, \tag{5.6}$$

namely that both $\|\eta_{\frac{n-3}{2},\frac{n-3}{2}}\|^2_{\mathrm{L}^2}$ and $\|v\|^2_{\mathrm{L}^2}$ are rational. For $\eta_{\frac{n-3}{2},\frac{n-3}{2}}$, this is already known after (5.5). We will now see that formally the same argument yields the case of v. Since the norm of v is constant, it is enough to consider the value at any $\psi_0 \in \mathbb{Q}$ in the smooth locus. The results in §3.3 and §3.5 show that $(\mathbb{V}_{\psi_0}, \mathcal{V}_{\psi_0})$ behaves like the pair formed by the rational Betti and algebraic de Rham primitive cohomologies in degree $n-1$ of a smooth projective algebraic variety defined over $\mathbb{Q}$, of dimension $n-1$.

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

In particular, we have a period isomorphism and a Poincaré type duality induced by the intersection form. From this, one derives the analogue of (5.4) for v: for any rational basis v' of $\det \mathbb{V}_{\psi_0}$, we have

$$\|v\|^2_{\mathrm{L}^2} \sim_{\mathbb{Q}^\times} (2\pi)^{(n-1)d} \|v'\|^2_{\mathrm{L}^2}, \quad d = \dim \mathbb{V}_{\psi_0}.$$

Now we use that the Hodge structure on $\mathbb{V}_{\psi_0}$ is concentrated in bidegree $((n-1)/2, (n-1)/2)$, by Proposition 3.13, and we take into account the Arakelov theoretic normalisation of the Kähler form. We readily deduce $\|v'\|^2_{\mathrm{L}^2} \sim_{\mathbb{Q}^\times} (2\pi)^{-(n-1)d}$. All in all, we conclude that $\|v\|^2_{\mathrm{L}^2} \sim_{\mathbb{Q}^\times} 1$ as desired.

Finally, plug (5.3) into (5.5) in the cases $2p \neq n-1$ and into (5.6) in the case $2p = n-1$. Also plug in the value of C_σ furnished by Theorem 2.3, and recall that Δ was determined only up to algebraic number. We conclude that C has the asserted shape. □

Corollary 5.2. *As $\psi \to \infty$, $\log \tau_{\mathrm{BCOV}}(Z_\psi)$ behaves as*

$$\log \tau_{\mathrm{BCOV}}(Z_\psi) = \kappa_\infty \log |\psi|^{-2} + \varrho_\infty \log\log |\psi|^{-2} + \textit{continuous}, \tag{5.7}$$

where

$$\kappa_\infty = (-1)^n \frac{n+1}{12} \left(\frac{(n-1)(n+2)}{2} + \frac{1-(-n)^{n+1}}{(n+1)^2} \right),$$

$$\varrho_\infty = (-1)^{n-1} \frac{(n-1)(n+1)}{12} \left(\frac{(-n)^{n+1}-1}{(n+1)^2} - 2n + 1 \right).$$

Proof. The general shape (5.7) was proven in [EFiMM21, Prop. 6.8]. The precise value of κ_∞ is $(n+1)(b-a)$ entirely due to the term $\left| \frac{(\psi^{n+1})^a}{(1-\psi^{n+1})^b} \right|$ in Theorem 5.1. Indeed, by Theorem 4.5, the sections η_k trivialise $R^k f_* \Omega^{n-1-k}_{\mathcal{Z}'/\mathbb{P}^1}(\log)_{\min}$ at infinity, and moreover the monodromy is unipotent there (Lemma 4.3). This entails that the functions $\log \|\eta_k\|^2_{\mathrm{L}^2}$ are $O(\log\log |\psi|^{-2})$ at infinity and hence do not contribute to κ_∞. For the subdominant term, the expression of [EFiMM21, Prop. 6.8] can be explicitly evaluated for the mirror family, thanks to the complete understanding of the limiting Hodge structure at infinity (again Lemma 4.3) and the known value of χ (Lemma 3.4). □

5.2. Canonical trivialisations of the Hodge bundles at the MUM point

The Picard–Fuchs equation of the mirror

For the mirror family $f\colon \mathcal{Z} \to U$, we review classical facts on the Picard–Fuchs equation of the local system of middle-degree cohomologies. The discussion serves as the basis for the construction of canonical trivialising sections of the middle-degree Hodge bundles close to the MUM point, which differ from the η_k by some periods.

The starting point is the construction of an invariant $(n-1)$-homological cycle at infinity for the mirror family $f\colon \mathcal{Z} \to \mathbb{P}^1$. Recall the Dwork pencil $h\colon \mathcal{X} \to \mathbb{P}^1$, which comes with a natural embedding in $\mathbb{P}^n \times \mathbb{P}^1$. We obtain a 'physical' n-cycle Γ in $\mathbb{P}^n$ as follows: we place ourselves in the affine piece $x_0 \neq 0$ and define Γ by the condition $|x_i/x_0| = 1$ for all i. If $\psi \in \mathbb{C}$ and $|\psi|^{-1}$ is small, then the fibre X_ψ does not encounter Γ. Therefore, Γ induces a constant family of cycles in $H_n(\mathbb{P}^n \setminus X_\psi, \mathbb{Q})$. Notice that these are clearly G-invariant cycles. The tube map $H_{n-1}(X_\psi, \mathbb{Q}) \to H_n(\mathbb{P}^n \setminus X_\psi, \mathbb{Q})$ is surjective and G-equivariant and induces an isomorphism $H_{n-1}(X_\psi, \mathbb{Q})_{\mathrm{prim}} \simeq H_n(\mathbb{P}^n \setminus X_\psi, \mathbb{Q})$ by [Gri69, Prop. 3.5]. Therefore, we can find a T-invariant cycle $\widetilde{\gamma}_0 \in H_{n-1}(X_\psi, \mathbb{Q})^G_{\mathrm{prim}}$ corresponding to Γ. Finally, through the isomorphism $H_{n-1}(X_\psi, \mathbb{Q})^G_{\mathrm{prim}} \simeq H_{n-1}(Z_\psi, \mathbb{Q})_{\min}$ deduced by duality from Lemma 3.11 and Proposition 3.13, $|G| \cdot \widetilde{\gamma}_0$ maps to a T-invariant cycle on Z_ψ, denoted γ_0. The convenience of multiplication by $|G|$ will be clear in a moment.

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

The period integral $I_0(\psi) := \int_{\gamma_0} \eta_0$ can be written as an absolutely convergent power series in ψ^{-1}. Indeed, taking into account the relationship between the cup-product on X_ψ and Z_ψ (see, e.g., Lemma 3.6) and the definition of η_0 (see Definition 3.15), we find

$$I_0(\psi) = \int_{\gamma_0} \eta_0 = -\frac{(n+1)\psi}{|G|} \int_{|G|\cdot\widetilde{\gamma}_0} \theta_0 = -(n+1)\psi \int_{\widetilde{\gamma}_0} \theta_0.$$

For the latter integral, we use that the residue map and the tube map are mutual adjoint and then perform an explicit computation: if $\mathbb{D} \subset \mathbb{C}$ is the unit disc around 0, we have

$$\begin{aligned} I_0(\psi) &= \frac{1}{(2\pi i)^n} \int_{(\partial\mathbb{D})^n} \frac{-(n+1)\psi dz_1 \wedge \ldots \wedge dz_n}{F_\psi(1, z_1, \ldots, z_n)} \\ &= \sum_{j\geq 0} \frac{1}{((n+1)\psi)^j} \frac{1}{(2\pi i)^n} \int_{(\partial\mathbb{D})^n} \left(1 + \sum_{l=1}^n z_l^{n+1}\right)^j \frac{dz_1}{z_1^{j+1}} \wedge \ldots \wedge \frac{dz_n}{z_n^{j+1}} \\ &= \sum_{k\geq 0} \frac{1}{((n+1)\psi)^{(n+1)k}} \frac{((n+1)k)!}{(k!)^{n+1}}. \end{aligned} \tag{5.8}$$

In these integrals, the parameters z_i are the affine coordinates x_i/x_0. To obtain the last equality, we expand the integrands in the second line with Newton's multinomial formula and then evaluate the resulting Cauchy integrals. We conclude that those with $j \neq (n+1)k$ for any k vanish, while those with $j = (n+1)k$ for some k equal $((n+1)k)!/(k!)^{n+1}$. Equation (5.8) is the period integral used in Theorem 4.5 to prove that η_0 trivialises $f_* K_{\mathcal{Z}/\mathbb{D}_\infty}(\log)$.

To the local system $R^{n-1}f_*\mathbb{C}$, there is an associated Picard–Fuchs equation, which coincides with that of $(R^{n-1}f_*\mathbb{C})_{\min}$ since the associated Hodge bundles of type $(n,0)$ are equal. We make the change of variable $z = \psi^{-(n+1)}$ so that I_0 becomes

$$I_0(z) = \sum_{k\geq 0} \frac{z^k}{(n+1)^{(n+1)k}} \frac{((n+1)k)!}{(k!)^{n+1}}.$$

Define the differential operators $\delta = z\frac{d}{dz}$ and

$$D = \delta^n - z\prod_{j=1}^n \left(\delta + \frac{j}{n+1}\right). \tag{5.9}$$

Differentiating $I_0(z)$ term by term and repeatedly, one checks $DI_0(z) = 0$. It is known (see [Gäh13, Thm. 6] and [CG11, Sec. 1]) that this is the Picard–Fuchs equation of $(R^{n-1}h_*\mathbb{C})_{\text{prim}}$, which necessarily coincides with that of $(R^{n-1}h_*\mathbb{C})^G_{\text{prim}}$, and hence $R^{n-1}f_*\mathbb{C}$.

We now exhibit all the solutions of the Picard–Fuchs equation. For dimension reasons, these will determine a multivalued basis of homology cycles. Following Zinger (see, e.g., [Zin08, pp. 1214–1215]), for $q = 0, \ldots, n-1$, we define an *a priori* formal series $I_{0,q}$ by

$$\sum_{q=0}^\infty I_{0,q}(t)w^q = e^{wt} \sum_{d=0}^\infty e^{dt} \frac{\prod_{r=1}^{(n+1)d}((n+1)w + r)}{\prod_{r=1}^d (w+r)^{n+1}} =: R(w,t).$$

Let us also define $F(w,t)$ for the infinite sum on the right-hand side so that $R(w,t) = e^{wt}F(w,t)$. Under the change of variable

$$e^t = (n+1)^{-(n+1)}z = ((n+1)\psi)^{-(n+1)}, \tag{5.10}$$

the series $I_{0,0}(t)$ becomes $I_0(z) = I_0(\psi)$ [Zin08, eq. (2–17)].

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

Proposition 5.3. *Under the change of variable in (5.10), the functions $I_{0,q}(z)$, $q = 0, \dots, n-1$, define a basis of multivalued holomorphic solutions of the Picard–Fuchs equation for the local system $R^{n-1}f_*\mathbb{C}$ on $0 < |z| < 1$.*

Proof. We first recall that the Picard–Fuchs equations of $R^{n-1}f_*\mathbb{C}$ and $(R^{n-1}f_*\mathbb{C})_{\min}$ coincide, and the latter is a local system of rank n.

After the change of variable, one checks that $F(w,z)$ is absolutely convergent on compact subsets in the region $|w| < 1$ and $|z| < 1$. This implies that the functions $I_{0,q}(z)$ are multivalued holomorphic functions on $0 < |z| < 1$. Again taking into account the change of variable, it is formal to verify that $R(w,t)$ solves the Picard-Fuchs (5.9), and hence so do the functions $I_{0,q}(z)$. To see that they form a basis of solutions, it is enough to notice that each $I_{0,q}(z)$ has a singularity of the form $(\log z)^q$ as $z \to 0$. □

An adapted basis of homological cycles

By Proposition 5.3, and because $(R^{n-1}f_*\mathbb{C})_{\min}$ has rank n, the functions $I_{0,q}(z)$ determine a flat multivalued basis of sections γ_q of $(R^{n-1}f_*\mathbb{C})^\vee_{\min}$ on $0 < |z| < 1$ by the recipe

$$I_{0,q}(z) = \int_{\gamma_q(z)} \eta_0.$$

See, for instance, [Voi99, Sec. 3.4 & Lemme 3.12] for a justification in an analogous situation. The notation is compatible with the invariant cycle γ_0 constructed above, as we already observed that $I_{0,0}(z) = I_0(z)$. The flat multivalued basis elements γ_q provide a basis of $(H_{n-1})_{\lim}$ whose underlying vector space is seen here as the (minimal) homology of the general fibre. We still denote by γ_q this basis of $(H_{n-1})_{\lim}$. We next prove that it is adapted to the homological weight filtration, recalled at the end of §4.1.

Proposition 5.4. *Let $W'_\bullet$ be the weight filtration of the limiting mixed Hodge structure on $(H_{n-1})_{\lim}$. Then $\gamma_q \in W'_{2q-2(n-1)} \setminus W'_{2q-1-2(n-1)}$.*

Proof. By [Sch73, Lemma (6.4)], the Poincaré duality induces an isomorphism between the weight filtration W_r on $H^{n-1}_{\lim}$ to the dual weight filtration $W'_{r-2(n-1)}$ on $(H_{n-1})_{\lim}$. Therefore, it is enough to establish $\gamma'_q \in W_{2q} \setminus W_{2q-1}$ for the Poincaré duals $\gamma'_q \in H^{n-1}_{\lim}$.

On each fibre Z_z, the Hodge decomposition and the Cauchy–Schwarz inequality imply

$$|I_{0,q}(z)| = \left| \int_{Z_z} \gamma'_q(z) \wedge \eta_0 \right| \leq (2\pi)^{n-1} \|\gamma'_q(z)\|_{\mathrm{L}^2} \|\eta_0\|_{\mathrm{L}^2}.$$

Now $|I_{0,q}(z)|$ grows like $(\log|z|^{-1})^q$ as $z \to 0$ along angular sectors (see proof of Proposition 5.3). Because the monodromy is maximally unipotent at infinity and η_0 is a basis of $f_*K_{\mathcal{Z}/\mathbb{D}_\infty}(\log)$, the L^2 norm $\|\eta_0\|_{\mathrm{L}^2}$ grows like $(\log|z|^{-1})^{(n-1)/2}$ (see [EFiMM18, Thm. A] or the more general [EFiMM21, Thm. 4.4]). We infer that as $z \to 0$, along angular sectors,

$$\|\gamma'_q(z)\|_{\mathrm{L}^2} \gtrsim (\log|z|^{-1})^{\frac{2q-(n-1)}{2}}.$$

By Schmid's metric characterisation of the limiting Hodge structure [Sch73, Thm. 6.6], we then see that $\gamma'_q \notin W_{2q-1}$.

It remains to show that $\gamma'_q \in W_{2q}$. First, starting with $q = n-1$, we already know $\gamma'_{n-1} \in W_{2n-2} \setminus W_{2n-3}$. We claim that $\gamma'_{n-2} \in W_{2n-4}$. Otherwise $\gamma'_{n-2} \in W_{2n-2} \setminus W_{2n-4}$. But the weight filtration has one-dimensional graded pieces in even degrees, and zero otherwise (see Lemma 4.3). It follows that $W_{2n-4} = W_{2n-3}$ and $\gamma'_{n-1} = \lambda\gamma'_{n-2} + \beta$, for some constant λ and some $\beta \in W_{2n-4}$. Integrating against η_0, this

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

relation entails

$$I_{0,n-1}(z) = \lambda I_{0,n-2}(z) + \int_{Z_z} \beta(z) \wedge \eta_0,$$

where $\beta(z)$ is the flat multivalued section corresponding to β. Let us examine the asymptotic behaviour of the right-hand side of this equality, as $z \to 0$, along angular sectors. We know that $|I_{0,n-2}(z)|$ grows like $(\log|z|^{-1})^{n-2}$. By the Hodge decomposition, the Cauchy–Schwarz inequality and Schmid's theorem, the integral grows at most like $(\log|z|^{-1})^{n-2}$. This contradicts that $|I_{0,n-1}(z)|$ grows like $(\log|z|^{-1})^{n-1}$. Hence $\gamma'_{n-2} \in W_{2n-4}$. Continuing inductively in this fashion, we conclude that $\gamma'_q \in W_{2q}$ for all q, as desired. □

A normalised basis of $R^{n-1}f_*\Omega^\bullet_{\mathcal{Z}/\mathbb{D}_\infty}(\log)_{\min}$

We construct a basis of holomorphic sections of $R^{n-1}f_*\Omega^\bullet_{\mathcal{Z}/\mathbb{D}_\infty}(\log)_{\min}$ close to infinity, which correspond to the period integrals $I_{p,q}(z)$. We proceed inductively:

1. Set $\widetilde{\vartheta}_0 = \eta_0$.
2. For $p \geq 1$, suppose that $\widetilde{\vartheta}_0, \dots, \widetilde{\vartheta}_{p-1}$ have been constructed. Define

$$I_{p-1,q}(z) = \int_{\gamma_q(z)} \widetilde{\vartheta}_{p-1}.$$

   This notation is consistent with the previous definition of $I_{0,q}$;
3. As by [Zin09, Prop. 3.1], in turn based on [ZZ08], the integral $I_{p-1,p-1}(z)$ is holomorphic and nonvanishing at $z = 0$, we can define $\widetilde{\vartheta}_p$ by

$$\widetilde{\vartheta}_p = \nabla_{zd/dz}\left(\frac{\widetilde{\vartheta}_{p-1}}{I_{p-1,p-1}(z)}\right). \tag{5.11}$$

One verifies, integrating (5.11) over $\gamma_q(z)$, that the period integrals $I_{p,q}(z) := \int_{\gamma_q(z)} \widetilde{\vartheta}_p$ satisfy the following recursion:

$$I_{p,q}(z) = z\frac{d}{dz}\left(\frac{I_{p-1,q}(z)}{I_{p-1,p-1}(z)}\right). \tag{5.12}$$

Taking into account the change of variable in (5.10), we see that this is the same recurrence relation as in [Zin08, eq. (2–18)] (see also [Zin09, eq. (0.16)]). Hence the $I_{p,q}(z)$ above coincides with the $I_{p,q}(t)$ in [Zin08]. We further normalise

$$\vartheta_p = \frac{\widetilde{\vartheta}_p}{I_{p,p}(z)}.$$

Proposition 5.5.

1. *For all k, the sections $\{\vartheta_j\}_{j=0,\dots,k}$, constitute a holomorphic basis of the filtered piece $\mathcal{F}^{n-1-k}R^{n-1}f_*\Omega^\bullet_{\mathcal{Z}/\mathbb{D}_\infty}(\log)_{\min}$.*
2. *The periods of ϑ_k satisfy*

$$\int_{\gamma_k} \vartheta_k = 1 \quad \textit{and} \quad \int_{\gamma_q} \vartheta_k = 0 \quad \textit{if} \quad q < k.$$

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

3. *The projection of ϑ_k to $R^k f_*\Omega^{n-1-k}_{\mathcal{Z}/\mathbb{D}_\infty}(\log)_{\min}$ relates to η_k by*

$$(\vartheta_k)^{n-1-k,k} = \frac{(-1)^k}{(n+1)^k}\frac{\eta_k}{\prod_{p=0}^k I_{p,p}(z)}.$$

4. *The sections $\{\vartheta_j\}_{j=0,\dots,n-1}$ are uniquely determined by properties in (1)-(2) above.*

Proof. We noticed that the period integrals $I_{p,p}(z)$ are holomorphic in z and nonvanishing at $z = 0$. With this observation at hand, the claims in (1)-(2) then follow from properties of the Gauss–Manin connection and Kodaira–Spencer maps, Lemma 3.17 and Theorem 4.5. From $\widetilde{\vartheta}_0 = \eta_0 = -(n+1)\psi\theta_0$ and the recursion (3.12) for θ_k, the definition in (5.11) further normalised gives

$$\vartheta_k = (-1)^{k-1}(n+1)\psi^{k+1}\frac{\theta_k}{\prod_{p=0}^k I_{p,p}(z)} \bmod \mathcal{F}^{n-1-(k-1)}R^{n-1}f_*\Omega^\bullet_{\mathcal{Z}/\mathbb{D}_\infty}(\log)_{\min}.$$

As θ_k maps to $\eta_k^\circ = -\frac{\eta_k}{(n+1)^{k+1}\psi^{k+1}}$ in the Hodge bundle $(R^k f_*\Omega^{n-1-k}_{\mathcal{Z}/U})_{\min}$, this proves (3). The uniqueness property in (4) is obtained by comparing two such bases adapted to the Hodge filtration as in (1) and then imposing the period relations in (2). □

Actually, the basis $\vartheta_\bullet = \{\vartheta_j\}_{j=0,\dots,n-1}$ is determined by the limiting Hodge structure $H^{n-1}_{\lim}$, up to constant, as we now show:

Proposition 5.6.

1. *Let $\gamma'_\bullet$ be an adapted basis of the weight filtration on $(H_{n-1})_{\lim}$, as in Proposition 5.4. Then there exists a unique holomorphic basis $\vartheta'_\bullet$ of $R^{n-1}f_*\Omega^\bullet_{\mathcal{Z}/\mathbb{D}_\infty}(\log)_{\min}$ satisfying the conditions analogous to (1)-(2) with respect to $\gamma'_\bullet$.*
2. *There exist nonzero constants $c_k \in \mathbb{C}$ such that $\vartheta'_k = c_k\vartheta_k$.*

Proof. We prove both assertions simultaneously. We write $\gamma_\bullet$ and $\gamma'_\bullet$ as column vectors. Since the graded pieces of the weight filtration on $(H_{n-1})_{\lim}$ are all one-dimensional, there exists a lower triangular matrix $A \in \mathsf{GL}_n(\mathbb{C})$ with $\gamma'_\bullet = A\gamma_\bullet$. If we decompose $A = D + L$, where D is diagonal and L is lower triangular, we see that the entries of the column vector $\vartheta'_\bullet := D^{-1}\vartheta_\bullet$ fulfill the requirements. □

Definition 5.7. We define the canonical trivialising section of $R^k f_*\Omega^{n-1-k}_{\mathcal{Z}/\mathbb{D}_\infty}(\log)_{\min}$ to be

$$\widetilde{\eta}_k = (\vartheta_k)^{n-1-k,k} = \frac{(-1)^k}{(n+1)^k}\frac{\eta_k}{\prod_{p=0}^k I_{p,p}(z)}.$$

By the previous proposition, up to constants, the sections $\widetilde{\eta}_k$ depend only on $(H_{n-1})_{\lim}$, or equivalently $H^{n-1}_{\lim}$ by Poincaré duality. These constructions are part of a wider framework about distinguished sections for degenerations of Hodge–Tate type. It is discussed in more detail in §6.2.

5.3. Generating series of Gromov–Witten invariants and Zinger's theorem

To state Zinger's theorem on generating series of Gromov–Witten invariants of genus one, and for coherence with the notations of this author, it is now convenient to work in the t variable instead of z. The mirror map in Zinger's normalisations is the change of variable

$$t \mapsto T = \frac{I_{0,1}(t)}{I_{0,0}(t)} = \frac{\int_{\gamma_1(t)}\eta_0}{\int_{\gamma_0(t)}\eta_0}. \tag{5.13}$$

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

Notice that this differs by a factor $2\pi i$ from the more standard Morrison's mirror map [Mor93] used in the introduction. The Jacobian of the mirror map is computed from (5.12)

$$\frac{dT}{dt} = I_{1,1}(t).$$

Let us introduce some last notations:

- X_{n+1} denotes a general degree $n+1$ hypersurface in $\mathbb{P}^n$.
- $N_1(0) = -\left(\frac{(n-1)(n+2)}{48} + \frac{1-(-n)^{n+1}}{24(n+1)^2}\right) = \frac{1}{24}\left(-\frac{n(n+1)}{2} + \frac{\chi(X_{n+1})}{n+1}\right)$.
- $N_1(d)$ is the genus 1 and degree d Gromov-Witten invariant of X_{n+1} ($d \geq 1$).

From these invariants, we build a generating series:

$$F_1^A(T) = N_1(0)T + \sum_{d=1}^{\infty} N_1(d)e^{dT}. \tag{5.14}$$

It follows from [Zin08, Thm. 2] that this generating series satisfies

$$F_1^A(T) = N_1(0)t + \frac{(n+1)^2 - 1 + (-n)^{n+1}}{24(n+1)} \log I_{0,0}(t)$$
$$- \begin{cases} \frac{n}{48}\log(1-(n+1)^{n+1}e^t) + \sum_{p=0}^{(n-2)/2} \frac{(n-2p^2)}{8} \log I_{p,p}(t), & \text{if } n \text{ even} \\ \frac{n-3}{48}\log(1-(n+1)^{n+1}e^t) + \sum_{p=0}^{(n-3)/2)} \frac{(n+1-2p)(n-1-2p)}{8} \log I_{p,p}(t) & \text{if } n \text{ odd}. \end{cases}$$

This identity has to be understood in the sense of formal series. As an application of relations between the hypergeometric series $I_{p,p}(t)$, studied in detail in [ZZ08], the following identity holds (for a version of this particular identity, see [Zin09, eq. (3.2)]):

$$\frac{n(3n-5)}{48}\log(1-(n+1)^{n+1}e^t) + \frac{1}{2}\sum_{p=0}^{n-2}\binom{n-p}{2}\log I_{p,p}(t)$$
$$= \begin{cases} \frac{n}{48}\log(1-(n+1)^{n+1}e^t) + \sum_{p=0}^{(n-2)/2} \frac{(n-2p^2)}{8} \log I_{p,p}(t), & \text{if } n \text{ even} \\ \frac{n-3}{48}\log(1-(n+1)^{n+1}e^t) + \sum_{p=0}^{(n-3)/2)} \frac{(n+1-2p)(n-1-2p)}{8} \log I_{p,p}(t) & \text{if } n \text{ odd}. \end{cases}$$

Consequently, Zinger's theorem takes the following pleasant form that we will use to simplify the task of recognising $F_1^A(T)$ in our expression for the BCOV invariant (see Theorem 5.1).

Theorem 5.8 (Zinger). *Under the change of variables $t \mapsto T$, the series $F_1^A(T)$ takes the form*

$$\begin{aligned} F_1^A(T) \ &= N_1(0)t + \frac{\chi(X_{n+1})}{24}\log I_{0,0}(t) \\ &\quad - \frac{n(3n-5)}{48}\log(1-(n+1)^{n+1}e^t) - \frac{1}{2}\sum_{p=0}^{n-2}\binom{n-p}{2}\log I_{p,p}(t). \end{aligned} \tag{5.15}$$

A final remark on the holomorphicity of $F_1^A(T)$ is in order. While Theorem 5.8 is *a priori* an identity of formal series, the right-hand side of (5.15) is actually a holomorphic function in t, for $\mathrm{Re}\, t \ll 0$. Then via the mirror map, $F_1^A(T)$ acquires the structure of a holomorphic function in T. One can check that the domain of definition is a half-plane $\mathrm{Re}\, T \ll 0$.

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

5.4. Genus one mirror symmetry and the BCOV invariant

We are now in position to show that the BCOV invariant of the mirror family $f\colon \mathcal{Z} \to U$ realises genus one mirror symmetry for Calabi–Yau hypersurfaces in projective space. That is, one can extract the generating series $F_1^A(T)$ from the function $\psi \mapsto \tau_{\mathrm{BCOV}}(Z_\psi)$. The precise recipe by which this is accomplished goes through expressing τ_{BCOV} in terms of the L^2 norms of the canonical sections $\widetilde{\eta}_k$ (see Definition 5.7). But first, we need to make $\tau_{\mathrm{BCOV}}(Z_\psi)$ and $F_1^A(T)$ depend on the same variable. To this end, we let

$$F_1^B(\psi) = F_1^A(T), \quad \text{for} \quad T = \frac{I_{0,1}(t)}{I_{0,0}(t)} \quad \text{and} \quad e^t = ((n+1)\psi)^{-(n+1)}. \tag{5.16}$$

Theorem 5.9. *In a neighbourhood of $\psi = \infty$, there is an equality*

$$\tau_{\mathrm{BCOV}}(Z_\psi) = C \left|\exp\left((-1)^{n-1} F_1^B(\psi)\right)\right|^4 \frac{\|\widetilde{\eta}_0\|_{\mathrm{L}^2}^{\chi/6}}{\left(\prod_{k=0}^{n-1} \|\widetilde{\eta}_k\|_{\mathrm{L}^2}^{2(n-1-k)}\right)^{(-1)^{n-1}}},$$

where $\chi = \chi(Z_\psi)$ and $C \in \pi^c \mathbb{Q}^\times_{>0}$, $c = \frac{1}{2}\sum_k (-1)^{k+1} k^2 b_k$.

Proof. The proof is a simple computation, which consists of changing the variable T to ψ, using (5.16), in the expression for $F_1^A(T)$ provided by Theorem 5.8. For the computation, recall that for a smooth hypersurface X_{n+1} in $\mathbb{P}^n$, $\chi(X_{n+1}) = (-1)^{n-1}\chi$. Modulo log of rational numbers, we find

$$\begin{aligned}
4F_1^A(T) &= \left(-\frac{n(n+1)}{12} + \frac{\chi(X_{n+1})}{6(n+1)}\right) t + \frac{\chi(X_{n+1})}{6} \log I_{0,0}(t) \\
&\quad - \frac{n(3n-5)}{12} \log(1-(n+1)^{n+1} e^t) - 2\sum_{p=0}^{n-2} \binom{n-p}{2} \log I_{p,p}(t) \\
&= \left(\frac{n(n+1)}{12} - \frac{\chi(X_{n+1})}{6(n+1)} + \frac{n(3n-5)}{12}\right) \log(\psi^{n+1}) \\
&\quad - \frac{n(3n-5)}{12} \log(\psi^{n+1}-1) + \frac{\chi(X_{n+1})}{6} \log I_{0,0}(t) - 2\sum_{p=0}^{n-2} \binom{n-p}{2} \log I_{p,p}(t) \\
&= (-1)^{n-1} \log \frac{(\psi^{n+1})^{2a}}{(\psi^{n+1}-1)^{2b}} + (-1)^{n-1} \frac{\chi}{6} \log I_{0,0}(t) - 2\sum_{p=0}^{n-2} \binom{n-p}{2} \log I_{p,p}(t).
\end{aligned}$$

Now, in terms of the canonical trivialising sections $\widetilde{\eta}_k$ given in Definition 5.7, Theorem 5.1 becomes

$$\tau_{\mathrm{BCOV}}(Z_\psi) = C \left|\frac{(\psi^{n+1})^a}{(1-\psi^{n+1})^b}\right|^2 \frac{|I_{0,0}(t)|^{\chi/6}}{\left(\prod_{p=0}^{n-2} |I_{p,p}(t)|^{2\binom{n-p}{2}}\right)^{(-1)^{n-1}}} \frac{\|\widetilde{\eta}_0\|_{\mathrm{L}^2}^{\chi/6}}{\left(\prod_{k=0}^{n-1} \|\widetilde{\eta}_k\|_{\mathrm{L}^2}^{2(n-1-k)}\right)^{(-1)^{n-1}}}. \qquad \square$$

Remark 5.10.

1. In relative dimension 3, we recover the main theorem of Fang–Lu–Yoshikawa [FLY08, Thm. 1.3]. Their result is presented in a slightly different form. The first formal discrepancy is in the choice of the trivialising sections. Their trivialisations can be related to ours via Kodaira–Spencer maps. The second discrepancy is explained by a different normalisation of F_1^A: they work with two times Zinger's generating series. This justifies why their expression for the BCOV invariant contains $|\exp(-F_1^B(\psi))|^2$, while our formula in dimension 3 specialises to $|\exp(-F_1^B(\psi))|^4$.

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

2. The norms of the sections $\widetilde{\eta}_k$ are independent of the choice of crepant resolution. It follows that the expression on the right-hand side in Theorem 5.9 is independent of the crepant resolution, except possibly for the constant C. In [EFiMM21, Conj. B], we conjectured that the BCOV invariant is a birational invariant. A proof of this conjecture has been announced in [Zha20, FZ20]. Thus C should be independent of the choice of crepant resolution.

Corollary 5.11.

1. *The invariant* $N_1(0)$ *satisfies*

$$N_1(0) = \frac{-1}{24}\int_{X_{n+1}} \mathrm{c}_{n-2}(X_{n+1}) \wedge H,$$

where H *is the hyperplane class in* $\mathbb{P}^n$.
2. *As* $\psi \to \infty$, $\log \tau_{\mathrm{BCOV}}(Z_\psi)$ *behaves as*

$$\log \tau_{\mathrm{BCOV}}(Z_\psi) = \left(\frac{(-1)^n}{12}\int_{X_{n+1}} \mathrm{c}_{n-2}(X_{n+1}) \wedge H\right) \log\left|\psi^{-(n+1)}\right|^2 + O(\log\log|\psi|). \tag{5.17}$$

Proof. The sought-for interpretation of $N_1(0)$, or equivalently for the coefficient κ_∞ in Corollary 5.2, is obtained by an explicit computation of and comparison to $\int_{X_{n+1}} \mathrm{c}_{n-1}(\Omega_{X_{n+1}}) \wedge H$. Indeed, by the cotangent exact sequence for the immersion of X_{n+1} into $\mathbb{P}^n$, this reduces to

$$\int_{X_{n+1}} \mathrm{c}_{n-2}(\Omega_{X_{n+1}}) \wedge H = \frac{(-1)^{n-1}}{n+1}\chi(X_{n+1}) - \int_{\mathbb{P}^n} \mathrm{c}_{n-1}(\Omega_{\mathbb{P}^n}) \wedge H,$$

and we have explicit formulas for both terms on the right. This settles both the first and second claims. □

Remark 5.12. The asymptotic expansion (5.17) has been written in the variable $\psi^{-(n+1)}$ on purpose since this is the natural parameter in a neighbourhood of the MUM point in the moduli space. In this form, the equation agrees with the predictions of genus one mirror symmetry (see [EFiMM21, Sec. 1.4] for a discussion).

6. The refined BCOV conjecture

In this section, we propose an alternative approach to genus one mirror symmetry for Calabi–Yau manifolds, which bypasses spectral theory and is closer in spirit to the genus zero picture. The counterpart of the Yukawa coupling on the mirror side will now be a Grothendieck–Riemann–Roch isomorphism (GRR) of line bundles built out of Hodge bundles. As in the case of the Yukawa coupling, one seeks canonical trivialisations of these Hodge bundles, and the expression of the GRR isomorphism in these trivialisations should then encapsulate the genus one Gromov–Witten invariants of the original Calabi–Yau manifold. This is our interpretation of the holomorphic limit of the BCOV invariant. We refer to this conjectural program as *the refined BCOV conjecture*.

6.1. The Grothendieck–Riemann–Roch isomorphism

Let $f\colon \mathcal{X} \to S$ be a projective morphism of connected complex manifolds whose fibres are Calabi–Yau manifolds. Recall from (2.6) that the BCOV bundle $\lambda_{\mathrm{BCOV}}(\mathcal{X}/S)$ is defined as a combination of determinants of Hodge bundles. Its formation commutes with arbitrary base change.

Conjecture 1. *For every projective family of Calabi–Yau manifolds* $f\colon \mathcal{X} \to S$ *as above, there exists a natural isomorphism of line bundles, compatible with any base change,*

$$\mathbf{GRR}(\mathcal{X}/S)\colon \lambda_{\mathrm{BCOV}}(\mathcal{X}/S)^{\otimes 12\kappa} \xrightarrow{\sim} (f_* K_{\mathcal{X}/S})^{\otimes \chi\kappa}. \tag{6.1}$$

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

Here χ is the Euler characteristic of any fibre of f, and κ only depends on the relative dimension of f.

Below, we present some arguments in favour of the conjecture.

- Applying this to the universal elliptic curve, the right-hand side becomes trivial in view of $\chi = 0$. This suggests that the left-hand side is trivial. It is indeed trivialised by the discriminant modular form Δ, with $\kappa = 1$. For higher-dimensional abelian varieties, both sides are trivial, and the identity provides a natural isomorphism.
- For $K3$ surfaces, both sides are identical, and the identity provides a natural isomorphism. See in particular Proposition 6.14. The referee kindly communicated to us a proof of the analogue of the conjecture for Enriques surfaces, relying on the works about analytic torsions and the Borcherds' Φ-function by Kawaguchi–Mukai–Yoshikawa [KMY18], Dai–Yoshikawa [DY20] and Yoshikawa [Yos04].
- In the category of schemes, a natural isomorphism of $\mathbb{Q}$-line bundles up to sign exists by work of Franke [Fra92] and the first author [Eri08]. It is compatible with the arithmetic Riemann–Roch theorem but is far more general and stronger.

The following proposition establishes a version of Conjecture 1 in the setting of arithmetic varieties (see Section 2.3). This is an application of the arithmetic Riemann–Roch theorem 2.3. Recall that an arithmetic ring A comes together with a finite collection of complex embeddings Σ, closed under complex conjugation. We will write $A^{\times,1}$ for the group of elements $u \in A^\times$ with $|\sigma(u)| = 1$ for all embedding $\sigma \in \Sigma$. For instance, if A is the ring of integers of a number field, then $A^{\times,1}$ is a finite group. If $A = \mathbb{Q}$ or $\mathbb{R}$, then $A^{\times,1} = \{\pm 1\}$. If $A = \mathbb{C}$, then $A^{\times,1}$ is the unit circle in $\mathbb{C}$.

Proposition 6.1. *Let $f\colon \mathcal{X} \to S$ be a smooth projective morphism of arithmetic varieties over an arithmetic ring A, with Calabi–Yau fibres. Let X_∞ be the generic fibre of f, and write $\chi = \chi(X_\infty)$. Assume that $S \to \operatorname{Spec} A$ is surjective and has geometrically connected fibres.*

1. *There exist an integer $\kappa \geq 1$ and an isomorphism of line bundles on S*

$$\mathbf{GRR}\colon \lambda_{\mathrm{BCOV}}(\mathcal{X}/S)^{\otimes 12\kappa} \xrightarrow{\sim} (f_* K_{\mathcal{X}/S})^{\otimes \chi\kappa},$$

   with the property of being an isometry for the Quillen-BCOV and L^2 metrics on $\lambda_{\mathrm{BCOV}}(\mathcal{X}/S)$ and $f_* K_{\mathcal{X}/S}$, respectively.
2. *If $\mathbf{GRR}'$ is another such isomorphism, for another choice of integer $\kappa' \geq 1$, then*

$$\mathbf{GRR}'^{\otimes\kappa} = \mathbf{GRR}^{\otimes\kappa'}$$

   up to multiplication by some $u \in A^{\times,1}$. Consequently, the formation of $\mathbf{GRR}$ is compatible with any base change between geometrically connected arithmetic varieties over A, up to the power κ and multiplication by a unit in $A^{\times,1}$.

Proof. The first claim is a restatement of the identity in (2.9) in $\widehat{\mathrm{CH}}^1(S)_{\mathbb{Q}}$, together with the isomorphism $\widehat{c}_1 : \widehat{\mathrm{Pic}}(S) \xrightarrow{\sim} \widehat{\mathrm{CH}}^1(S)$ and the very definition of $\widehat{\mathrm{Pic}}(S)$ as the group of isomorphism classes of hermitian line bundles over S.

For the second claim, notice that both $\mathbf{GRR}'^{\otimes\kappa}$ and $\mathbf{GRR}^{\otimes\kappa'}$ induce isometries between the hermitian line bundles $\lambda_{\mathrm{BCOV}}(\mathcal{X}/S)^{\otimes 12\kappa\kappa'}$ and $(f_* K_{\mathcal{X}/S})^{\otimes\chi\kappa\kappa'}$, endowed with the Quillen-BCOV and L^2 metrics, respectively. These isomorphisms differ by multiplication by a unit $u \in \Gamma(S, \mathcal{O}_S^\times)$. The isometry property guarantees that the induced holomorphic function on S^{an} has modulus one and is constant on the connected components. Hence, if we fix $\sigma \in \mathbb{C}$, u is constant on S_σ^{an}. If we see u in $\Gamma(S_\sigma, \mathcal{O}_{S_\sigma}^\times)$, we infer from this that it satisfies the descent condition with respect to $S_\sigma \to \operatorname{Spec}\mathbb{C}$. It follows that u already satisfies the descent condition with respect to $S \to \operatorname{Spec} A$. This is easily seen if S is affine by the flatness of $S \to \operatorname{Spec} A$, and in general one may replace S by the disjoint union of the open subsets of an affine covering of S. Because $S \to \operatorname{Spec} A$ is actually faithfully flat by assumption, we conclude

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

that $u \in A^\times$. Now u has modulus one as a function on S^{an}, which exactly means $u \in A^{\times,1}$. The base change property then follows from the compatibility of $\lambda_{\mathrm{BCOV}}(\mathcal{X}/S)$ and $f_*K_{\mathcal{X}/S}$ with base change and the fact that the Quillen and Hodge metrics are preserved as well. □

Remark 6.2.

1. If $A^{\times,1}$ is a finite group of order d, then the second claim of the corollary entails
$$\mathbf{GRR}'^{\otimes d\kappa} = \mathbf{GRR}^{\otimes d\kappa'}.$$
Therefore, after possibly adjusting κ, the isomorphism is uniquely determined.
2. The proposition applies to the mirror family of Calabi–Yau hypersurfaces studied in Section 3. Here $A = \mathbb{Q}$, and therefore the resulting isomorphism is determined by the previous remark.

6.2. Strongly unipotent monodromy and distinguished sections

The below discussion is based on [Del] and [Mor97, §6.3, §7.1].

Hodge–Tate structures
Let $(V, F^\bullet, W_\bullet)$ be a mixed Hodge structure on a $\mathbb{Q}$-vector space V, where $F^\bullet$ is the decreasing Hodge filtration of $V_\mathbb{C}$ and $W_\bullet$ is the increasing weight filtration of V. We also write $W_\bullet$ for the induced filtrations on $V_\mathbb{R}$ and $V_\mathbb{C}$.

Definition 6.3. A mixed Hodge structure is Hodge–Tate if the Hodge filtration is opposite to the weight filtration, in the sense that for any k, the natural map
$$F^k \oplus W_{2k-2} \to V_\mathbb{C}$$
is an isomorphism. Equivalently if the following two conditions are satisfied:

1. $\mathrm{Gr}^W_{2k} V = W_{2k}/W_{2k-1}$ is isomorphic to a sum of Tate twists $\mathbb{Q}(-k)$. In other words, it is purely of type (k,k).
2. $\mathrm{Gr}^W_{2k+1} V = \{0\}$.

According to [Del, §6], if a limiting Hodge structure has this property, it should be viewed as maximally degenerate. An example of this situation is $H^{n-1}_{\lim}$ for the mirror family around ∞, as explained in the proof of Lemma 4.3. For Calabi–Yau degenerations over $\mathbb{D}^\times$, this condition on the limiting middle Hodge structure implies that the monodromy is maximally unipotent.

It follows from the definition of Hodge–Tate mixed Hodge structure that the natural map
$$F^p/F^{p+1} \hookrightarrow V_\mathbb{C}/F^{p+1} = (F^{p+1} \oplus W_{2p})/F^{p+1} \to W_{2p} \to W_{2p}/W_{2p-2} \tag{6.2}$$
is an isomorphism and that there are natural isomorphisms
$$F^p \cap W_{2p} \simeq \mathrm{Gr}^W_{2p}(V_\mathbb{C}) \tag{6.3}$$
and
$$F^p \cap W_{2p} \simeq \mathrm{Gr}^p_F(V_\mathbb{C}) \tag{6.4}$$
compatible with the isomorphism in (6.2).

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

Distinguished sections

Suppose now that we are provided a variation of integrally polarised Hodge structures $(\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^\bullet)$ of weight w over $\mathbf{D}^\times = (\mathbb{D}^\times)^d$. Here $\mathbb{V}_{\mathbb{Z}}$ is an integral local system, and $\mathcal{F}^p$ is the Hodge filtration of $\mathcal{V} := \mathbb{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbf{D}^\times}$. Denote by ∇ the flat connection on $\mathcal{V}$, and suppose the local monodromies are unipotent. Denote by T_j the endomorphism of the local system $\mathbb{V}_{\mathbb{Z}}$ given by the monodromy around the coordinate axis $(s_j = 0)$ of $(\mathbb{D}^\times)^d$. Consider the family of operators $N_j := \log T_j$ over $\mathbf{D}^\times$. Let $\mathbb{W}_k$ be the associated increasing weight monodromy filtration of $\mathbb{V}_{\mathbb{Q}}$, and also consider the associated vector bundle $\mathcal{W}_k = \mathbb{W}_k \otimes \mathcal{O}_{\mathbf{D}^\times}$. The bundle $\mathcal{W}_k$ is preserved by ∇ and satisfies $N_j \mathbb{W}_k \subseteq \mathbb{W}_{k-2}$, and for any positive real numbers $a_j > 0$, with $N = \sum a_j N_j$, we have an isomorphism

$$N^k : \mathrm{Gr}^W_{w+k} \mathbb{V}_{\mathbb{R}} \to \mathrm{Gr}^W_{w-k} \mathbb{V}_{\mathbb{R}}.$$

By the results of Schmid [Sch73], associated to $(\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^\bullet)$ and for any base point $s \in \mathbf{D}^\times$, there is a limiting mixed Hodge structure $V_{\lim}$ on $\mathbb{V}_{\mathbb{Q},s}$. Its weight filtration is given by $\mathbb{W}_{k,s}$. The following lemma can be found in [Del, Sec. 6]:

Lemma 6.4. *If $V_{\lim}$ is Hodge–Tate, then, after possibly shrinking $\mathbf{D}^\times$, $(\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^p, \mathbb{W}_k)$ is a variation of mixed Hodge structures over $\mathbf{D}^\times$ with the same Hodge numbers as $V_{\lim}$. In particular, the natural morphism $\mathcal{F}^p \oplus \mathcal{W}_{2p-2} \to \mathcal{V}$ is an isomorphism over $\mathbf{D}^\times$.*

Remark 6.5. The Hodge and the weight filtrations on $\mathcal{V}$ extend as subvector bundles of the Deligne extension $\widetilde{\mathcal{V}}$ over $\mathbf{D}$. The extended filtrations continue to be opposite in the sense of Lemma 6.4.

From now on, we suppose the limiting Hodge structure is Hodge–Tate. In this setting, after Lemma 6.4, we have the analogues of the isomorphisms in (6.2), (6.3) and (6.4), namely isomorphisms

$$\mathcal{F}^p / \mathcal{F}^{p+1} \to \mathcal{W}_{2p} / \mathcal{W}_{2p-2}, \tag{6.5}$$

$$\mathcal{F}^p \cap \mathcal{W}_{2p} \simeq \mathrm{Gr}^W_{2p} \mathcal{V} \tag{6.6}$$

and

$$\mathcal{F}^p \cap \mathcal{W}_{2p} \simeq \mathrm{Gr}^p_F \mathcal{V}. \tag{6.7}$$

Since $N_j \mathbb{W}_{2p} \subseteq \mathbb{W}_{2p-2} \subseteq \mathbb{W}_{2p-1}$, N_j and hence the monodromy act trivially on the local system $\mathrm{Gr}^W_{2p} \mathbb{V}_{\mathbb{Q}}$, which is thus constant. In particular, $\mathrm{Gr}^W_{2p} \mathcal{V}$ is a trivial flat vector bundle. We abusively often identify this trivial local system with the vector space $H^0(\mathbf{D}^\times, \mathrm{Gr}^W_{2p} \mathbb{V}_{\mathbb{Q}})$.

Definition 6.6. The distinguished sections of $\mathcal{F}^p / \mathcal{F}^{p+1}$ are the global sections corresponding to $\mathrm{Gr}^W_{2p} \mathbb{V}_{\mathbb{C}} \subseteq \mathrm{Gr}^W_{2p}(\mathcal{V})$ under the isomorphism in (6.5). A basis of distinguished sections will be called a distinguished basis. It is unique up to a matrix transformation with constant complex coefficients.

For the determinant bundles, we have the following corollary:

Corollary 6.7. *Keep the notations and assumptions of Definition 6.6. The exterior products of a distinguished basis provides a local frame for* $\det(\mathcal{F}^p / \mathcal{F}^{p+1})$.

Any frame as in the corollary will be called a distinguished trivialisation. It is unique up to a scalar constant in $\mathbb{C}^\times$.

Remark 6.8. Using the integral lattice $\mathbb{V}_{\mathbb{Z}}$, one naturally defines an integral structure $\mathrm{Gr}^W_{2k} \mathbb{V}_{\mathbb{Z}}$ on $\mathrm{Gr}^W_{2k} \mathbb{V}_{\mathbb{Q}}$. Accordingly, there are integral distinguished sections and trivialisations. The corresponding integral distinguished trivialisation of $\det(\mathcal{F}^p / \mathcal{F}^{p+1})$ is unique up to sign.

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

Suppose now that we are in the geometric case of a projective family of complex manifolds $f : \mathcal{X} \to \mathbf{D}^\times$, endowed with a relatively ample line bundle. We assume that for any k, $R^k f_* \mathbb{Z}$ is a local system with unipotent local monodromies. Each variation $(R^k f_* \mathbb{Q})_{\text{prim}}$ is integrally polarised, and Schmid's theory in [Sch73] recalled above applies. By the Lefschetz decomposition, $R^k f_* \mathbb{Q}$ admits a limiting Hodge structure and in particular a monodromy weight filtration $\mathbb{W}_\bullet$. The constructions are independent of the choice of ample line bundle.

In the geometric case, we can provide an alternative description of the distinguished sections in terms of the behaviour of periods:

Lemma 6.9. *If the local monodromies of $R^k f_* \mathbb{C}$ are unipotent and the limiting Hodge structure is Hodge–Tate, the distinguished sections of $\mathcal{F}^p / \mathcal{F}^{p+1}$ uniquely correspond to elements $\eta \in \mathcal{F}^p$ such that*

1. *$\int_\gamma \eta = 0$ for all multivalued flat homology cycles γ in $\mathbb{W}'_{-2p-2} \subseteq (R^k f_* \mathbb{C})^\vee$.*
2. *$\int_\gamma \eta$ is constant for all multivalued flat homology cycles γ in $\mathbb{W}'_{-2p} \subseteq (R^k f_* \mathbb{C})^\vee$.*

Proof. By definition of the dual weight filtration recalled at the end of §4.1 and (6.7), we can identify $\mathrm{Gr}_F^p \mathcal{V}$ with the set of sections of $\mathcal{F}^p$ whose periods along cycles in $\mathbb{W}'_{-2p-2}$ vanish. Moreover, by (6.6), $\eta \in \mathcal{F}^p \cap \mathcal{W}_{2p}$ corresponds to a distinguished section exactly when $\nabla\eta \in \mathcal{W}_{2p-2} \otimes \Omega^1_{\mathbf{D}^\times}$. This in turn is equivalent to $\int_\gamma \nabla\eta = 0$ for all γ of $\mathbb{W}'_{-2p}$. The statement then follows from the formula

$$d\left(\int_\gamma \eta\right) = \int_\gamma \nabla\eta$$

for flat multivalued homology sections γ. □

Strongly unipotent monodromy degenerations

Definition 6.10. We say that a projective family $f : \mathcal{X} \to \mathbf{D}^\times$ of complex manifolds is of strongly unipotent monodromy if

1. for all $k \geq 0$, all local monodromies $R^k f_* \mathbb{Q}$ are unipotent;
2. for all $k \geq 0$, the variations of Hodge structures associated to the local systems $R^k f_* \mathbb{Q}$ have limits at 0 that are Hodge–Tate.

In the situation of a family of Calabi–Yau manifolds in relative dimension 3 with one-dimensional complex moduli, with $h^{1,0} = h^{2,0} = 0$ and unipotent monodromies, strongly unipotent monodromy is equivalent to $N^3 \neq 0$ on $H^3_{\lim}$. This is the usual definition of maximally unipotent monodromy. In general, for a family of Calabi–Yau manifolds $f\colon \mathcal{X} \to \mathbf{D}^\times$ of relative dimension n, the definition of strongly maximally unipotent monodromy is stronger than imposing that $N^n \neq 0$ on $R^n f_* \mathbb{C}$.

We now show that our results on the mirror family $f\colon \mathcal{Z} \to \mathbb{D}^\times_\infty$ provide an example of the previous phenomena and constructions. In preparation for the discussion, recall the minimal decomposition introduced in Proposition 3.13, and in particular the local system $\mathbb{V}$ and its associated flat vector bundle $\mathcal{V}$.

Lemma 6.11. *The mirror family $f\colon \mathcal{Z} \to \mathbb{D}^\times_\infty$ has strongly unipotent monodromy.*

Proof. Outside the middle cohomology $n-1$, being Hodge–Tate follows from the fact that the variation of Hodge structures associated to $R^{2p} f_* \mathbb{Q}$ is purely of type (p, p) (see Lemma 3.4), and the monodromy is trivial by Lemma 4.1.

In the middle cohomology, by the Lefschetz decomposition, it is enough to deal with the local system $(R^{n-1} f_* \mathbb{Q})_{\text{prim}}$. This is a sum of $(R^{n-1} f_* \mathbb{Q})_{\min}$ and $\mathbb{V}$. The limiting Hodge structure associated to $\mathbb{V}$ is Hodge–Tate by Proposition 3.13 and Lemma 4.2. That $(R^{n-1} f_* \mathbb{Q})_{\min}$ is unipotent and has limiting mixed Hodge structure that is Hodge–Tate follows from Lemma 4.3 and its proof. □

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

The following proposition summarises the results of §4.2 and §5.2, to the effect of describing distinguished trivialisations of the Hodge bundles.

Proposition 6.12. *The distinguished trivialisations of the determinants of the Hodge bundles* $R^q\Omega^p_{\mathcal{Z}/\mathbb{D}^\times_\infty}$ *are described as follows:*

1. *Suppose* $2p \neq n-1$. *Any basis of the trivial local system* $R^{2p}f_*\mathbb{C}$ *provides a distinguished trivialisation of* $\det R^p f_*\Omega^p_{\mathcal{Z}/\mathbb{D}^\times_\infty}$.
2. *Suppose* $2k \neq n-1$. *Then* $\widetilde{\eta}_k$ *are distinguished trivialisations of* $R^k\Omega^{n-1-k}_{\mathcal{Z}/\mathbb{D}^\times_\infty}$.
3. *Suppose* $2k = n-1$. *For any polarisation L, any basis u (respectively, v) of the trivial local systems* $R^{2n-4}f_*\mathbb{C}$ *(respectively,* $\mathbb{V}$*), the section* $\widetilde{\eta}_k \wedge (\det Lu) \wedge \det v$ *is a distinguished trivialisation of* $\det R^k f_*\Omega^k_{\mathcal{Z}/\mathbb{D}^\times_\infty}$.

6.3. Relationship with mirror symmetry

We now present our refinement of the BCOV conjecture for degenerating families of Calabi–Yau manifolds with strongly unipotent monodromy. The statement predicts that **GRR** realises genus one mirror symmetry. We then show that the conjecture holds for the case of mirrors of hypersurfaces in projective space as a consequence of our previous main theorems. The case of K3 surfaces is not covered by those considerations, but a proof is also provided.

To prepare for the formulation of the conjecture, let $f\colon \mathcal{X} \to \mathbf{D}^\times = (\mathbb{D}^\times)^d$ be a projective morphism of Calabi–Yau n-folds, with $d = h^{1,n-1}$, the dimension of the deformation space of the fibres, effectively parametrised and with strongly unipotent monodromy. We denote by $\widetilde{\eta}_{p,q}$ an integral distinguished trivialisation of $\det R^q f_*\Omega^p_{\mathcal{X}/\mathbf{D}^\times}$, which is unique up to sign (see Corollary 6.7 and Remark 6.8). Using these, both bundles appearing in the conjectural Grothendieck–Riemann–Roch isomorphism in (6.1) admit canonical trivialisations. Precisely, up to sign, the BCOV bundle is canonically trivialised by

$$\widetilde{\eta}_{\mathrm{BCOV}} := \bigotimes_{p,q} \widetilde{\eta}_{p,q}^{\otimes (-1)^{p+q}p}. \tag{6.8}$$

Likewise, $\widetilde{\eta}_{n,0}$ trivialises the $f_*K_{\mathcal{X}/\mathbf{D}^\times}$. Expressed in these trivialisations, we can write

$$\mathbf{GRR}(\mathcal{X}/\mathbf{D}^\times)\colon\ \widetilde{\eta}_{\mathrm{BCOV}}^{12\kappa} \mapsto \mathbf{GRR}(s)\cdot \widetilde{\eta}_{n,0}^{\chi\kappa} \tag{6.9}$$

for an invertible holomorphic function $s \mapsto \mathbf{GRR}(s)$ on $\mathbf{D}^\times$.

In the above situation, it is expected that there are canonical mirror coordinates $\mathbf{q} = (q_1, \ldots, q_d)$ on $\mathbf{D}$. In [Mor93], for one-dimensional moduli, this is constructed through exponentials of quotients of well-selected periods. For mirrors of hypersurfaces, this amounts to Zinger's mirror map recalled in (5.13). A general alternative construction of mirror coordinates in the Hodge–Tate setting is suggested in [Del, Sec. 14].

Conjecture 2. *Let* $f\colon \mathcal{X} \to \mathbf{D}^\times = (\mathbb{D}^\times)^d$ *be a projective morphism of Calabi–Yau n-folds, with* $d = h^{1,n-1}$, *the dimension of the deformation space of the fibres, effectively parametrised with strongly unipotent monodromy. In the mirror coordinates* $\mathbf{q} = (q_1, \ldots, q_d)$ *of* $\mathbf{D}$, *the function defined in (6.9) becomes*

$$\mathbf{GRR}(\mathbf{q}) = C \cdot \exp\left((-1)^n F_1^A(\mathbf{q})\right)^{24\kappa},$$

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

where C is a constant,

$$F_1^A(\mathbf{q}) = -\frac{1}{24}\sum_{k=1}^{d}\left(\int_{X^\vee} \mathrm{c}_{n-1}(X^\vee)\wedge\omega_k\right)\log q_k + \sum_{\beta\in H_2(X^\vee,\mathbb{Z})} \mathrm{GW}_1(X^\vee,\beta)\,\mathbf{q}^{\langle\underline{\omega},\beta\rangle}$$

is a generating series of genus one Gromov–Witten invariants on a mirror Calabi–Yau manifold $X^\vee$, and

- *$\underline{\omega} = (\omega_1,\dots,\omega_d)$ is some basis of $H^{1,1}(X^\vee)\cap H^2(X^\vee,\mathbb{Z})$ formed by ample classes.*
- *$\mathrm{GW}_1(X^\vee,\beta)$ is the genus one Gromov–Witten invariant on $X^\vee$ associated to the class β.*
- *$\mathbf{q}^{\langle\underline{\omega},\beta\rangle} = \prod_k q_k^{\langle\omega_k,\beta\rangle}$.*

As supporting evidence, we consider the case of the mirror family of Calabi–Yau hypersurfaces in $\mathbb{P}^n$ and settle the second part of the Main Theorem in the introduction:

Theorem 6.13. *Let $n\geq 4$. Then Conjecture 1 and Conjecture 2 are true, up to a constant, for the mirror family $f\colon \mathcal{Z}\to\mathbb{D}_\infty^\times$ in a neighbourhood of the MUM point.*

Proof. First, the existence of a natural isomorphism as in Conjecture 1 is provided by Proposition 6.1 and Remark 6.2 (2). Secondly, for Conjecture 2, consider $\widetilde{\eta}_{\mathrm{BCOV}}$ defined as in (6.8). Since distinguished trivialisations are equal up to a constant, for the purpose of proving Conjecture 2, we can suppose that the sections $\widetilde{\eta}_{p,q}$ are actually those determined by Proposition 6.12. By the isometry property of **GRR** and the very definition of the BCOV invariant, we have

$$\tau_{\mathrm{BCOV}}^{12\kappa} = \frac{\|\,\mathbf{GRR}(\widetilde{\eta}_{\mathrm{BCOV}}^{12\kappa})\|_{\mathrm{L}^2}^2}{\|\widetilde{\eta}_{\mathrm{BCOV}}\|_{\mathrm{L}^2,\mathrm{BCOV}}^{24\kappa}}.$$

In other words,

$$\tau_{\mathrm{BCOV}} = |\,\mathbf{GRR}(\mathbf{q})|^{1/6\kappa}\,\frac{\|\widetilde{\eta}_{n-1,0}\|_{\mathrm{L}^2}^{\chi/6}}{\|\widetilde{\eta}_{\mathrm{BCOV}}\|_{\mathrm{L}^2,\mathrm{BCOV}}^2}. \tag{6.10}$$

As in the proof of Theorem 5.1 (see also [EFiMM21, Prop. 4.2]), the quantity $\|\widetilde{\eta}_{\mathrm{BCOV}}\|_{\mathrm{L}^2,\mathrm{BCOV}}$ coincides with the factor $\prod_{k=0}^{n-1}\|\widetilde{\eta}_k\|_{\mathrm{L}^2}^{n-1-k}$ up to a constant. We conclude by comparing (6.10) with Theorem 5.9. □

The cases of one- and two-dimensional Calabi–Yau varieties are not covered by the above result. The one-dimensional case essentially corresponds to the Kronecker limit formula recalled in §1.5. We now study the case of $K3$ surfaces. Since $h^{1,1}=20$ for a $K3$ surface, our one-dimensional Dwork-type family cannot be a mirror family. It is still expected that the mirror of a K3 surface is a K3 surface; a systematic construction in terms of polarised lattices can be found in, for example, [Dol96]. We will assume this below.

Proposition 6.14. *Conjecture 1 and Conjecture 2 are true, up to a constant, for any mirror family of a K3 surface. Moreover, $\kappa=1$.*

Proof. The BCOV line takes a particularly simple form for a $K3$ surface X: its square can be written as

$$\lambda_{\mathrm{BCOV}}(X)^{\otimes 2} = \det H^{2,0}(X)^{\otimes 4}\otimes\det H^{1,1}(X)^{\otimes 2}\otimes\det H^{2,2}(X)^{\otimes 4}\simeq\det H^{2,0}(X)^{\otimes 4}. \tag{6.11}$$

The isomorphisms $\det H^{1,1}(X)^{\otimes 2}\simeq\mathbb{C}$ and $\det H^{2,2}(X)\simeq\mathbb{C}$ are both induced by Serre duality and are thus isometries for the L^2 norms and standard metric on $\mathbb{C}$. Since $\chi(X)=24$, the square of the right-hand side of Conjecture 1 is provided by the same object.

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

Let $f\colon \mathcal{X} \to \mathbf{D}^\times$ be a family of K3 surfaces. The previous construction globalises to an isomorphism of line bundles

$$\lambda_{\mathrm{BCOV}}(\mathcal{X}/\mathbf{D}^\times)^{\otimes 2} \xrightarrow{\sim} (f_* K_{\mathcal{X}/\mathbf{D}^\times})^{\otimes 4}$$

compatible with base change. Taking 6th powers and setting $\kappa = 1$, this proves Conjecture 1 in this case. We hence propose that **GRR** is induced by (6.11).

Following the proof of Theorem 6.13, to prove Conjecture 2, we need to construct distinguished trivialisations of both sides. For $H^{1,1}$, we choose the section of $\det R^1 f_* \Omega^1_{\mathcal{X}/\mathbf{D}^\times} = (\det R^2 f_* \mathbb{C}) \otimes \mathcal{O}_{\mathbf{D}^\times}$ induced by a generator of $\det R^2 f_* \mathbb{Z}$, and analogously for $\det R^2 f_* \Omega^2_{\mathcal{X}/\mathbf{D}^\times} = (\det R^4 f_* \mathbb{C}) \otimes \mathcal{O}_{\mathbf{D}\times}$. Their L^2 norms are locally constant by [EFiMM21, Prop. 4.2]. Picking any distinguished section $\widetilde{\eta}_{2,0}$ of $f_* K_{\mathcal{X}/\mathbf{D}^\times}$, it allows us to write down the section $\widetilde{\eta}_{\mathrm{BCOV}}$ of (6.8).

The analogous formula to (6.10) becomes, in this case,

$$\tau_{\mathrm{BCOV}} = |\,\mathbf{GRR}(\mathbf{q})|^{1/6} \frac{\|\widetilde{\eta}_{2,0}\|^4_{\mathrm{L}^2}}{\|\widetilde{\eta}_{\mathrm{BCOV}}\|^2_{\mathrm{L}^2,\mathrm{BCOV}}} = C|\,\mathbf{GRR}(\mathbf{q})|^{1/6},$$

for a constant $C > 0$. By triviality of the Gromov–Witten invariants for K3 surfaces (see, for example, [LP07, Cor. 3.3]), to prove Conjecture 2, we need to prove that τ_{BCOV} is constant. This is the content of [EFiMM21, Thm. 5.12]. □

7. A Chowla–Selberg formula for the BCOV invariant

In this section, we discuss an example of using the arithmetic Riemann–Roch theorem to evaluate the BCOV invariant of a Calabi–Yau manifold with complex multiplication, similar to the derivation of the Chowla–Selberg formula from the Kronecker limit formula for elliptic curves. In such situations, or more generally for Calabi–Yau manifolds whose Hodge structures have some extra symmetries, we expect that the BCOV invariant can be evaluated in terms of special values of Γ-functions or other special functions.

Let $p \geq 5$ be a prime number, and define $n = p - 1$. We consider the mirror family $f\colon \mathcal{Z} \to U$ to Calabi–Yau hypersurfaces of degree p in $\mathbb{P}^n$. The restriction on the dimension here has been made to simplify the exposition. The special fibre Z_0 is a crepant resolution of X_0/G, where X_0 is now the Fermat hypersurface

$$x_0^p + \ldots + x_n^p = 0.$$

The quotient X_0/G has an extra action of $\mu_p \subset \mathbb{C}$: a pth root of unity $\xi \in \mathbb{C}$ sends a point $(x_0\colon \ldots \colon x_n)$ to $(x_0\colon \ldots \colon x_{n-1}\colon \xi x_n)$. This action induces a $\mathbb{Q}$-linear action of $K = \mathbb{Q}(\mu_p) \subset \mathbb{C}$ on $H^{n-1}(X_0, \mathbb{Q})^G$. As a rational Hodge structure, the latter is isomorphic to $H^{n-1}(Z_0, \mathbb{Q})$. For this, see §3.3, and especially Lemma 3.11 and Proposition 3.13 (we are in odd dimension, and all the cohomology is primitive now). Hence $H^{n-1}(Z_0, \mathbb{Q})$ inherits a $\mathbb{Q}$-linear action of K. Observe that $[K\colon \mathbb{Q}] = p - 1$, which is exactly the dimension of $H^{n-1}(Z_0, \mathbb{Q})$. We say that Z_0 has complex multiplication by K. Similarly, the algebraic de Rham cohomology $H^{n-1}(Z_0, \Omega^\bullet_{Z_0/\mathbb{Q}})$ affords a $\mathbb{Q}$-linear action of K. Indeed, this is clear for $H^{n-1}(X_0, \Omega^\bullet_{X_0/\mathbb{Q}})^G$, since the action of μ_p on X_0 by automorphisms can be defined over $\mathbb{Q}$ and commutes with the G action. Then we transfer this to the cohomology of Z_0 via Lemma 3.11, which in this case provides an isomorphism $H^{n-1}(Z_0, \Omega^\bullet_{Z_0/\mathbb{Q}}) \simeq H^{n-1}(X_0, \Omega^\bullet_{X_0/\mathbb{Q}})^G$.

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

Let us fix a nontrivial $\xi \in \mu_p$. If we base change $H^{n-1}(Z_0, \mathbb{Q})$ to K, we have an eigenspace decomposition

$$H^{n-1}(Z_0, K) = \bigoplus_{k=0}^{p-1} H^{n-1}(Z_0, K)_{\xi^k}.$$

Hence, ξ acts by multiplication by ξ^k on $H^{n-1}(Z_0, K)_{\xi^k}$. Similarly, for algebraic de Rham cohomology,

$$H^{n-1}(Z_0, \Omega^\bullet_{Z_0/K}) = \bigoplus_{k=0}^{p-1} H^{n-1}(Z_0, \Omega^\bullet_{Z_0/K})_{\xi^k}.$$

If we compare with $H^{n-1}(X_0, \Omega^\bullet_{X_0/K})^G$, and we recall the construction of the sections θ_k and η_k° (see §3.5), we see by inspection that ξ acts on η_k° by multiplication by ξ^{k+1}. Therefore, we infer that the nontrivial eigenspaces only occur when $1 \leq k \leq p-1$ and

$$H^{n-1}(Z_0, \Omega^\bullet_{Z_0/K})_{\xi^k} = K\eta^\circ_{k-1} = H^{k-1}(Z_0, \Omega^{n-k}_{Z_0/K}).$$

Hence, the eigenspace $H^{n-1}(Z_0, \Omega^\bullet_{Z_0/K})_{\xi^k}$ has Hodge type $(n-k, k-1)$.

The period isomorphism relating algebraic de Rham and Betti cohomologies decomposes into eigenspaces as well. We obtain refined period isomorphisms

$$\mathrm{per}_k : H^{n-1}(Z_0, \Omega^\bullet_{Z_0/K})_{\xi^k} \otimes_K \mathbb{C} \xrightarrow{\sim} H^{n-1}(Z_0, K)_{\xi^k} \otimes_K \mathbb{C}.$$

Evaluating the isomorphism on K-bases of both sides, we obtain a period, still denoted $\mathrm{per}_k \in \mathbb{C}^\times/K^\times$.

Lemma 7.1. *Fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ in $\mathbb{C}$. Then there is an equality in $\mathbb{C}^\times/\overline{\mathbb{Q}}^\times$*

$$\mathrm{per}_k = \frac{1}{\pi}\Gamma\left(\frac{k+1}{p}\right)^p.$$

Proof. The claim is equivalent to the analogous computation on X_0. Hidden behind this phrase is the comparison of cup products on X_0 and Z_0 accounted for by Lemma 3.6. On X_0, the formula for the period is well-known and given, for instance, in [Gro78, Sec. 4, p. 206] (see more generally [DMOS82, Chap. I, Sec. 7]). Notice that the author would rather work with the Fermat hypersurface $x_0^p + \ldots + x_{n-1}^p = x_n^p$. However, as we compute periods up to algebraic numbers, by applying the obvious isomorphism of varieties defined over $\overline{\mathbb{Q}}$, the result is the same. Also, we have used standard properties of the Γ-function to transform [Gro78] in our stated form. □

Theorem 7.2. *For Z_0 of dimension $p-2$, with $p \geq 5$ prime, the BCOV invariant satisfies*

$$\tau_{\mathrm{BCOV}}(Z_0) = \frac{1}{\pi^\sigma}\left(\Gamma\left(\frac{1}{p}\right)^{\chi(Z_0)/12}\prod_{k=1}^{p-1}\Gamma\left(\frac{k}{p}\right)^{p-k-1}\right)^{2p} \quad \text{in} \quad \mathbb{R}^\times/\mathbb{R}\cap\overline{\mathbb{Q}}^\times,$$

where

$$\sigma = p\left(\frac{\chi(Z_0)}{12} + \frac{(p-1)(p-2)}{2}\right) + \frac{1}{2}\sum_k (-1)^k k^2 b_k.$$

Proof. We apply Theorem 5.1, written in terms of the sections η_k° instead of η_k (which vanish at 0). Up to rational number, this has the effect of letting down the term $(\psi^{n+1})^a$ in that statement. We are thus

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

led to evaluate the L^2 norms of the sections η_k°. By [MR04, Lemma 3.4], the L^2 norms satisfy

$$\|\eta_k^\circ\|_{L^2}^2 = (2\pi)^{-(p-2)} |\operatorname{per}_k|^2.$$

It is now enough to plug this expression into Theorem 5.1, as well as the value of per_k provided by Lemma 7.1. □

Combining Theorem 2.3 and the conjecture of Gross–Deligne (see [Fre17, MR04] for up-to-date discussions and positive results), one can propose a general conjecture for the values of the BCOV invariants of some Calabi–Yau varieties with complex multiplication. For this to be plausible, however, it seems necessary to impose further conditions on the Hodge structure. Other recent examples of Calabi–Yau manifolds whose BCOV invariants should adopt a special form are given in [CdlOEvS20].

Acknowledgements. The authors extend their heartfelt gratitude to Ken-Ichi Yoshikawa for generously sharing his ideas and insights into BCOV invariants. Special thanks are also extended to Nicholas Shepherd–Barron, who explained Proposition 3.1 to us and allowed us to include its proof in the article. We thank the referee for the diligent reading and criticism of the article. In particular, it helped us write and expand Section 6.2 and correct many inaccuracies that appeared in the first version. The first author thanks Michael Björklund and Hjalmar Rosengren for discussions relating to Picard–Fuchs equations and their solutions.

Conflicts of Interest. None.

Financial support. The first author was supported by the 'Svensk-Franska stiftelsen', funding visits to France in 2019. The second author was supported by the French research grant ANR PERGAMO (ANR-18-CE40-0017), as well as the Knut och Alice Wallenberg foundation from Sweden (guest researcher program), funding stays in France and Sweden from 2019 to 2022. Finally, this research was also supported by grants from the 'Stiftelsen G S Magnusons fond', grant 'Higher-dimensional genus 1 mirror symmetry' (MG2019 0003), funding visits of the second and third authors to Sweden in 2019 and 2021.

References

[Bat94] V. V. Batyrev. Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties. *J. Algebraic Geom.*, **3**(3): 493–535, 1994.

[BB96] V. V. Batyrev and L. A. Borisov. Mirror duality and string-theoretic Hodge numbers. *Invent. Math.*, **126**(1), 1996.

[BCOV94] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa. Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes. *Comm. Math. Phys.*, **165**(2): 311–427, 1994.

[BD96] V. V. Batyrev and D. I. Dais. Strong McKay correspondence, string-theoretic Hodge numbers and mirror symmetry. *Topology*, **35**(4): 901–929, 1996.

[Beh97] K. Behrend. Gromov-Witten invariants in algebraic geometry. *Invent. Math.*, **127**(3): 601–617, 1997.

[BG14] G. Bini and A. Garbagnati. Quotients of the Dwork pencil. *J. Geom. Phys.*, **75**: 173–198, 2014.

[BvS95] V. V. Batyrev and D. van Straten. Generalized hypergeometric functions and rational curves on Calabi-Yau complete intersections in toric varieties. *Comm. Math. Phys.*, **168**(3): 493–533, 1995.

[CdlOEvS20] P. Candelas, X. de la Ossa, M. Elmi, and D. van Straten. A one parameter family of Calabi-Yau manifolds with attractor points of rank two. *J. High Energy Phys.*, (10):**202**, 73, 2020.

[CG11] A. Corti and V. Golyshev. Hypergeometric equations and weighted projective spaces. *Sci. China Math.*, **54**(8): 1577–1590, 2011.

[CL12] K. Costello and S. Li. Quantum BCOV theory on Calabi-Yau manifolds and the higher genus B-model. *arXiv e-prints*, page arXiv:1201.4501, January 2012.

[Del] P. Deligne. Local behavior of Hodge structures at infinity. In *Mirror symmetry, II*, volume **1** of *AMS/IP Stud. Adv. Math.*, pages 683–699.

[Del71] P. Deligne. Théorie de Hodge. *II. Inst. Hautes Études Sci. Publ. Math.*, (**40**): 5–57, 1971.

[Del87] P. Deligne. Le déterminant de la cohomologie. In *Current trends in arithmetical algebraic geometry (Arcata, Calif., 1985)*, volume **67** of *Contemp. Math.*, pages 93–177. Amer. Math. Soc., Providence, RI, 1987.

[DHZ98] D. I. Dais, M. Henk, and G. M. Ziegler. All abelian quotient C.I.-singularities admit projective crepant resolutions in all dimensions. *Adv. Math.*, **139**(2): 194–239, 1998.

[DHZ06] D. I. Dais, M. Henk, and G. M. Ziegler. On the existence of crepant resolutions of Gorenstein abelian quotient singularities in dimensions ≥ 4. In *Algebraic and geometric combinatorics*, volume **423** of Contemp. Math., pages 125–193. Amer. Math. Soc., Providence, RI, 2006.

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

[DMOS82] P. Deligne, J. S. Milne, A. Ogus, and K.-Y. Shih. *Hodge cycles, motives, and Shimura varieties*, volume **900** of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1982.

[Dol96] I. V. Dolgachev. Mirror symmetry for lattice polarized $K3$ surfaces. volume **81**, pages 2599–2630. 1996. *Algebraic geometry*, 4.

[DY20] X. Dai and K.-I. Yoshikawa. Analytic torsion for log-Enriques surfaces and Borcherds product. *arXiv e-prints*, page arXiv:2009.10302, September 2020.

[EFiMM18] D. Eriksson, G. Freixas i Montplet, and C. Mourougane. Singularities of metrics on Hodge bundles and their topological invariants. *Algebraic Geometry*, **5**: 1–34, 2018.

[EFiMM21] D. Eriksson, G. Freixas i Montplet, and C. Mourougane. BCOV invariants of Calabi-Yau manifolds and degenerations of Hodge structures. *Duke Math. J.*, **170**(3): 379–454, 2021.

[Eri08] D. Eriksson. *Formule de Lefschetz fonctorielle et applications géométriques*. PhD thesis, Université Paris-Sud 11, 2008.

[EZT14] F. El Zein and Lê Dũng Tráng. Mixed Hodge structures. In *Hodge theory, volume 49 of Math. Notes*, pages 123–216. Princeton Univ. Press, Princeton, NJ, 2014.

[FLY08] H. Fang, Z. Lu, and K.-I. Yoshikawa. Analytic torsion for Calabi-Yau threefolds. *J. Differential Geom.*, **80**(2): 175–259, 2008.

[Fra92] J. Franke. Riemann–Roch in functorial form. Unpublished, 1992.

[Fre17] J. Fresán. Periods of Hodge structures and special values of the gamma function. *Invent. Math.*, **208**(1): 247–282, 2017.

[FZ20] L. Fu and Y. Zhang. Motivic integration and the birational invariance of BCOV invariants. *arXiv e-prints*, page arXiv:2007.04835, July 2020.

[Gäh13] S. Gährs. Picard-Fuchs equations of special one-parameter families of invertible polynomials. In *Arithmetic and geometry of K3 surfaces and Calabi-Yau threefolds*, volume **67** of *Fields Inst. Commun.*, pages 285–310. Springer, New York, 2013.

[GKZ08] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky. *Discriminants, resultants and multidimensional determinants*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2008.

[Gri69] P. A. Griffiths. On the periods of certain rational integrals. I, II. *Ann. of Math. (2) 90 (*1969*),* 460–495*; ibid. (2)*, 90:496–541, 1969.

[Gro78] B. H. Gross. On the periods of abelian integrals and a formula of Chowla and Selberg. *Invent. Math.*, **45**(2): 193–211, 1978. With an appendix by David E. Rohrlich.

[GS90a] H. Gillet and C. Soulé. Arithmetic intersection theory. *Inst. Hautes Études Sci. Publ. Math.*, (72): 93–174 (1991), 1990.

[GS90b] H. Gillet and C. Soulé. Characteristic classes for algebraic vector bundles with Hermitian metric. I. *Ann. of Math. (2)*, **131**(1): 163–203, 1990.

[GS90c] H. Gillet and C. Soulé. Characteristic classes for algebraic vector bundles with Hermitian metric. II. *Ann. of Math. (2)*, **131**(2): 205–238, 1990.

[GS92] H. Gillet and C. Soulé. An arithmetic Riemann-Roch theorem. *Invent. Math.*, **110**(3): 473–543, 1992.

[Har66] R. Hartshorne. *Residues and duality*. Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. **20**. Springer-Verlag, Berlin-New York, 1966.

[HSBT10] M. Harris, N. Shepherd-Barron, and R. Taylor. A family of Calabi-Yau varieties and potential automorphy. *Ann. of Math. (2)*, **171**(2): 779–813, 2010.

[Huy05] D. Huybrechts. *Complex geometry*. Universitext. Springer-Verlag, Berlin, 2005. An introduction.

[Ill94] L. Illusie. Autour du théorème de monodromie locale. *Astérisque*, (223): 9–57, 1994. Périodes p-adiques (Bures-sur-Yvette, 1988).

[KMY18] S. Kawaguchi, S. Mukai, and K.-I. Yoshikawa. Resultants and the Borcherds Phi-function. *Amer. J. Math.*, **140**(6): 1471–1519, 2018.

[KO68] N. M. Katz and T. Oda. On the differentiation of de Rham cohomology classes with respect to parameters. *J. Math. Kyoto Univ.*, **8**: 199–213, 1968.

[KP08] A. Klemm and R. Pandharipande. Enumerative geometry of Calabi-Yau 4-folds. *Comm. Math. Phys.*, **281**(3): 621–653, 2008.

[LP07] J. Lee and T. Parker. A structure theorem for the Gromov-Witten invariants of Kähler surfaces. *J. Differential Geom.*, **77**(3): 483–513, 2007.

[Mor93] D. R. Morrison. Mirror symmetry and rational curves on quintic threefolds: a guide for mathematicians. *J. Amer. Math. Soc.*, **6**(1): 223–247, 1993.

[Mor97] D. R. Morrison. Mathematical aspects of mirror symmetry. In *Complex algebraic geometry (Park City, UT, 1993)*, volume **3** of *IAS/Park City Math. Ser.*, pages 265–327. Amer. Math. Soc., Providence, RI, 1997.

[MR04] V. Maillot and D. Roessler. On the periods of motives with complex multiplication and a conjecture of Gross-Deligne. *Ann. of Math. (2)*, **160**(2): 727–754, 2004.

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press

[MR12] V. Maillot and D. Rössler. On the birational invariance of the BCOV torsion of Calabi-Yau threefolds. *Comm. Math. Phys.*, **311**(2): 301–316, 2012.

[SC67] A. Selberg and S. Chowla. On Epstein's zeta-function. *J. Reine Angew. Math.*, **227**: 86–110, 1967.

[Sch73] W. Schmid. Variation of Hodge structure: the singularities of the period mapping. *Invent. Math.*, **22**: 211–319, 1973.

[Ste77] J. Steenbrink. Mixed Hodge structure on the vanishing cohomology. In *Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976)*, pages 525–563. Sijthoff and Noordhoff, Alphen aan den Rijn, 1977.

[Ste76] J. Steenbrink. Limits of Hodge structures. *Invent. Math.*, **31**(3): 229–257, 1975/76.

[Voi99] C. Voisin. *Mirror symmetry*, volume **1** of *SMF/AMS Texts and Monographs*. American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 1999. Translated from the 1996 French original by Roger Cooke.

[Voi07] C. Voisin. *Hodge theory and complex algebraic geometry. II, volume 77 of Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, english edition, 2007.

[Yas04] T. Yasuda. Twisted jets, motivic measures and orbifold cohomology. *Compos. Math.*, **140**(2): 396–422, 2004.

[Yos99] K.-I. Yoshikawa. Discriminant of theta divisors and Quillen metrics. *J. Differential Geom.*, **52**(1): 73–115, 1999.

[Yos04] K.-I. Yoshikawa. $K3$ surfaces with involution, equivariant analytic torsion, and automorphic forms on the moduli space. *Invent. Math.*, **156**(1): 53–117, 2004.

[Yos17] K.-I. Yoshikawa. Analytic torsion for Borcea–Voisin threefolds. In *Geometry, Analysis and Probability, volume 310 of Progress in Math.*, pages 279–361. Springer International Publisher, 2017.

[Zha20] Y. Zhang. BCOV invariant and blow-up. *arXiv e-prints*, page arXiv:2003.03805, March 2020.

[Zin08] A. Zinger. Standard versus reduced genus-one Gromov-Witten invariants. *Geom. Topol.*, **12**(2): 1203–1241, 2008.

[Zin09] A. Zinger. The reduced genus 1 Gromov-Witten invariants of Calabi-Yau hypersurfaces. *J. Amer. Math. Soc.*, **22**(3): 691–737, 2009.

[ZZ08] D. Zagier and A. Zinger. Some properties of hypergeometric series associated with mirror symmetry. In *Modular forms and string duality*, volume **54** of *Fields Inst. Commun.*, pages 163–177. Amer. Math. Soc., Providence, RI, 2008.

https://doi.org/10.1017/fmp.2022.13 Published online by Cambridge University Press