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QUANTITATIVE PROPERTIES OF MEROMORPHIC SOLUTIONS TO SOME DIFFERENTIAL-DIFFERENCE EQUATIONS

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Abstract

We investigate several quantitative properties of entire and meromorphic solutions to some differential-difference equations and generalised delay differential-difference equations. Our results are sharp in a certain sense as illustrated by several examples.

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1. Introduction and main results

Laine investigated the differential equation

$$f^{(n)}(z) + a_{n-1}(z)f^{(n-1)}(z) + \cdots + a_1(z)f'(z) + a_0(z)f(z) = P_1(z)e^{P_0(z)}, \tag{1.1}$$

with $a_0 \not\equiv 0, a_1, \ldots, a_{n-1}, P_0, P_1 \not\equiv 0$ polynomials, in his monograph [12] on Nevanlinna theory and complex differential equations. In particular, he proved the following important result.

Theorem 1.1 [12, Theorem 8.8]. *Suppose that*

$$\beta := \deg P_0 \leq \gamma := 1 + \max_{j=0,1,\ldots,n-1}\left\{\frac{\deg a_j}{n-j}\right\}$$

in (1.1). Then all meromorphic solutions f to (1.1) satisfy $\beta \leq \sigma(f) \leq \gamma$. If, in addition, $\sigma(f) > \beta$, then $\sigma(f) = \lambda(f) = \bar{\lambda}(f)$.

Our first results are related to this theorem of Laine. We assume the reader is familiar with the basics of the Nevanlinna theory of meromorphic functions in the complex plane $\mathbb{C}$ as in [11], such as the first and second main theorems, and the

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standard notations, such as the characteristic function $T(r, f)$, the proximity function $m(r, f)$, and the integrated counting functions $N(r, f)$ and $\bar{N}(r, f)$. As usual, $S(r, f)$ denotes any quantity satisfying $S(r, f) = o(T(r, f))$ when $r \to +\infty$ outside a possible exceptional set of finite logarithmic measure. We use $\sigma(f)$, $\varsigma(f)$, and $\lambda(f)$ and $\bar{\lambda}(f)$ to denote the order, the hyper-order, and the exponent of convergence of zeros and that of the distinct zeros for a meromorphic function f in $\mathbb{C}$, respectively.

Our first result concerns some quantitative properties of (finite-order) entire solutions to the following differential-difference equation, associated with (1.1),

$$h_n(z)f^{(n)}(z+\eta_n) + \cdots + h_1(z)f'(z+\eta_1) + h_0(z)f(z+\eta_0) = P_1(z)e^{P_0(z)}f^{(k)}(z+c). \quad (1.2)$$

Here, $k > 0$ is an integer, P_0 is entire, $P_1, h_0 \not\equiv 0, h_1, \ldots, h_n$ are small functions of f (that is, they belong to the family $\mathcal{S}_f$ of meromorphic functions $a(z)$ in $\mathbb{C}$ with $T(r, a) = S(r, f)$), and $c, \eta_0, \eta_1, \ldots, \eta_n$ are constants.

Theorem 1.2. *Assume f is a transcendental entire solution to (1.2) with hyper-order $\varsigma(f)$ strictly less than 1. Then, the following three conclusions hold.*

(i) $\sigma(f) \geq \sigma(e^{P_0})$.

(ii) *If f has a finite Borel value b and either $\sigma(f) > \sigma(e^{P_0})$ or $\sigma(f) = \sigma(e^{P_0})$ yet $\varsigma(f) > \sigma(P_0)$, then $b = 0$.*

(iii) *If f, e^{P_0} have finite orders with $\sigma(f) > \sigma(e^{P_0}) + 1$, $P_1, h_0 \not\equiv 0, h_1, \ldots, h_n$ have finite orders that are strictly less than $\sigma(f) - 1$, and $c, \eta_0, \eta_1, \ldots, \eta_n$ are pairwise distinct, then $\lambda(f - a) \geq \sigma(f) - 1$ for every finite value $a \in \mathbb{C}$.*

Theorem 1.2 provides a deep extension to the main results of Liu and Song [13, Theorems 1–3]. The following examples show that our theorem is best possible in a certain sense.

Example 1.3. $f(z) = e^z \sin z$ is a transcendental entire solution of finite order to the equation $f''(z + 2\pi i) + 2f(z) = 2f'(z)$.

Example 1.4. $f(z) = e^z$ is a transcendental entire solution of finite order to the equation $f''(z + 2\pi i) + f(z) = 2f'(z)$.

Example 1.5. $f(z) = e^{z^3}$ is a transcendental entire solution of finite order to the equation $3z^2 f(z + \eta_0) = e^{3\eta_0 z^2 + 3\eta_0^2 z + \eta_0^3} f'(z)$ with $P_0(z) = 3\eta_0 z^2 + 3\eta_0^2 z + \eta_0^3$.

Example 1.3 illustrates $\sigma(f) > \sigma(e^{P_0})$; however, we do not know whether the equality $\sigma(f) = \sigma(e^{P_0})$ can occur in case (i). Via Example 1.4, conclusion (ii) arises. We can also observe $\sigma(f) = \sigma(e^{P_0}) + 1$ but $\lambda(f) < \sigma(f) - 1$ in Example 1.5, so that the condition $\sigma(f) > \sigma(e^{P_0}) + 1$ cannot be further reduced.

On the other hand, if $P_1 e^{P_0} \equiv 0$ in (1.1), Voorhoeve *et al.* [15] observed in 1975 that every exponential polynomial is a solution to this new equation. In addition, when the coefficients of this new equation are exponential polynomials and exactly one coefficient has order strictly larger than those of the others, Wen *et al.* [16] recently proved that a transcendental exponential polynomial solution to such an equation has a specific dual relation to the maximum order coefficient.

An exponential polynomial has the form

$$f(z) = P_1(z)e^{Q_1(z)} + \cdots + P_k(z)e^{Q_k(z)},$$

where the P_j and Q_j are polynomials and $P_j \not\equiv 0$ for $j = 1, 2, \ldots, k$. We denote $q = \max\{\deg Q_j : Q_j \not\equiv 0\}$, and let $\omega_1, \omega_2, \ldots, \omega_m$ be the pairwise distinct, nonzero leading coefficients of those polynomials Q_j that attain the maximum degree q. Then f can be rewritten as

$$f(z) = F_0(z) + F_1(z)e^{\omega_1 z^q} + \cdots + F_m(z)e^{\omega_m z^q},$$

where by construction F_0 and $F_l \not\equiv 0$ for $l = 1, 2, \ldots, m$ are either exponential polynomials of orders strictly less than q or polynomials. Note that $q = 0$ means f is a polynomial.

Motivated by the consideration of transcendental exponential polynomials as in [15, 16], we also discuss exponential polynomial solutions to the equation

$$a_n(z)f^{(n)}(z+\eta_n) + \cdots + a_1(z)f'(z+\eta_1) + a_0(z)f(z+\eta_0) = P_1(z)e^{P_0(z)}f(z+c), \quad (1.3)$$

and arrive at the following conclusion.

Theorem 1.6. *Every transcendental exponential polynomial solution f to (1.3) satisfies $\sigma(f) \geq \deg P_0 + 1$. Here, P_0 is a nonconstant polynomial, $P_1, a_0, a_1, \ldots, a_n$ are exponential polynomials of orders at most $\deg P_0 - 1$, and $c, \eta_0, \eta_1, \ldots, \eta_n$ are constants.*

Finally, we consider the generalised delay differential-difference equation

$$a_1(z)\omega(z+c_1) + \cdots + a_n(z)\omega(z+c_n) + a(z)\frac{\omega'(z)}{H(z,\omega(z))} = \frac{P(z,\omega(z))}{Q(z,\omega(z))}, \quad (1.4)$$

which is related to the difference Painlevé equation recently studied by Halburd and Korhonen [7]. There have been many studies of discrete (or difference) Painlevé equations. One way in which difference Painlevé equations arise is in the study of difference equations admitting meromorphic solutions of slow growth in the sense of Nevanlinna theory. The idea that the existence of sufficiently many finite-order meromorphic solutions could be considered as a version of the Painlevé property for difference equations was first advocated by Ablowitz *et al.* [1]. This is, however, a very restrictive property, as demonstrated by the relatively short list of possible equations obtained in [5, 6] of the form

$$\omega(z+1) + \omega(z-1) = R(z, \omega(z)),$$

where R is rational in ω with meromorphic coefficients in z, while ω is assumed to be of finite order but to grow faster than the coefficients. It was later shown in [8] that the same list arises if the finite-order hypothesis is replaced with that of hyper-order strictly less than 1.

Some reductions of integrable differential-difference equations are known to generate delay differential equations with formal continuum limits to differential Painlevé equations. For example, Quispel *et al*. [14] obtained the equation

$$\omega(z)(\omega(z+1)-\omega(z-1))+a\omega'(z)=b\omega(z), \tag{1.5}$$

with a and b constants, as a symmetry reduction of the Kac–van Moerbeke equation. They also observed that (1.5) admits a formal continuum limit to the first Painlevé equation, and obtained an associated linear problem by extending the symmetry reduction to the Lax pair for the Kac–van Moerbeke equation. Painlevé(-type) delay differential equations were also considered in Grammaticos *et al*. [3] from the point of view of a kind of singularity confinement.

Recently, Halburd and Korhonen [7] considered an extended version of (1.5) and studied the delay differential-difference equation

$$\omega(z+1)-\omega(z-1)+a(z)\frac{\omega'(z)}{\omega(z)}=R(z,\omega(z))=\frac{P(z,\omega(z))}{Q(z,\omega(z))}. \tag{1.6}$$

In particular, they proved the following result.

Theorem 1.7 [7, Theorem 1.1]. *Assume ω is a transcendental meromorphic solution to (1.6), where a is a rational function, $P(z,\omega(z))$ is a polynomial in ω with rational coefficients in z, and $Q(z,\omega(z))$ is a polynomial in ω having zeros that are nonzero rational functions in z but are not zeros of $P(z,\omega(z))$. If the hyper-order of ω is strictly less than 1, then*

$$\deg_\omega(P)=\deg_\omega(Q)+1\le 3 \quad \text{or} \quad \deg_\omega(R)\le 1.$$

Here $\deg_\omega(P)=\deg_\omega(P(z,\omega(z)))$ denotes the degree of P as a polynomial in ω, while $\deg_\omega(R)=\max\{\deg_\omega(P),\deg_\omega(Q)\}$ denotes the degree of R as a rational function in ω.

We study the more general equation (1.4) and derive the following result.

Theorem 1.8. *Assume ω is a transcendental meromorphic solution to (1.4), where $a,a_1,a_2,\dots,a_n$ are rational functions and $c_1,c_2,\dots,c_n$ are pairwise distinct, nonzero constants, $H(z,\omega(z))$ and $P(z,\omega(z))$ are polynomials in ω with rational coefficients in z, and $Q(z,\omega(z))$ is a polynomial in ω having zeros that are nonzero rational functions in z but are not zeros of $P(z,\omega(z))$. If the hyper-order of ω is strictly less than 1 and $\deg_\omega(H)\ge 1$, then*

$$\deg_\omega(P)\le\deg_\omega(Q)+1.$$

Two examples are given below to exhibit the sharpness of Theorem 1.8.

Example 1.9. $\omega(z)=ze^{\pi iz}$ is a finite-order transcendental entire solution to

$$\omega(z+1)-\omega(z-1)+\frac{\omega'(z)}{\pi i\omega^2(z)}=\frac{-2\omega^2(z)+z}{z\omega(z)}.$$

Example 1.10. $\omega(z)=e^{\pi iz}$ is a finite-order transcendental entire solution to

$$\omega(z+1)-\omega(z-1)+\frac{\omega'(z)}{\pi i\omega^2(z)}=\frac{1}{\omega(z)}.$$

2. Preliminary results

This section summarises several key results needed subsequently in this paper.

LEMMA 2.1 [18, Theorem 1.62]. *Take an integer $n \geq 3$, and let $f_1, f_2, \ldots, f_n$ be meromorphic functions in $\mathbb{C}$ with $f_1, f_2, \ldots, f_{n-1}$ nonconstant and $f_n \not\equiv 0$. Assume $\sum_{j=1}^{n} f_j \equiv 1$ and*

$$\sum_{j=1}^{n} N\Big(r, \frac{1}{f_j}\Big) + (n-1)\sum_{j=1}^{n} \bar{N}(r, f_j) \leq (\lambda + o(1))T(r, f_k)$$

with some constant $\lambda < 1$ for all $r \in I$ and $k = 1, 2, \ldots, n-1$. Then $f_n \equiv 1$. Here, I denotes a subset of positive real numbers of infinite Lebesgue measure.

LEMMA 2.2 [18, Theorem 1.51]. *Let f_j be meromorphic functions and g_j be entire functions in $\mathbb{C}$ for $j = 1, 2, \ldots, n$ with an integer $n \geq 2$. Assume*

(i) $\sum_{j=1}^{n} f_j e^{g_j} \equiv 0$;
(ii) *$g_k - g_l$ is not a constant whenever $1 \leq k \neq l \leq n$;*
(iii) *$T(r, f_j) = S(r, e^{g_k - g_l})$ for every $1 \leq j \leq n$ and all $1 \leq k \neq l \leq n$.*

Then $f_j \equiv 0$ for $j = 1, 2, \ldots, n$.

LEMMA 2.3 [8, Theorem 5.1 and Lemma 8.3]. *Let η_1, η_2 be two complex numbers, and let f be a meromorphic function in $\mathbb{C}$ satisfying $\varsigma = \varsigma(f) < 1$. Then, for every sufficiently small $\varepsilon > 0$ and all $s \in (0, +\infty)$,*

$$m\Big(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\Big) = o\Big(\frac{T(r, f)}{r^{1-\varsigma-\varepsilon}}\Big) = S(r, f)$$

and

$$T(r+s, f) = T(r, f) + o\Big(\frac{T(r, f)}{r^{1-\varsigma-\varepsilon}}\Big) = T(r, f) + S(r, f).$$

Here, $r \in (0, +\infty)$ outside of an exceptional set of finite logarithmic measure.

For the preceding result, one may also consult Chiang and Feng [2, Theorem 2.1 and Corollary 2.6] for the finite-order situation.

A version of the differential-difference analogue of Clunie's theorem is given next; see also the interesting result by Halburd and Korhonen [4, Theorem 3.1].

PROPOSITION 2.4 [12, Lemma 2.4.2]. *Let f be a transcendental meromorphic solution with $\varsigma(f) < 1$ to*

$$f^n(z)D_1(z, f(z)) = D_2(z, f(z)),$$

where $D_1(z, f(z)), D_2(z, f(z))$ are differential-difference polynomials in f with small meromorphic coefficients, that is, the coefficients $a(z)$ satisfy $T(r, a) = S(r, f)$. If the degree of $D_2(z, f(z))$, as a polynomial in f, its derivatives and its shifts, is at most n, then

$$m(r, D_1(z, f(z))) = S(r, f).$$

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Lemma 2.5 [17, Corollary 4.2]. *Let q be a positive integer, let $a_0, a_1, \ldots, a_n$ be either exponential polynomials having orders strictly less than q or polynomials, and let $b_1, b_2, \ldots, b_n \in \mathbb{C} \setminus \{0\}$ be pairwise distinct constants. If $\sum_{j=1}^{n} a_j(z)e^{b_j z^q} \equiv a_0(z)$, then $a_j \equiv 0$ for $j = 0, 1, \ldots, n$.*

3. Proof of Theorem 1.2

Proof of Conclusion (i). First, we rewrite (1.2) in the form

$$\frac{h_n(z)f^{(n)}(z+\eta_n) + \cdots + h_1(z)f'(z+\eta_1) + h_0(z)f(z+\eta_0)}{P_1(z)f^{(k)}(z+c)} = e^{P_0(z)}.$$

Since $\varsigma(f) < 1$ by assumption, applying the first main theorem [11, Theorem 1.2], the lemma of the logarithmic derivative [11, Lemma 2.3] and Lemma 2.3 above yields

$$\begin{aligned} T(r, e^{P_0}) &\le T\Big(r, \sum_{j=0}^{n} h_j f^{(j)}(z+\eta_j)\Big) + T\Big(r, \frac{1}{f^{(k)}(z+c)}\Big) + T\Big(r, \frac{1}{P_1}\Big) \\ &\le \sum_{j=0}^{n} T(r, f^{(j)}(z+\eta_j)) + T(r, f^{(k)}(z+c)) + \sum_{j=0}^{n} T(r, h_j) + T(r, P_1) + O(1) \\ &\le \sum_{j=0}^{n} T(r, f^{(j)}) + T(r, f^{(k)}) + S(r, f) \le (n+2)T(r, f) + S(r, f), \end{aligned}$$

which implies $\sigma(e^{P_0}) \le \sigma(f)$ so that conclusion (i) is verified. □

Proof of Conclusion (ii). Assume b is the second Borel value of the entire function f. From [18, Theorems 1.42 and 2.11], f can be rewritten in the form

$$f(z) = g(z)e^{Q(z)} + b. \tag{3.1}$$

Here, g is an entire function with $T(r, g) = S(r, f)$ and Q is an entire function with $T(r, e^Q) = T(r, f) + S(r, f)$ (so that $\sigma(e^Q) = \sigma(f)$ and $\sigma(Q) = \varsigma(f)$). Observe that

$$f^{(t)}(z) = (g(z)e^{Q(z)} + b)^{(t)} = (g(z)e^{Q(z)})^{(t)} = \varphi_t(z)e^{Q(z)}, \tag{3.2}$$

where the φ_t are polynomials formed by g, Q and their derivatives. It follows that $T(r, \varphi_t) = S(r, f)$ for $t = 1, 2, \ldots, n$ and $t = k$. Substituting (3.1) and (3.2) into (1.2),

$$\begin{aligned} &h_n(z)\varphi_n(z+\eta_n)e^{Q(z+\eta_n)} + \cdots + h_1(z)\varphi_1(z+\eta_1)e^{Q(z+\eta_1)} \\ &\quad + h_0(z)g(z+\eta_0)e^{Q(z+\eta_0)} + bh_0(z) = P_1(z)\varphi_k(z+c)e^{P_0(z)+Q(z+c)}. \end{aligned} \tag{3.3}$$

Suppose $b \neq 0$. When $n = 0$, we get a contradiction immediately via (3.3) by comparing the zeros of $g(z+\eta_0)e^{Q(z+\eta_0)} + b$ and $P_1(z)\varphi_k(z+c)$. When $n \ge 1$,

$$\begin{aligned} &\frac{h_n(z)\varphi_n(z+\eta_n)e^{Q(z+\eta_n)}}{-bh_0(z)} + \cdots + \frac{h_1(z)\varphi_1(z+\eta_1)e^{Q(z+\eta_1)}}{-bh_0(z)} \\ &\quad + \frac{g(z+\eta_0)e^{Q(z+\eta_0)}}{-b} + \frac{P_1(z)\varphi_k(z+c)e^{P_0(z)+Q(z+c)}}{bh_0(z)} = 1, \end{aligned}$$

in view of the hypothesis $h_0 \not\equiv 0$. This gives $f_1 + \cdots + f_n + f_{n+1} + f_{n+2} = 1$, where

$$f_1 = \frac{h_n(z)\varphi_n(z+\eta_n)e^{Q(z+\eta_n)}}{-bh_0(z)},$$
$$\cdots,$$
$$f_n = \frac{h_1(z)\varphi_1(z+\eta_1)e^{Q(z+\eta_1)}}{-bh_0(z)},$$
$$f_{n+1} = \frac{g(z+\eta_0)e^{Q(z+\eta_0)}}{-b},$$
$$f_{n+2} = \frac{P_1(z)\varphi_k(z+c)e^{P_0(z)+Q(z+c)}}{bh_0(z)}.$$

Because either $\sigma(e^Q) > \sigma(e^{P_0})$ or $\sigma(Q) > \sigma(P_0)$ by assumption, it is easily seen that

$$T(r, f_l) = T(r, e^Q) + S(r, f) = T(r, f) + S(r, f)$$

for $l = 1, 2, \ldots, n+1$, while f_{n+2} cannot be a constant. In addition,

$$\sum_{l=1}^{n+2} N\Big(r, \frac{1}{f_l}\Big) + (n+1)\sum_{l=1}^{n+2} \bar{N}(r, f_l) = S(r, f_s)$$

for $s = 1, 2, \ldots, n+1$. Lemma 2.1 immediately leads to a contradiction. □

Proof of Conclusion (iii). Suppose f, e^{P_0} have finite orders with $\sigma(e^{P_0}) < \sigma(f) - 1$, and there is a finite value $a \in \mathbb{C}$ with $\lambda(f - a) < \sigma(f) - 1$. Then a is a Borel value of the entire function f, and according to the arguments in part (ii),

$$f(z) = g(z)e^{Q(z)} + a. \tag{3.4}$$

Here, Q is a polynomial with $\sigma(f) = \deg Q = q$, and g is an entire function with $\sigma(g) = \lambda(g) = \lambda(f - a) < \sigma(f) - 1$. Besides, (3.2) holds with $\sigma(\varphi_t) < \sigma(f) - 1$ for $t = 1, 2, \ldots, n$ and $t = k$. Substituting (3.4) and (3.2) into (1.2), just as for (3.3), yields

$$\begin{aligned} h_n(z)\varphi_n(z+\eta_n)e^{Q(z+\eta_n)} + \cdots + h_1(z)\varphi_1(z+\eta_1)e^{Q(z+\eta_1)} \\ + h_0(z)g(z+\eta_0)e^{Q(z+\eta_0)} + ah_0(z) = P_1(z)\varphi_k(z+c)e^{P_0(z)+Q(z+c)}. \end{aligned} \tag{3.5}$$

The assumption $\sigma(f) > \sigma(e^{P_0}) + 1 > 1$ implies that $\sigma(f) = \deg Q = q \ge 2$ and $\sigma(e^{P_0}) < q - 1$, and as $c, \eta_0, \eta_1, \ldots, \eta_n$ are pairwise distinct, it follows that

$$\sigma(e^{Q(z+\eta_u)-Q(z+\eta_v)}) = \sigma(e^{Q(z+\eta_u)-P_0(z)-Q(z+c)}) = q - 1$$

for $0 \le u \ne v \le n$. Since $\sigma(h_j\varphi_j) < q - 1$ and $\sigma(P_1\varphi_k) < q - 1$, for $j = 0, 1, \ldots, n$ (with $\varphi_0 := g$) and $0 \le u \ne v \le n$,

$$\max\{T(h_j\varphi_j), T(P_1\varphi_k)\} = S(r, e^{Q(z+\eta_u)-Q(z+\eta_v)})$$

and

$$\max\{T(h_j\varphi_j), T(P_1\varphi_k)\} = S(r, e^{Q(z+\eta_u)-P_0(z)-Q(z+c)}).$$

When $a = 0$, in view of (3.5) and Lemma 2.2, $h_0 g(z+\eta_0) \equiv 0$, which is impossible. When $a \ne 0$ and $n = 0$, a contradiction follows immediately from (3.5) by comparing the zeros of $g(z+\eta_0)e^{Q(z+\eta_0)} + a$ and $P_1(z)\varphi_k(z+c)$; when $a \ne 0$ and $n \ge 1$, we can employ the same argument using Lemma 2.1 as in part (ii) to arrive at a contradiction. This covers all possibilities, so that $\lambda(f - a) \ge \sigma(f) - 1$ is verified. □

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4. Proof of Theorem 1.6

Let f be a transcendental exponential polynomial solution to (1.3) with finite order $\sigma(f)$. Then $\deg P_0 \leq \sigma(f)$ by analysis parallel to that in the proof of part (i) of Theorem 1.2, since by hypothesis $T(r, P_1), T(r, a_j) = S(r, e^{P_0})$ for $j = 0, 1, \ldots, n$. Now suppose that $\deg P_0 = \sigma(f) = q \geq 1$. Then

$$f(z) = F_{0,0}(z) + F_{1,0}(z)e^{\omega_1 z^q} + F_{2,0}(z)e^{\omega_2 z^q} + \cdots + F_{m,0}(z)e^{\omega_m z^q}$$

with $\sigma(F_{0,0}), \sigma(F_{l,0}) < q$ and $F_{l,0} \not\equiv 0$ for $l = 1, 2, \ldots, m$. This leads to

$$f^{(t)}(z) = F_{0,t}(z) + F_{1,t}(z)e^{\omega_1 z^q} + F_{2,t}(z)e^{\omega_2 z^q} + \cdots + F_{m,t}(z)e^{\omega_m z^q},$$

so that

$$\begin{aligned} f^{(t)}(z+d) = F_{0,t}(z+d) &+ F_{1,t}(z+d)e^{H_{1,d}(z)}e^{\omega_1 z^q} \\ &+ F_{2,t}(z+d)e^{H_{2,d}(z)}e^{\omega_2 z^q} + \cdots + F_{m,t}(z+d)e^{H_{m,d}(z)}e^{\omega_m z^q}. \end{aligned} \tag{4.1}$$

Here, $\omega_l \neq 0$ are pairwise distinct constants, $H_{l,d}$ are polynomials of degrees at most $q-1$ that depend on d (note that $H_{l,d} \equiv 0$ if $d = 0$), $d \in \{c, \eta_0, \eta_1, \ldots, \eta_n\}$, and $F_{0,t}, F_{l,t}$ are either exponential polynomials of orders strictly less than q or polynomials such that

$$F_{0,t}(z) = F'_{0,t-1}(z) \quad \text{and} \quad F_{l,t}(z) = F'_{l,t-1}(z) + q\omega_l z^{q-1} F_{l,t-1}(z) \tag{4.2}$$

for $l = 1, 2, \ldots, m$ and $t = 1, 2, \ldots, n$.

Write $P_0(z) = b_q z^q + G(z)$ with $b_q \neq 0$ and $\deg G \leq q-1$. Substituting (4.1) and (4.2) into (1.3) yields,

$$\begin{aligned} &\sum_{j=0}^{n} a_j(z)F_{0,j}(z+\eta_j) + \sum_{j=0}^{n}\{a_j(z)F_{1,j}(z+\eta_j)e^{H_{1,\eta_j}(z)}\}e^{\omega_1 z^q} \\ &\quad + \sum_{j=0}^{n}\{a_j(z)F_{2,j}(z+\eta_j)e^{H_{2,\eta_j}(z)}\}e^{\omega_2 z^q} + \cdots + \sum_{j=0}^{n}\{a_j(z)F_{m,j}(z+\eta_j)e^{H_{m,\eta_j}(z)}\}e^{\omega_m z^q} \\ &= P_1(z)F_{0,0}(z+c)e^{G(z)}e^{b_q z^q} + \sum_{l=1}^{m} P_1(z)F_{l,0}(z+c)e^{G(z)+H_{l,c}(z)}e^{(b_q+\omega_l)z^q}. \end{aligned}$$

For the sake of convenience, rewrite this equality as

$$\tau_0(z) + \sum_{l=1}^{m} \tau_l(z)e^{\omega_l z^q} = \zeta_0(z)e^{b_q z^q} + \sum_{l=1}^{m} \zeta_l(z)e^{(b_q+\omega_l)z^q}, \tag{4.3}$$

where τ_0, τ_l and ζ_0, ζ_l for $l = 1, 2, \ldots, m$ have the obvious meanings.

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When $m = 1$, from (4.3),

$$\tau_0(z) + \tau_1(z)e^{\omega_1 z^q} = \zeta_0(z)e^{b_q z^q} + \zeta_1(z)e^{(b_q+\omega_1)z^q}.$$

Note that $b_q\omega_1 \neq 0$. If $b_q = \omega_1$, then Lemma 2.5 yields $\zeta_1 = P_1F_{1,0}(z + c)e^{G+H_{1,c}} \equiv 0$ since $b_q + \omega_1 \neq 0$, which is impossible. If instead $b_q \neq \omega_1$, then Lemma 2.5 yields $\zeta_0 = P_1F_{0,0}(z + c)e^G \equiv 0$, which is true provided $F_{0,0} \equiv 0$; this further leads to $\tau_0 \equiv 0$ by (4.2) so that $\zeta_1 \equiv 0$ again.

When $m \geq 2$ and $b_q \in \{\omega_1, \omega_2, \ldots, \omega_m\}$, say, $b_q = \omega_1$ in (4.3), then

$$\tau_0(z) + \sum_{l=2}^{m} \tau_l(z)e^{\omega_l z^q} = \sum_{l=1}^{m} \zeta_l(z)e^{(b_q+\omega_l)z^q}. \tag{4.4}$$

If $b_q + \omega_l \neq 0$ for every $l = 2, 3, \ldots, m$ (notice that $b_q = \omega_1$ and hence $b_q + \omega_1 \neq 0$), then there exists an index, say, $l = 2$, such that $b_q + \omega_2$ stands alone in (4.4) without a match; this unfortunately yields $\zeta_2 = P_1F_{2,0}(z + c)e^{G+H_{2,c}} \equiv 0$, which is impossible. So, $b_q + \omega_l = 0$ for exactly one index (which certainly cannot be $l = 1$), say, $l = 2$. Then $\omega_1 = -\omega_2$ and (4.4) becomes

$$\sum_{l=2}^{m} \tau_l(z)e^{\omega_l z^q} = \zeta_1(z)e^{2\omega_1 z^q} + \sum_{l=3}^{m} \zeta_l(z)e^{(\omega_1+\omega_l)z^q},$$

so that

$$\{\omega_2, \omega_3, \ldots, \omega_m\} = \{-2\omega_2, \omega_3 - \omega_2, \ldots, \omega_m - \omega_2\},$$

which implies $\omega_2 + \sum_{l=3}^{m} \omega_l = -m\omega_2 + \sum_{l=3}^{m} \omega_l$. This holds only if $\omega_2 = 0$, which is absurd.

When $m \geq 2$ but $b_q \notin \{\omega_1, \omega_2, \ldots, \omega_m\}$ in (4.3), then $\zeta_0 \equiv \tau_0 \equiv 0$ follows with an analogous argument, so that

$$\sum_{l=1}^{m} \tau_l(z)e^{\omega_l z^q} = \sum_{l=1}^{m} \zeta_l(z)e^{(b_q+\omega_l)z^q}.$$

If $b_q + \omega_l = 0$ for some $l = 1, 2, \ldots, m$, say, $l = 1$, then $\zeta_1 \equiv 0$, a contradiction. So, $b_q + \omega_l \neq 0$ for every $l = 1, 2, \ldots, m$, and matching of common terms leads to

$$\{\omega_1, \omega_2, \ldots, \omega_m\} = \{\omega_1 + b_q, \omega_2 + b_q, \ldots, \omega_m + b_q\}.$$

In particular, $\sum_{l=1}^{m} \omega_l = mb_q + \sum_{l=1}^{m} \omega_l$. This holds only if $b_q = 0$, which is impossible.

So, $\sigma(f) \geq \deg P_0 + 1$ follows and the proof of Theorem 1.6 is complete.

5. Proof of Theorem 1.8

Suppose $\deg_\omega(P) = p \geq \deg_\omega(Q) + 2$ and $\deg_\omega(H) = h \geq 1$, while $P(z, \omega(z))$ and $H(z, \omega(z))$ can be written explicitly as

$$P(z, \omega(z)) = b_p(z)\omega^p(z) + \cdots + b_1(z)\omega(z) + b_0(z),$$
$$H(z, \omega(z)) = d_h(z)\omega^h(z) + \cdots + d_1(z)\omega(z) + d_0(z).$$

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Here, $b_0, b_1, \ldots, b_p$ and $d_0, d_1, \ldots, d_h$ are rational functions in z. Then we can rewrite (1.4) as

$$\begin{aligned}\bigg\{\bigg[\sum_{j=1}^{n} a_j(z)\omega(z+c_j)\bigg]&H(z,\omega(z)) + a(z)\omega'(z)\bigg\}Q(z,\omega(z))\\ &- \{b_{p-1}(z)\omega^{p-1}(z) + \cdots + b_1(z)\omega(z) + b_0(z)\}H(z,\omega(z))\\ &- \{d_{h-1}(z)\omega^{h-1}(z) + \cdots + d_1(z)\omega(z) + d_0(z)\}b_p(z)\omega^p(z) = b_p(z)d_h(z)\omega^{p+h}(z).\end{aligned}$$

The degree of the left-hand side of this equality, as a polynomial in ω, ω' and its shifts with rational coefficients, is at most $p + h - 1$. From Lemma 2.4,

$$m(r,\omega) = S(r,\omega).$$

As a result, ω has infinitely many poles.

Let z_1 be a pole of ω with multiplicity $k_1 \geq 1$. If z_1 is neither a zero nor a pole of the rational function a in the term $a\omega'/H(z,\omega)$, one of the following holds:

(a) z_1 is a simple pole of $a(z)\omega'(z)/H(z,\omega(z))$ when $\deg_\omega(H) = 1$;
(b) z_1 is a regular point of $a(z)\omega'(z)/H(z,\omega(z))$ when $\deg_\omega(H) \geq 2$.

Applying these observations and noting that $\deg_\omega(P) - \deg_\omega(Q) \geq 2$, we see that ω has a pole at one of the points $z_1 + c_1, z_1 + c_2, \ldots, z_1 + c_n$, which is neither a zero nor a pole of the rational function coefficients of (1.4). Denote by z_2 this pole of ω with multiplicity $k_2 \geq k_1(\deg_\omega(P) - \deg_\omega(Q))$. Substituting z_2 into (1.4), by a parallel discussion to that above, ω has a pole at one of the points $z_2 + c_1, z_2 + c_2, \ldots, z_2 + c_n$, say, z_3, which is neither a zero nor a pole of the rational function coefficients of (1.4) and has multiplicity

$$k_3 \geq k_2(\deg_\omega(P) - \deg_\omega(Q)) \geq k_1(\deg_\omega(P) - \deg_\omega(Q))^2.$$

This iteration process can be repeated to generate a sequence $\{z_m : m \geq 1\}$ of poles of ω with multiplicities $\{k_m : m \geq 1\}$, which are neither zeros nor poles of the given rational functions, such that

$$k_m \geq k_1(\deg_\omega(P) - \deg_\omega(Q))^{m-1}.$$

(It may be helpful to mention that similar iteration techniques (but with more complexity) can be found in Han [9] or [10, Proposition 2.2], and the many references therein.)

We now estimate the growth of the counting function $n(r,\omega)$. Set

$$r_m := |z_1| + (m-1)\alpha$$

for $\alpha := \max\{|c_1|, |c_2|, \ldots, |c_n|\}$. It is geometrically straightforward to see that

$$z_m \in B(z_1, (m-1)\alpha) \subsetneq B(0, |z_1| + (m-1)\alpha) = B(0, r_m),$$

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which in particular implies, for sufficiently large m, that

$$n(r_m, \omega) \geq k_1(\deg_\omega(P) - \deg_\omega(Q))^{m-1}.$$

As a consequence,

$$\begin{aligned}\varsigma(\omega) &\geq \limsup_{r\to+\infty} \frac{\log\log n(r,\omega)}{\log r} \geq \limsup_{m\to+\infty} \frac{\log\log n(r_m,\omega)}{\log r_m} \\ &\geq \limsup_{m\to+\infty} \frac{\log\log k_1(\deg_\omega(P) - \deg_\omega(Q))^{m-1}}{\log r_m} = 1.\end{aligned}$$

However, this contradicts the condition $\varsigma(\omega) < 1$. So $\deg_\omega(P) \leq \deg_\omega(Q) + 1$ follows.

References

[1] M. J. Ablowitz, R. Halburd and B. Herbst, 'On the extension of the Painlevé property to difference equations', *Nonlinearity* **13** (2000), 889–905.

[2] Y. M. Chiang and S. Feng, 'On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane', *Ramanujan J.* **16** (2008), 105–129.

[3] B. Grammaticos, A. Ramani and I. C. Moreira, 'Delay-differential equations and the Painlevé transcendents', *Phys.* A **196** (1993), 574–590.

[4] R. Halburd and R. Korhonen, 'Difference analogue of the lemma on the logarithmic derivative with applications to difference equations', *J. Math. Anal. Appl.* **314** (2006), 477–487.

[5] R. Halburd and R. Korhonen, 'Finite-order meromorphic solutions and the discrete Painlevé equations', *Proc. Lond. Math. Soc. (3)* **94** (2007), 443–474.

[6] R. Halburd and R. Korhonen, 'Meromorphic solutions of difference equations, integrability and the discrete Painlevé equations', *J. Phys.* A **40** (2007), R1–R38.

[7] R. Halburd and R. Korhonen, 'Growth of meromorphic solutions of delay differential equations', *Proc. Amer. Math. Soc.* **145** (2017), 2513–2526.

[8] R. Halburd, R. Korhonen and K. Tohge, 'Holomorphic curves with shift-invariant hyperplane preimages', *Trans. Amer. Math. Soc.* **366** (2014), 4267–4298.

[9] Q. Han, 'Addendum to: Positive solutions of elliptic problems involving both critical Sobolev nonlinearities on exterior regions', *Monatsh. Math.* **177** (2015), 325–327.

[10] Q. Han, 'On the first exterior p-harmonic Steklov eigenvalue', *J. Math. Anal. Appl.* **434** (2016), 1182–1193.

[11] W. K. Hayman, *Meromorphic Functions* (Clarendon Press, Oxford, 1964).

[12] I. Laine, *Nevanlinna Theory and Complex Differential Equations* (Walter de Gruyter & Co., Berlin, 1993).

[13] K. Liu and C. J. Song, 'Meromorphic solutions of complex differential-difference equations', *Results Math.* **72** (2017), 1759–1771.

[14] G. R. W. Quispel, H. W. Capel and R. Sahadevan, 'Continuous symmetries of differential-difference equations: the Kac–van Moerbeke equation and Painlevé reduction', *Phys. Lett.* A **170** (1992), 379–383.

[15] M. Voorhoeve, A. J. Van der Poorten and R. Tijdeman, 'On the number of zeros of certain functions', *Indag. Math.* **37** (1975), 407–416.

[16] Z. T. Wen, G. G. Gundersen and J. Heittokangas, 'Dual exponential polynomials and linear differential equations', *J. Differential Equations* **264** (2018), 98–114.

[17] Z. T. Wen, J. Heittokangas and I. Laine, 'Exponential polynomials as solutions of certain nonlinear difference equations', *Acta Math. Sin. (Engl. Ser.)* **28** (2012), 1295–1306.

[18] H. X. Yi and C. C. Yang, *Uniqueness Theory of Meromorphic Functions* (China Science Publishing & Media Ltd, Beijing, 1995).

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