SOME LOCAL LIMIT RESULTS IN FLUCTUATION THEORY

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Let X_i, $i = 1, 2, 3, \cdots$ be a sequence of independent and identically distributed random variables and write $S_0 = 0$, $S_n = \sum_{i=1}^{n} X_i$, $n \geqq 1$. N_n is the number of positive terms in the sequence $S_0, S_1, S_2, \cdots, S_n$, $n \geqq 0$. It has been shown by Spitzer [7] that the limiting distribution

$$\lim_{n\to\infty} Pr(N_n \leqq nx) = F(x), \qquad -\infty < x < \infty,$$

exists if and only if

$$\text{(1)} \qquad \lim_{n\to\infty} \frac{Pr(S_1 > 0) + \cdots + Pr(S_n > 0)}{n} = \alpha, \qquad 0 \leqq \alpha \leqq 1$$

exists, and that $F(x)$ is then related to α by

$$\text{(2)} \qquad \begin{aligned} F(x) = F_\alpha(x) &= \frac{\sin \pi\alpha}{\pi} \int_0^x t^{\alpha-1}(1-t)^{-\alpha} dt, \quad \text{if } 0 < \alpha < 1,\ 0 \leqq x \leqq 1, \\ F_0(x) &= 0 \quad \text{if } x < 0,\ 1 \quad \text{if } x \geqq 0, \\ F_1(x) &= 0 \quad \text{if } x < 1,\ 1 \quad \text{if } x \geqq 1. \end{aligned}$$

Special cases of this result had previously been obtained by quite a number of authors (see Sparre Andersen [6]).

In his book [8], Spitzer gives local limit results for the probabilities $Pr(N_n = k)$ in the case where the X_i are symmetric or have zero mean and finite variance (his results, however, actually go through for the case $\sum n^{-1}[\frac{1}{2} - Pr(S_n > 0)]$ convergent). In this paper we shall generalize the work of Spitzer to obtain local limit results for all cases in which the limit in (1) exists. This provides an illuminating synthesis of the limit behaviour; it will be shown that the result (2) is an immediate byproduct of the local limit results in the case $0 < \alpha < 1$.

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Firstly, we shall need some results due to Sparre Andersen. These are the following:

$$Pr(N_n = k) = Pr(N_k = k)Pr(N_{n-k} = 0), \qquad 0 \leqq k \leqq n, \tag{3}$$

and for $0 \leqq t < 1$,

$$\sum_{k=0}^{\infty} Pr(N_k = 0)t^k = \exp\left\{\sum_1^{\infty} \frac{t^k}{k} Pr(S_k \leqq 0)\right\}, \tag{4}$$

$$\sum_{k=0}^{\infty} Pr(N_k = k)t^k = \exp\left\{\sum_1^{\infty} \frac{t^k}{k} Pr(S_k > 0)\right\} \tag{5}$$

((3) appeared in [5] and (4) and (5) in [6]; see also [8], 219).

Now let us suppose that

$$\lim_{n\to\infty} \frac{Pr(S_1 > 0) + \cdots + Pr(S_n > 0)}{n} = \alpha, \qquad 0 \leqq \alpha \leqq 1. \tag{6}$$

With this in view the obvious thing to do is to modify (5) to look at

$$(1-t)^{\alpha} \sum_{k=0}^{\infty} Pr(N_k = k)t^k = \exp\left\{\sum_1^{\infty} \frac{t^k}{k} [Pr(S_k > 0)-\alpha]\right\}.$$

However, as we shall see, the limit $\lim_{t\uparrow 1} \sum_1^{\infty} t^k k^{-1}[Pr(S_k > 0)-\alpha]$ need not exist under the condition (6) so we must delve further. Let us write

$$u_k = Pr(S_k > 0)-\alpha;$$

we shall examine the function

$$L\left(\frac{1}{1-t}\right) = \exp\left\{\sum_1^{\infty} \frac{t^k}{k} u_k\right\}, \qquad 0 \leqq t < 1. \tag{7}$$

Put

$$\varepsilon(x) = \frac{1}{x} \sum_1^{\infty} \left(1-\frac{1}{x}\right)^{k-1} u_k, \qquad x \geqq 1.$$

We have

$$\int_1^x \frac{\varepsilon(y)}{y}\, dy = \int_1^x \frac{1}{y^2} \sum_1^{\infty} \left(1-\frac{1}{y}\right)^{k-1} u_k dy$$
$$= \sum_1^{\infty} \left(1-\frac{1}{x}\right)^k \frac{u_k}{k}$$

and

$$\lim_{x\to\infty} \varepsilon(x) = \lim_{x\to\infty} \left\{\frac{1}{x} \sum_1^{\infty} \left(1-\frac{1}{x}\right)^{k-1} Pr(S_k > 0) - \frac{\alpha}{x} \sum_1^{\infty} \left(1-\frac{1}{x}\right)^{k-1}\right\}$$
$$= \lim_{y\uparrow 1} (1-y) \sum_1^{\infty} y^{k-1} Pr(S_k > 0)-\alpha$$
$$= 0$$

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using Feller [2], Theorem 5, 423. It then follows from the Corollary, 274 of [2] that

$$L(u) = \exp\left\{\sum_1^\infty \left(1-\frac{1}{u}\right)^k \frac{u_k}{k}\right\}, \qquad u \geqq 1,$$

is a slowly varying function in the sense of Karamata. The upshot of this is, of course, the existence of a class of functions of slow variation L having the same asymptotic behaviour as that defined by (7) above and such that

$$\lim_{t\uparrow 1} (1-t)^\alpha \left[L\left(\frac{1}{1-t}\right)\right]^{-1} \sum_{k=0}^\infty Pr(N_k = k)t^k = 1. \tag{8}$$

Applying a Tauberian theorem, Theorem 5, 423 of [2] to (8) we see that it is equivalent to the relation

$$\sum_{k=0}^n Pr(N_k = k) \sim \frac{1}{\Gamma(\alpha+1)} n^\alpha L(n) \qquad \text{as } n \to \infty, \tag{9}$$

and furthermore, as $Pr(N_k = k)$ is monotone decreasing in k, we have for $\alpha > 0$ that (9) is equivalent to

$$Pr(N_n = n) \sim \frac{1}{\Gamma(\alpha)} n^{\alpha-1} L(n) \qquad \text{as } n \to \infty. \tag{10}$$

In order to calculate the asymptotic value for $Pr(N_n = 0)$ we follow exactly the same principles, working with (4) and rewriting (6) in the form

$$\lim_{n\to\infty} \frac{Pr(S_1 \leqq 0) + \cdots + Pr(S_n \leqq 0)}{n} = 1-\alpha, \qquad 0 \leqq \alpha \leqq 1.$$

Then, with L as defined by (7),

$$\begin{aligned}(1-t)^\alpha L\left(\frac{1}{1-t}\right) &\sum_{k=0}^\infty Pr(N_k = 0)t^k \\ &= \exp\left\{\sum_1^\infty \frac{t^k}{k} [Pr(S_k \leqq 0) - (1-\alpha)]\right\} \\ &= \exp\left\{-\sum_1^\infty \frac{t^k}{k} [Pr(S_k > 0) - \alpha]\right\}\end{aligned}$$

and we obtain, as before,

$$\sum_{k=0}^n Pr(N_k = 0) \sim \frac{1}{\Gamma(2-\alpha)} \frac{n^{1-\alpha}}{L(n)} \qquad \text{as } n \to \infty, \tag{11}$$

which if $\alpha < 1$ is equivalent to

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$$(12) \qquad Pr(N_n = 0) \sim \frac{1}{\Gamma(1-\alpha)} \frac{1}{n^\alpha L(n)} \qquad \text{as } n \to \infty.$$

Let us now formalize our results. We have, in effect, established the following theorem.

THEOREM 1. *Let* $X_i, i = 1, 2, 3, \cdots$ *be independent and identically distributed random variables for which*

$$\lim_{n\to\infty} \frac{Pr(S_1 > 0) + \cdots + Pr(S_n > 0)}{n} = \alpha, \qquad 0 \leqq \alpha \leqq 1.$$

It is always possible to find a function of slow variation L such that

$$(13) \qquad \begin{aligned} &\lim_{n\to\infty} n^{1-\alpha} [L(n)]^{-1} Pr(N_n = n) = \frac{1}{\Gamma(\alpha)}, && 1 \geqq \alpha > 0,\\ &\lim_{n\to\infty} [L(n)]^{-1} \sum_{k=0}^{n} Pr(N_k = k) = 1, && \alpha = 0, \end{aligned}$$

and

$$(14) \qquad \begin{aligned} &\lim_{n\to\infty} n^{\alpha} L(n) Pr(N_n = 0) = \frac{1}{\Gamma(1-\alpha)}, && 1 > \alpha \geqq 0,\\ &\lim_{n\to\infty} L(n) \sum_{k=0}^{n} Pr(N_k = 0) = 1, && \alpha = 1. \end{aligned}$$

Furthermore, the existence of a limit of the form (13) *or* (14) *implies the existence of the other.*

The function L of the theorem has the same asymptotic behaviour as that defined by (7).

It is plain to see that in the case $\alpha = 0$, $L(n)$ must be bounded away from zero while in the case $\alpha = 1$, $L(n)$ must be bounded above.

We proceed immediately to the next theorem.

THEOREM 2. *Under the conditions of Theorem* 1 *we have for* $0 < \alpha < 1$,

$$\text{(a)} \qquad k^{1-\alpha}(n-k)^\alpha \frac{L(n-k)}{L(k)} Pr(N_n = k) = \frac{\sin \pi\alpha}{\pi} + o(k, n)$$

where $o(k, n)$ *tends to zero uniformly in k and n as* $\min(k, n-k) \to \infty$.

(b) *For* $0 \leqq x \leqq 1$,

$$\lim_{n\to\infty} Pr(N_n \leqq nx) = \frac{\sin \pi\alpha}{\pi} \int_0^x t^{\alpha-1}(1-t)^{-\alpha} dt.$$

PROOF. In view of (3) and the results of Theorem 1 we have for $0 < \alpha < 1$,

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$$\begin{aligned}
&k^{1-\alpha}(n-k)^{\alpha}\frac{L(n-k)}{L(k)}Pr(N_n=k)\\
&\qquad=\frac{k^{1-\alpha}}{L(k)}Pr(N_k=k)\cdot(n-k)^{\alpha}L(n-k)Pr(N_{n-k}=0)\\
&\qquad=\left[\frac{1}{\Gamma(\alpha)}+o_1(k)\right]\left[\frac{1}{\Gamma(1-\alpha)}+o_2(n-k)\right]\\
&\qquad=\frac{\sin\pi\alpha}{\pi}+o(k,n)
\end{aligned}$$

where $o(k, n)$ has the required property, namely that given any $\varepsilon > 0$ there is some $N = N(\varepsilon)$ such that $o(k, n) < \varepsilon$ when $k > N$ and $n-k > N$.

In order to obtain (b) from (a) we write

$$\begin{aligned}
Pr(N_n \leqq nx) &= Pr(N_n \leqq [nx])\\
&= \frac{1}{n}\sum_{k=0}^{[nx]}\left(\frac{k}{n}\right)^{-1+\alpha}\left(1-\frac{k}{n}\right)^{-\alpha}\frac{L\{n(1-k/n)\}}{L\{n\cdot k/n\}}\left(\frac{\sin\pi\alpha}{\pi}+o(k,n)\right)
\end{aligned}$$

We shall interpret the limit of this as the approximation to a Riemann integral. For arbitrarily small $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n\to\infty}\frac{\sin\pi\alpha}{\pi n}\sum_{k=[n\varepsilon]}^{[nx]}\left(\frac{k}{n}\right)^{-1+\alpha}\left(1-\frac{k}{n}\right)^{-\alpha}\frac{L\{n(1-k/n)\}}{L\{n\cdot k/n\}}\\
&\qquad=\frac{\sin\pi\alpha}{\pi}\int_{\varepsilon}^{x}t^{-1+\alpha}(1-t)^{-\alpha}\,dt
\end{aligned}$$

in view of the properties of slowly varying functions. We then let $\varepsilon\to 0$ and note that as the integral exists the error term must go to zero and the proof is complete. The result is a natural generalization of T2, 226-227, of [8].

3. Remarks

As regards the applicability of the above results, we should make the following comments although the substance of them is well-known. If $M_n = \max(0, S_1, S_2, \cdots, S_n)$, then N_n has the same distribution as $T_n = \min[k \mid 0 \leqq k \leqq n;\ S_k = M_n]$. This follows as

$$Pr(T_n = 0) = Pr(S_1 \leqq 0, S_2 \leqq 0, \cdots, S_n \leqq 0) = Pr(N_n = 0),$$

$$\begin{aligned}
Pr(T_n = k) &= Pr(S_k > 0, S_k > S_1, \cdots, S_k > S_{k-1}, S_k \geqq S_{k+1}, \cdots, S_k \geqq S_n)\\
&= Pr\Big(\sum_1^k X_i > 0, \sum_2^k X_i > 0, \cdots, X_k > 0, X_{k+1} \leqq 0, \cdots, \sum_{k+1}^n X_i \leqq 0\Big)\\
&= Pr(N_k = k)Pr(N_{n-k} = 0)\\
&= Pr(N_n = k),\ k > 0,
\end{aligned}$$

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in view of (3) and using the fact that the X_i are independent and identically distributed. Further, we readily see that

$$Pr(N_n = 0) = Pr(M_n = 0).$$

The quantity $Pr(M_n = 0)$ is of some interest, for example, in the theory of the $GI/G/1$ queue where it has the interpretation as the probability that the n-th arriving customer finds the queue empty and receives immediate service.

4. Examples

Since comparatively little is known about the behaviour of sequences of probabilities $\{Pr(S_n < 0), n = 1, 2, 3, \cdots\}$ upon which the results of the paper depend, it seems necessary to give some examples. What we shall do is elucidate the relation between

$$\lim_{n\to\infty} \frac{Pr(S_1 < 0) + \cdots + Pr(S_n < 0)}{n} = \alpha \tag{15}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n} [Pr(S_n < 0) - \alpha] \tag{16}$$

for a special class of random variables. We note that the well-known result $Pr(S_n = 0) \leqq c/\sqrt{n}$ for some c independent of n (see [4] for example) takes care of the $S_n = 0$ case; it is useful to incorporate this in the interests of symmetry.

Firstly, let us go over what is known. It has been shown ([8], 228-230) that any α, $0 \leqq \alpha \leqq 1$ is realizable as a limit in (15). Further, ([8], 199), if $EX_i = 0$, $EX_i^2 < \infty$ then (16) with $\alpha = \frac{1}{2}$ converges. (This convergence has been further elucidated in [4], [1], [3] but these additional results are not pertinent to the present discussion).

As a source of examples we shall follow Spitzer and look at the random variables X_i whose distribution is carried by the integers such that

$$Pr(X_i = j) = p_j, \qquad j = 0, \pm 1, \pm 2, \cdots$$

satisfies

$$p_j = 0 \qquad \text{for } j < -1, \ p_{-1} > 0.$$

(Spitzer calls the associated random walks left continuous in this case). In view of the comments above we shall interest ourselves solely in the case $EX_i = 0$, $EX_i^2 = \infty$. Now the apparatus for studying whether (15) is true has been set up in [8] (particularly 227-230) and we shall make the necessary additions to look at (16) in the same light. We extract the following information:

$$(17)\qquad 1-r(t) = \exp\left\{-\sum_1^\infty \frac{t^k}{k} Pr(S_k < 0)\right\}, \qquad 0 \leqq t < 1,$$

where $r(t)$ is the unique positive solution (less than one) of the equation

$$(18)\qquad \psi[r(t)] = t^{-1}.$$

Here,

$$(19)\qquad \psi(z) = E(z^{X_i}) = \sum_{j=-1}^\infty p_j z^j, \qquad |z| < 1,$$

and we have the obvious properties

$$\psi(1) = 1, \quad \psi'(1) = 0, \quad \psi''(1) = \infty,$$

together with the property $r = r(t) \to 1$ as $t \to 1$. The relation (15) holds if and only if

$$(20)\qquad \lim_{t\uparrow 1} \frac{-1}{\psi'(r)} \frac{\psi(r)-1}{1-r} = \alpha.$$

Now let us look at the question of the convergence of (16). We have

$$\frac{(1-t)^\alpha}{1-r(t)} = \exp\left\{\sum_1^\infty \frac{t^n}{n} [Pr(S_n < 0)-\alpha]\right\}$$

and (16) will converge if and only if

$$0 < \lim_{t\uparrow 1} \frac{(1-t)^\alpha}{1-r(t)} < \infty,$$

or equivalently, from (18).

$$(21)\qquad 0 < \lim_{r\uparrow 1} \frac{[\psi(r)-1]^\alpha}{1-r} < \infty.$$

The behaviour of $\psi(r)$ and $\psi'(r)$ near $r = 1$ is best determined by partial summation. Put

$$(22)\qquad u_n = \sum_{j>n} j p_j, \qquad n \geqq -2;$$

then $np_n = u_{n-1}-u_n$, $u_{-2} = 0$ and

$$(23)\qquad \begin{aligned} \psi'(r) &= \sum_{n=-1}^\infty (u_{n-1}-u_n) r^n = \sum_{n=-2}^\infty u_n r^{n+1} - \sum_{n=-1}^\infty u_n r^n \\ &= -(1-r) \sum_{n=-1}^\infty u_n r^n. \end{aligned}$$

Similarly, putting

$$v_n = \sum_{j>n} p_j, \qquad n \geqq -2,$$

we have

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$$\psi(r) = \sum_{n=-1}^{\infty} (v_{n-1} - v_n) r^n = r^{-1} - (1-r) \sum_{n=-1}^{\infty} v_n r^n.$$

Let us further introduce

$$w_n = \sum_{m \geqq n} v_m = \sum_{n \geqq m} \sum_{j > m} p_j = \sum_{j > n} (j-n) p_j, \qquad n \geqq -1. \tag{24}$$

We have

$$w_{-1} = \sum_{j > -1} (j+1) p_j = \sum_{j \geqq -1} (j+1) p_j = 1$$

and

$$\begin{aligned}\sum_{n=-1}^{\infty} v_n r^n &= \sum_{n=-1}^{\infty} (w_n - w_{n+1}) r^n \\ &= \sum_{n=-1}^{\infty} w_n r^n - \sum_{n=0}^{\infty} w_n r^{n-1} \\ &= r^{-1} - (1-r) \sum_{n=0}^{\infty} w_n r^{n-1}\end{aligned}$$

so that

$$\psi(r) - 1 = (1-r)^2 \sum_{n=0}^{\infty} w_n r^{n-1}. \tag{25}$$

From (20), (23) and (25), we then have

$$\lim_{r \uparrow 1} \frac{-1}{\psi'(r)} \frac{\psi(r)-1}{1-r} = \lim_{r \uparrow 1} \frac{\sum_{n=0}^{\infty} w_n r^n}{\sum_{n=-1}^{\infty} u_n r^n} = \alpha \tag{26}$$

and

$$\lim_{r \uparrow 1} \frac{[\psi(r)-1]^{\alpha}}{1-r} = \lim_{r \uparrow 1} \left[(1-r)^{2-1/\alpha} \sum_{n=0}^{\infty} w_n r^{n-1} \right]^{\alpha}. \tag{27}$$

Clearly,

$$\sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} \sum_{j > n} j p_j = \sum_{j=1}^{\infty} j^2 p_j,$$

$$\sum_{n=0}^{\infty} w_n = \sum_{n=0}^{\infty} \sum_{j > n} (j-n) p_j = \sum_{j=1}^{\infty} \frac{j(j+1)}{2} p_j,$$

and both are infinite if (and only if) $EX_i^2 = \infty$.

As a simple example take

$$p_n = \frac{c}{n^{2+\gamma}} \text{ for } n > N \qquad (0 < \gamma \leqq 1,\ c > 0).$$

Then, for $n > N$,

$$u_n = \sum_{j > n} \frac{c}{j^{1+\gamma}} \sim \frac{1}{\gamma} \frac{c}{n^{\gamma}},$$

$$u_n - w_n = n \sum_{j > n} \frac{c}{j^{2+\gamma}} \sim \frac{1}{1+\gamma} \frac{c}{n^{\gamma}},$$

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so that

$$w_n \sim \frac{1}{\gamma(1+\gamma)} \frac{c}{n^\gamma} \sim \frac{1}{1+\gamma} u_n,$$

and (26) holds with $\alpha = 1/1+\gamma$. Further, making use of Theorem 5, 423 of Feller [2], we see that for $\gamma < 1$ the limit in (27) is finite and positive. If $\gamma = 1$, on the other hand (so that $\alpha = \frac{1}{2}$), the limit in (27) becomes

$$\lim_{r \uparrow 1} \left[\sum_{n=0}^{\infty} w_n r^{n-1} \right]^{\frac{1}{2}}$$

which is infinite since $w_n \sim c/2n$. In this particular case it is not difficult to show that the X_i belong to the domain of attraction of the normal distribution with norming constants proportional to $(n \log n)^{\frac{1}{2}}$ (see [2], 303-304). This is rather interesting as it shows that the convergence result of Spitzer cited previously does not extend from the domain of normal attraction of the normal distribution to its entire domain of attraction. For $\gamma < 1$, on the other hand, the X_i belong to the domain of normal attraction of a stable law with index $1+\gamma$. It seems reasonable to conjecture that the series (16) will converge in all such cases.

As another example, take

$$p_n = \frac{c}{n^{2+\gamma} \log n} \text{ for } n > N \qquad (0 < \gamma < 1,\ c > 0)$$

Then, for $n > N$, using Theorem 1, 273 of Feller [2], we have

$$u_n = \sum_{j>n} \frac{c}{j^{1+\gamma} \log j} \sim \frac{c}{\gamma n^\gamma \log n},$$

$$u_n - w_n = n \sum_{j>n} \frac{c}{j^{2+\gamma} \log j} \sim \frac{c}{(1+\gamma) n^\gamma \log n},$$

so that

$$w_n \sim \frac{1}{\gamma(1+\gamma)} \frac{c}{n^\gamma \log n} \sim \frac{1}{1+\gamma} u_n,$$

and again (26) holds with $\alpha = 1/1+\gamma$. The limit in (27) is no longer finite; we have instead, using Theorem 5, 423 of [2],

$$0 < \lim_{r \uparrow 1} \left\{ (1-r)^{2-1/\alpha} \left[\log \left(\frac{1}{1-r} \right) \right]^{-1} \sum_{n=0}^{\infty} w_n r^{n-1} \right\}^{\alpha} < \infty.$$

It is, however, not easy to extract the explicit function of slow variation which will arise in our Theorem 1. We note in passing that the X_i here belong to the domain of attraction (but not that of normal attraction) of a stable law with index $1+\gamma$. The norming constants in this case are proportional to $(n/\log n)^{1/(1+\gamma)}$.

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