

ON CONSTANT-SIGN PERIODIC SOLUTIONS IN MODELLING THE SPREAD OF INTERDEPENDENT EPIDEMICS

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Abstract

We consider the following model that describes the *spread of n types of epidemics which are interdependent on each other*:

$$u_i(t) = \int_{t-\tau}^t g_i(t, s) f_i(s, u_1(s), u_2(s), \dots, u_n(s)) ds, \quad t \in \mathbb{R}, \quad 1 \leq i \leq n.$$

Our aim is to establish criteria such that the above system has one or multiple *constant-sign periodic solutions* (u_1, u_2, \dots, u_n) , that is, for each $1 \leq i \leq n$, u_i is periodic and $\theta_i u_i \geq 0$ where $\theta_i \in \{1, -1\}$ is fixed. Examples are also included to illustrate the results obtained.

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1. Introduction

In this paper we shall consider the following system of integral equations that describes the *spread of n types of epidemics which are interdependent on each other*:

$$u_i(t) = \int_{t-\tau}^t g_i(t, s) f_i(s, u_1(s), u_2(s), \dots, u_n(s)) ds, \quad t \in \mathbb{R}, \quad 1 \leq i \leq n. \quad (M1)$$

Here, for $1 \leq i \leq n$, we let $u_i(t)$ represent the proportion of the population infected with type i disease at time t , $f_i(t, u_1(t), u_2(t), \dots, u_n(t))$ denotes the proportion of the population newly infected with type i disease per unit time, $g_i(t, s)$ is a certain per

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unit time environmental factor associated with type i disease at time t , and $\tau \in \mathbb{R}^+$ is the length of time an individual remains infectious with type i disease.

Throughout we shall denote $u = (u_1, u_2, \dots, u_n)$. We say that u is a solution of *constant sign* of (M1) if for each $1 \leq i \leq n$, we have $\theta_i u_i(t) \geq 0$ for $t \in \mathbb{R}$, where $\theta_i \in \{1, -1\}$ is fixed. Let $0 < \omega < \infty$. A solution u of (M1) is said to be ω -periodic if u_i is ω -periodic for each $1 \leq i \leq n$. More precisely, we mean $u \in (A_\omega(\mathbb{R}))^n = A_\omega(\mathbb{R}) \times A_\omega(\mathbb{R}) \times \dots \times A_\omega(\mathbb{R})$ (n times) where

$$A_\omega(\mathbb{R}) = \{y \in BC(\mathbb{R}) \mid y(t) = y(t + \omega) \text{ for all } t \in \mathbb{R}\}$$

and $BC(\mathbb{R})$ is the space of bounded and continuous functions on \mathbb{R} with values in \mathbb{R} .

A simplified model of (M1) when $n = 1$ and $g_i(t, s) = 1$, namely

$$y(t) = \int_{t-\tau}^t f(s, y(s)) ds, \quad t \in [\tau, \infty) \quad (\text{M2})$$

was first introduced in 1976 by Cooke and Kaplan [9] and Smith [19]. Using Krasnosel'skii's fixed point theorem, it is shown in [9] that (M2) has a nontrivial periodic solution provided the effective contact rate $a(t) = \lim_{y \rightarrow 0} f(t, y)/y$ exceeds a certain threshold level. On the other hand, Nussbaum [17] and Smith [19] have verified the existence of a nontrivial periodic solution when τ exceeds some threshold value. In the eighties, Leggett and Williams [15, 20] employed their own fixed point theorems to obtain existence criteria for (M2). Their criteria complement the threshold-type results in [9, 17, 19].

In the present work, we have generalised the well-known (M2) to the system (M1) which not only models the spread of *interdependent* epidemics but also incorporates *environmental factors* in the modelling, and therefore is more robust for real-world applications. Further, by employing a variety of fixed point theorems we shall establish *new* existence results for *constant-sign periodic solutions*, which, when reduced to a special case, improve and generalise those in [9, 12, 15]. As a side remark, much work has been carried out on the related system of the form

$$u_i(t) = \int_0^1 g_i(t, s) f_i(s, u_1(s), u_2(s), \dots, u_n(s)) ds, \quad t \in [0, 1], \quad 1 \leq i \leq n.$$

The reader may refer to [2–7] which are motivated by the vast amount of literature on the existence of positive solutions [1, 8, 10, 11, 16, 18].

The paper is organised as follows. In Section 2, we state some well-known results in the literature which will be used to obtain the main theorems. Existence results for (M1) are developed in Section 3. We also include examples to illustrate the importance of the results obtained.

2. Preliminaries

We require the definition of a *Carathéodory function*.

DEFINITION 2.1. A function $f : I \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a *Carathéodory function* if the following conditions hold:

- (a) the map $t \mapsto f(t, u)$ is measurable for all $u \in \mathbb{R}^n$;
- (b) the map $u \mapsto f(t, u)$ is continuous for almost all $t \in I$.

The next three theorems are fixed point theorems; they are the main tools used in later sections. Theorem 2.2 is usually called *Krasnosel'skii's fixed point theorem in a cone*, Theorem 2.3 is known as the *Leray-Schauder alternative*, and Theorem 2.4 is from Leggett and Williams [14].

THEOREM 2.2 ([13]). Let $B = (B, \|\cdot\|)$ be a Banach space, and let $C \subset B$ be a cone in B . Assume Ω_1, Ω_2 are open bounded subsets of B with $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$, and let $S : C \cap (\overline{\Omega}_2 \setminus \Omega_1) \rightarrow C$ be a continuous and completely continuous operator such that either

- (a) $\|Su\| \leq \|u\|, u \in C \cap \partial\Omega_1$, and $\|Su\| \geq \|u\|, u \in C \cap \partial\Omega_2$, or
- (b) $\|Su\| \geq \|u\|, u \in C \cap \partial\Omega_1$, and $\|Su\| \leq \|u\|, u \in C \cap \partial\Omega_2$.

Then S has a fixed point in $C \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

THEOREM 2.3 ([18]). Let B be a Banach space with $E \subseteq B$ closed and convex. Assume U is a relatively open subset of E with $0 \in U$ and $S : \overline{U} \rightarrow E$ is a continuous and compact map. Then either

- (a) S has a fixed point in \overline{U} , or
- (b) there exists $u \in \partial U$ and $\lambda \in (0, 1)$ such that $u = \lambda Su$.

THEOREM 2.4 ([14]). Let $B = (B, \|\cdot\|)$ be a Banach space, $C \subset B$ a cone in $B, r_1 > 0, r_2 > 0, r_1 \neq r_2$ with $R = \max\{r_1, r_2\}$ and $r = \min\{r_1, r_2\}$. Define $\overline{C}_\eta = \{u \in C \mid \|u\| \leq \eta\}$ and

$$\begin{aligned} C(u_0) &= \{u \in C \mid \text{there exists } \lambda > 0 \text{ such that } u \geq \lambda u_0\} \\ &= \{u \in C \mid \text{there exists } \lambda > 0 \text{ such that } u - \lambda u_0 \in C\}. \end{aligned}$$

Let $S : \overline{C}_R \rightarrow C$ be a continuous and compact map such that

- (a) there exists $u_0 \in C \setminus \{0\}$ with $Su \not\leq u$ (equivalently $u - Su \notin C$) for $u \in \partial\overline{C}_{r_2} \cap C(u_0)$, and
- (b) $\|Su\| \leq \|u\|$ for $u \in \partial\overline{C}_{r_1}$.

Then S has at least one fixed point $u \in C$ with $r \leq \|u\| \leq R$.

3. Existence Results for (M1)

Criteria for the existence of one or more *constant-sign periodic solutions* of model (M1) are presented in this section. These results will be developed via various fixed point theorems.

To begin, let $0 < \omega < \infty$ and let the Banach space $B = (A_\omega(\mathbb{R}))^n$ be equipped with the norm $\|u\|_\omega = \max_{1 \leq i \leq n} \sup_{t \in [0, \omega]} |u_i(t)| = \max_{1 \leq i \leq n} \|u_i\|_\omega = \sup_{t \in [0, \omega]} |u_i(t)|$, $1 \leq i \leq n$.

Define the operator $S : B \rightarrow (C(\mathbb{R}))^n$ by

$$Su(t) = (S_1u(t), S_2u(t), \dots, S_nu(t)), \quad t \in \mathbb{R}, \quad (3.1)$$

where

$$S_iu(t) = \int_{t-\tau}^t g_i(t, s) f_i(s, u(s)) ds, \quad t \in \mathbb{R}, \quad 1 \leq i \leq n. \quad (3.2)$$

Clearly, a fixed point of the operator S is a solution of the system (M1).

With $\theta_i \in \{1, -1\}$, $1 \leq i \leq n$ fixed, define

$$[0, \infty)_i = \begin{cases} [0, \infty), & \theta_i = 1, \\ (-\infty, 0], & \theta_i = -1, \end{cases}$$

$$\tilde{K} = \{u \in B \mid \theta_i u_i(t) \geq 0, \quad t \in \mathbb{R}, \quad 1 \leq i \leq n\}$$

and

$$\begin{aligned} K &= \{u \in \tilde{K} \mid \theta_j u_j(t) > 0 \text{ for some } j \in \{1, \dots, n\} \text{ and some } t \in \mathbb{R}\} \\ &= \tilde{K} \setminus \{0\}. \end{aligned}$$

Our first result uses Krasnosel'skii's fixed point theorem (Theorem 2.2).

THEOREM 3.1. *Let $1 \leq p \leq \infty$, q be such that $1/p + 1/q = 1$, $0 < \omega < \infty$ and let $\theta_i \in \{1, -1\}$, $1 \leq i \leq n$ be fixed. Assume the following hold for each $1 \leq i \leq n$:*

$$g'_i(s) \equiv g_i(t, s) \in L^q[0, \omega] \text{ for each } t \in [0, \omega]; \quad (3.3)$$

$$\text{the map } t \mapsto g'_i \text{ is continuous from } [0, \omega] \text{ to } L^q[0, \omega]; \quad (3.4)$$

$$g_i(t, s) \geq 0 \text{ for all } t \in [0, \omega] \text{ and a.e. } s \in [0, \omega]; \quad (3.5)$$

$$g_i(t, s + \omega) = g_i(t, s) \text{ for all } t \in \mathbb{R} \text{ and a.e. } s \in \mathbb{R}; \quad (3.6)$$

$$g_i(t + \omega, s) = g_i(t, s) \text{ for all } t \in \mathbb{R} \text{ and a.e. } s \in \mathbb{R}; \quad (3.7)$$

$$f_i : [0, \omega] \times \mathbb{R}^n \rightarrow \mathbb{R} \text{ is a Carathéodory function}; \quad (3.8)$$

$$\theta_i f_i(t, u) \geq 0 \text{ for a.e. } t \in [0, \omega] \text{ and all } u \in \tilde{K}; \quad (3.9)$$

$$f_i(t + \omega, u) = f_i(t, u) \text{ for a.e. } t \in \mathbb{R} \text{ and all } u \in \tilde{K}; \quad (3.10)$$

$$\left\{ \begin{array}{l} \text{for each } r > 0, \text{ there exists } \mu_r^i \in L^p[0, \omega] \text{ such that } |u_j| \leq r, \\ 1 \leq j \leq n, \text{ implies } |f_i(t, u_1, u_2, \dots, u_n)| \leq \mu_r^i(t) \text{ for a.e.} \\ t \in [0, \omega]; \end{array} \right. \quad (3.11)$$

$$\left\{ \begin{array}{l} \text{there exists a function } \psi_i : \prod_{j=1}^n [0, \infty)_j \rightarrow [0, \infty) \text{ continuous and} \\ \text{'nondecreasing' in the sense that for each } 1 \leq j \leq n, \text{ if } |u_j| \leq |v_j|, \\ \text{then } \psi_i(u_1, \dots, u_{j-1}, u_j, u_{j+1}, \dots, u_n) \leq \psi_i(u_1, \dots, u_{j-1}, v_j, u_{j+1}, \\ \dots, u_n), \text{ a constant } \rho_i, 0 < \rho_i \leq 1, \text{ and a function } b_i : \mathbb{R} \rightarrow [0, \infty), \\ b_i \in L^p[0, \omega], b_i(t + \omega) = b_i(t) \text{ for a.e. } t \in \mathbb{R}, \text{ with } \rho_i b_i(t) \psi_i(u) \leq \\ \theta_i f_i(t, u) \leq b_i(t) \psi_i(u) \text{ for a.e. } t \in [0, \omega] \text{ and all } u \in \tilde{K}; \end{array} \right. \quad (3.12)$$

$$\left\{ \begin{array}{l} \text{there exists a continuous function } \phi_i : (0, 1) \rightarrow (0, \infty) \text{ such that for} \\ \text{any } 0 < m < 1 \text{ and } u \in [0, \infty)^n, \text{ we have } \psi_i(mu) \geq \phi_i(m) \psi_i(u); \end{array} \right. \quad (3.13)$$

$$K_{2,i} = \inf_{t \in [0, \omega]} \int_{t-\tau}^t g_i(t, s) b_i(s) ds > 0; \quad (3.14)$$

$$\left\{ \begin{array}{l} \text{there exists } 0 < M_i < 1 \text{ with } M_i \leq \rho_i (K_{2,i} / K_{1,i}) \phi_i(M_0), \text{ where} \\ K_{1,i} = \sup_{t \in [0, \omega]} \int_{t-\tau}^t g_i(t, s) b_i(s) ds \text{ and } M_0 = \min_{1 \leq j \leq n} M_j \in (0, 1). \end{array} \right. \quad (3.15)$$

Moreover, suppose

$$\text{there exists } \alpha > 0 \text{ with } \max_{1 \leq j \leq n} \psi_j(\alpha, \alpha, \dots, \alpha) K_{1,j} < \alpha \quad (3.16)$$

and

$$\left\{ \begin{array}{l} \text{there exists } \beta > 0, \beta \neq \alpha, \text{ so that for each } z \in \{1, 2, \dots, n\}, \text{ there} \\ \text{exists } j_z \in \{1, 2, \dots, n\} \text{ such that for any } u \in [0, \infty)^n, \text{ we have} \\ \psi_{j_z}(u) \geq a_{j_z,k}(u_k) \text{ for each } 1 \leq k \leq n, \text{ where } a_{j_z,k} : [0, \infty) \rightarrow [0, \infty) \\ \text{is continuous, } a_{j_z,k}(x) > 0 \text{ if } x > 0, \text{ and } \beta < \rho_{j_z} a_{j_z,z}(M_0 \beta) K_{2,j_z}. \end{array} \right. \quad (3.17)$$

Then (M1) has at least one constant-sign solution $u \in (A_\omega(\mathbb{R}))^n$ satisfying

(a) $0 < \alpha < |u|_\omega < \beta$ and $\theta_i u_i(t) > M_i \alpha$ for all $t \in \mathbb{R}$ and some $i \in \{1, \dots, n\}$, if $\alpha < \beta$;

(b) $0 < \beta < |u|_\omega < \alpha$ and $\theta_i u_i(t) > M_i \beta$ for all $t \in \mathbb{R}$ and some $i \in \{1, \dots, n\}$, if $\beta < \alpha$.

PROOF. We shall employ Theorem 2.2. Define a cone C in B as

$$\begin{aligned} C &= \{u \in B \mid \theta_i u_i(t) \geq 0 \text{ and } \theta_i u_i(t) \geq M_i |u|_\omega \text{ for } 1 \leq i \leq n, t \in [0, \omega]\} \\ &= \{u \in B \mid \theta_i u_i(t) \geq 0 \text{ and } \theta_i u_i(t) \geq M_i |u|_\omega \text{ for } 1 \leq i \leq n, t \in \mathbb{R}\}, \end{aligned} \quad (3.18)$$

where M_i is defined in (3.15). Note that $C \subseteq \tilde{K}$.

First, we shall show that S maps C into $(A_\omega(\mathbb{R}))^n$, that is, that

$$S_i : C \rightarrow A_\omega(\mathbb{R}), \quad 1 \leq i \leq n. \tag{3.19}$$

Let $u \in C$. Then, using (3.6), (3.7) and (3.10), we get for $t \in \mathbb{R}$ and $1 \leq i \leq n$,

$$\begin{aligned} S_i u(t + \omega) &= \int_{t+\omega-\tau}^{t+\omega} g_i(t + \omega, s) f_i(s, u(s)) ds \\ &= \int_{t-\tau}^t g_i(t + \omega, x + \omega) f_i(x + \omega, u(x + \omega)) dx \\ &= \int_{t-\tau}^t g_i(t + \omega, x + \omega) f_i(x + \omega, u(x)) dx \\ &= \int_{t-\tau}^t g_i(t, x) f_i(x, u(x)) dx = S_i u(t). \end{aligned} \tag{3.20}$$

We shall next show that $S_i u \in C[0, \omega]$, $1 \leq i \leq n$. Since $u \in B \subset (BC(\mathbb{R}))^n$, there exists $r > 0$ with $|u|_\omega \leq r$. Therefore (3.11) guarantees, for each $1 \leq i \leq n$, the existence of $\mu_r^i \in L^p[0, \omega]$, such that $|f_i(s, u(s))| \leq \mu_r^i(s)$, a.e. $s \in [0, \omega]$. This, together with (3.10), implies that there exists $\eta_r^i \in L^p[-\tau, \omega]$, $1 \leq i \leq n$, with

$$|f_i(s, u(s))| \leq \eta_r^i(s), \quad \text{a.e. } s \in [-\tau, \omega]. \tag{3.21}$$

(Alternatively we could assume without loss of generality that $\mu_r^i(t + \omega) = \mu_r^i(t)$ for a.e. $t \in \mathbb{R}$ and take $\eta_r^i = \mu_r^i$.) Now, for $t_1, t_2 \in [0, \omega]$ with $t_2 > t_1$ and $t_2 - \tau < t_1$, we have for each $1 \leq i \leq n$,

$$\begin{aligned} S_i u(t_1) - S_i u(t_2) &= \int_{t_1-\tau}^{t_1} g_i(t_1, s) f_i(s, u(s)) ds - \int_{t_2-\tau}^{t_2} g_i(t_2, s) f_i(s, u(s)) ds \\ &= \int_{t_1-\tau}^{t_2-\tau} g_i(t_1, s) f_i(s, u(s)) ds + \int_{t_2-\tau}^{t_1} [g_i(t_1, s) - g_i(t_2, s)] f_i(s, u(s)) ds \\ &\quad - \int_{t_1}^{t_2} g_i(t_2, s) f_i(s, u(s)) ds \end{aligned}$$

and so

$$\begin{aligned} |S_i u(t_1) - S_i u(t_2)| &\leq \left(\int_{t_1-\tau}^{t_2-\tau} |g_i^{t_1}(s)|^q ds \right)^{1/q} \left(\int_{t_1-\tau}^{t_2-\tau} |f_i(s, u(s))|^p ds \right)^{1/p} \\ &\quad + \left(\int_{t_2-\tau}^{t_1} |g_i^{t_1}(s) - g_i^{t_2}(s)|^q ds \right)^{1/q} \left(\int_{t_2-\tau}^{t_1} |f_i(s, u(s))|^p ds \right)^{1/p} \end{aligned}$$

$$+ \left(\int_{t_1}^{t_2} |g_i^{t_2}(s)|^q ds \right)^{1/q} \left(\int_{t_1}^{t_2} |f_i(s, u(s))|^p ds \right)^{1/p}.$$

Using (3.21), it then follows that

$$\begin{aligned} & |S_i u(t_1) - S_i u(t_2)| \\ & \leq \sup_{t \in [0, \omega]} \left(\int_{-t}^{w-t} |g_i^t(s)|^q ds \right)^{1/q} \left(\int_{t_1-t}^{t_2-t} [\eta_r^i(s)]^p ds \right)^{1/p} \\ & \quad + \left(\int_{-t}^w |g_i^{t_1}(s) - g_i^{t_2}(s)|^q ds \right)^{1/q} \left(\int_{-t}^w [\eta_r^i(s)]^p ds \right)^{1/p} \\ & \quad + \sup_{t \in [0, \omega]} \left(\int_0^w |g_i^t(s)|^q ds \right)^{1/q} \left(\int_{t_1}^{t_2} [\mu_r^i(s)]^p ds \right)^{1/p}. \end{aligned} \tag{3.22}$$

We also note that (3.3) and (3.6) guarantee that

$$g_i^t \in L^q[-\tau, \omega] \quad \text{for } t \in [0, \omega]. \tag{3.23}$$

In view of (3.4) and (3.23), it is clear from (3.22) that $|S_i u(t_1) - S_i u(t_2)| \rightarrow 0$ as $t_1 \rightarrow t_2$. Hence

$$S_i u \in C[0, \omega], \quad 1 \leq i \leq n. \tag{3.24}$$

Further, noting (3.23) and (3.21), we have for $t \in [0, \omega]$ and $1 \leq i \leq n$,

$$|S_i u(t)| \leq \int_{t-\tau}^t |g_i(t, s)| \eta_r^i(s) ds \leq \left(\int_{-\tau}^w |g_i^t(s)|^q ds \right)^{1/q} \left(\int_{-\tau}^w [\eta_r^i(s)]^p ds \right)^{1/p} < \infty.$$

Thus

$$S_i u \text{ is bounded, } 1 \leq i \leq n. \tag{3.25}$$

Having established (3.20), (3.24) and (3.25), we have completed the proof of (3.19).

Next, we shall verify that

$$S : C \rightarrow (A_\omega(\mathbb{R}))^n \text{ is continuous.} \tag{3.26}$$

Let $\{u^m\}$, $u \in C$ ($\{u^m\}$ is a sequence) with $u^m \rightarrow u$. Since $u^m, u \in B \subset (BC(\mathbb{R}))^n$, there exists $r > 0$ with $|u^m|_\omega, |u|_\omega \leq r$. Therefore, as in a previous argument, (3.10) and (3.11) lead to the existence of $\eta_r^i \in L^p[-\tau, \omega]$, $1 \leq i \leq n$, such that (3.21) and (3.21)| $_{u=u^m}$ hold. Hence

$$|f_i(s, u^m(s)) - f_i(s, u(s))| \leq 2\eta_r^i(s), \quad \text{a.e. } s \in [-\tau, \omega] \tag{3.27}$$

and also from (3.8) and (3.10) we have

$$|f_i(s, u^m(s)) - f_i(s, u(s))| \rightarrow 0 \quad \text{pointwise for a.e. } s \in [-\tau, \omega]. \tag{3.28}$$

Now, for $t \in [0, \omega]$ and $1 \leq i \leq n$, we find

$$\begin{aligned} & |S_i u^m(t) - S_i u(t)| \\ & \leq \int_{t-\tau}^t |g_i(t, s)[f_i(s, u^m(s)) - f_i(s, u(s))]| ds \\ & \leq \left(\int_{t-\tau}^t |g_i'(s)|^q ds \right)^{1/q} \left(\int_{t-\tau}^t |f_i(s, u^m(s)) - f_i(s, u(s))|^p ds \right)^{1/p} \\ & \leq \sup_{x \in [0, \omega]} \left(\int_{-\tau}^{\omega} |g_i^x(s)|^q ds \right)^{1/q} \left(\int_{-\tau}^{\omega} |f_i(s, u^m(s)) - f_i(s, u(s))|^p ds \right)^{1/p} \\ & \xrightarrow{m \rightarrow \infty} 0, \end{aligned}$$

where we have used (3.23), (3.27) and (3.28). It follows that

$$\|S u^m - S u\|_{\omega} = \max_{1 \leq i \leq n} \|S_i u^m - S_i u\|_{\omega} = \max_{1 \leq i \leq n} \sup_{t \in [0, \omega]} |S_i u^m(t) - S_i u(t)| \xrightarrow{m \rightarrow \infty} 0.$$

Hence we have proved (3.26).

Next, we shall show that

$$S : C \rightarrow (A_{\omega}(\mathbb{R}))^n \text{ is completely continuous.} \tag{3.29}$$

Let Ω be a bounded set in C , that is, there exists $r > 0$ such that $|u|_{\omega} \leq r$ for all $u \in \Omega$. We shall prove that $S_i \Omega$, $1 \leq i \leq n$, is relatively compact in $A_{\omega}(\mathbb{R})$. Let $\{u^m\}$ be a sequence in Ω . Then, for each $1 \leq i \leq n$, $\{S_i u^m\}$ is a sequence in $S_i \Omega$. Now, using once again a previous argument, (3.10) and (3.11) lead to the existence of $\eta_r^i \in L^p[-\tau, \omega]$, $1 \leq i \leq n$, such that (3.21) $_{|u=u^m}$ holds. Hence, we get for $t \in [0, \omega]$ and $1 \leq i \leq n$,

$$\begin{aligned} |S_i u^m(t)| & \leq \left(\int_{t-\tau}^t |g_i'(s)|^q ds \right)^{1/q} \left(\int_{t-\tau}^t |f_i(s, u^m(s))|^p ds \right)^{1/p} \\ & \leq \sup_{x \in [0, \omega]} \left(\int_{-\tau}^{\omega} |g_i^x(s)|^q ds \right)^{1/q} \left(\int_{-\tau}^{\omega} [\eta_r^i(s)]^p ds \right)^{1/p} < \infty, \end{aligned}$$

where we have also used (3.23). Thus $\{S_i u^m\}$ is a uniformly bounded sequence in $C[0, \omega]$. An argument similar to that in (3.22) guarantees that $\{S_i u^m\}$ is equicontinuous on $[0, \omega]$. The Arzela-Ascoli theorem guarantees an $S_i u \in C[0, \omega]$ and a subsequence $\{S_i u^{m_k}\}$ of $\{S_i u^m\}$ which converges uniformly on $[0, \omega]$ to $S_i u$. Since $S_i u^{m_k}(t + \omega) =$

$S_i u^{m_k}(t)$, by letting $k \rightarrow \infty$ we have $S_i u(t + \omega) = S_i u(t)$. Thus $S_i u \in A_\omega(\mathbb{R})$ and $S_i u^{m_k} \rightarrow S_i u$ in $A_\omega(\mathbb{R})$. So $S_i : C \rightarrow A_\omega(\mathbb{R})$ is completely continuous, $1 \leq i \leq n$. This completes the proof of (3.29).

We also need to show that

$$S : C \rightarrow C. \tag{3.30}$$

Let $u \in C$. We note that for a fixed $t \in [0, \omega]$, any $s \in [t - \tau, t]$ can be written as $s + z\omega = s_\omega$, where z is some integer and $s_\omega \in [0, \omega]$. Therefore, in view of (3.5), (3.6), (3.9), (3.10) and (3.12), we get for $t \in [0, \omega]$, a.e. $s \in [t - \tau, t]$ and $1 \leq i \leq n$,

$$g_i(t, s) = g_i(t, s_\omega) \geq 0, \tag{3.31}$$

$$\theta_i f_i(s, u(s)) = \theta_i f_i(s_\omega, u(s)) \geq 0, \tag{3.32}$$

$$\theta_i f_i(s, u(s)) = \theta_i f_i(s_\omega, u(s)) \leq b_i(s_\omega)\psi_i(u(s)) = b_i(s)\psi_i(u(s)) \tag{3.33}$$

and

$$\theta_i f_i(s, u(s)) = \theta_i f_i(s_\omega, u(s)) \geq \rho_i b_i(s_\omega)\psi_i(u(s)) = \rho_i b_i(s)\psi_i(u(s)), \tag{3.34}$$

where we have also used the fact that $b_i(x + \omega) = b_i(x)$ for a.e. $x \in \mathbb{R}$. Then, noting (3.31) and (3.32), it is clear that

$$\theta_i S_i u(t) = \int_{t-\tau}^t g_i(t, s)\theta_i f_i(s, u(s)) ds \geq 0, \quad t \in [0, \omega], \quad 1 \leq i \leq n. \tag{3.35}$$

Now, using (3.12), (3.31), (3.33) and (3.35), we find for $t \in [0, \omega]$ and $1 \leq i \leq n$,

$$\begin{aligned} |S_i u(t)| &= \theta_i S_i u(t) \leq \int_{t-\tau}^t g_i(t, s)b_i(s)\psi_i(u(s)) ds \\ &\leq \psi_i(|u_1|_\omega, |u_2|_\omega, \dots, |u_n|_\omega) \int_{t-\tau}^t g_i(t, s)b_i(s) ds \\ &\leq \psi_i(|u_1|_\omega, |u_2|_\omega, \dots, |u_n|_\omega) K_{1,i}. \end{aligned}$$

This yields

$$|S_i u|_\omega \leq K_{1,i}\psi_i(|u_1|_\omega, |u_2|_\omega, \dots, |u_n|_\omega), \quad 1 \leq i \leq n. \tag{3.36}$$

On the other hand, since $u \in C$, we have

$$|u_i(t)| = \theta_i u_i(t) \geq M_i |u_i|_\omega \geq M_0 |u_i|_\omega, \quad t \in \mathbb{R}, \quad 1 \leq i \leq n, \tag{3.37}$$

where $M_0 = \min_{1 \leq j \leq n} M_j$. Therefore (3.13)–(3.15), (3.31) and (3.34)–(3.37) give for $t \in [0, \omega]$ and $1 \leq i \leq n$,

$$\theta_i S_i u(t) \geq \rho_i \int_{t-\tau}^t g_i(t, s)b_i(s)\psi_i(u(s)) ds$$

$$\begin{aligned}
 &\geq \rho_i \psi_i(M_0|u_1|_\omega, M_0|u_2|_\omega, \dots, M_0|u_n|_\omega) \int_{t-\tau}^t g_i(t, s)b_i(s) ds \\
 &\geq \rho_i \phi_i(M_0)\psi_i(|u_1|_\omega, |u_2|_\omega, \dots, |u_n|_\omega) \int_{t-\tau}^t g_i(t, s)b_i(s) ds \\
 &\geq \rho_i \phi_i(M_0)\psi_i(|u_1|_\omega, |u_2|_\omega, \dots, |u_n|_\omega)K_{2,i} \\
 &\geq \rho_i \phi_i(M_0)|S_i u|_\omega \frac{K_{2,i}}{K_{1,i}} \geq M_i |S_i u|_\omega.
 \end{aligned}$$

Together with (3.35), we have shown that $Su \in C$ and hence (3.30) is proved.

Let

$$\Omega_\alpha = \{u \in B \mid |u|_\omega < \alpha\} \quad \text{and} \quad \Omega_\beta = \{u \in B \mid |u|_\omega < \beta\}. \tag{3.38}$$

We shall now show that

$$|Su|_\omega < |u|_\omega \quad \text{for } u \in C \cap \partial\Omega_\alpha \tag{3.39}$$

and

$$|Su|_\omega > |u|_\omega \quad \text{for } u \in C \cap \partial\Omega_\beta. \tag{3.40}$$

To verify (3.39), let $u \in C \cap \partial\Omega_\alpha$. Then $|u|_\omega = \alpha$. For $t \in [0, \omega]$, using (3.12), (3.31), (3.33) and (3.35) we find, for $1 \leq i \leq n$,

$$|S_i u(t)| \leq \psi_i(|u|_\omega, |u|_\omega, \dots, |u|_\omega) \int_{t-\tau}^t g_i(t, s)b_i(s) ds \leq \psi_i(\alpha, \alpha, \dots, \alpha)K_{1,i},$$

which, together with (3.16), leads to

$$|Su|_\omega = \max_{1 \leq i \leq n} |S_i u|_\omega \leq \max_{1 \leq i \leq n} \psi_i(\alpha, \alpha, \dots, \alpha)K_{1,i} < \alpha = |u|_\omega. \tag{3.41}$$

Thus (3.39) is proved.

Next, let $u \in C \cap \partial\Omega_\beta$. Then $|u|_\omega = \beta = |u_z|_\omega$ for some $z \in \{1, 2, \dots, n\}$. For $t \in [0, \omega]$, using (3.17), (3.31), (3.34), (3.35) and (3.37), we get for some $j_z \in \{1, 2, \dots, n\}$,

$$\begin{aligned}
 |S_{j_z} u(t)| &\geq \rho_{j_z} \int_{t-\tau}^t g_{j_z}(t, s)b_{j_z}(s)\psi_{j_z}(u(s)) ds \\
 &\geq \rho_{j_z} \psi_{j_z}(M_0|u_1|_\omega, M_0|u_2|_\omega, \dots, M_0|u_n|_\omega) \int_{t-\tau}^t g_{j_z}(t, s)b_{j_z}(s) ds \\
 &\geq \rho_{j_z} a_{j_z,z}(M_0|u_z|_\omega)K_{2,j_z} = \rho_{j_z} a_{j_z,z}(M_0\beta)K_{2,j_z} > \beta = |u|_\omega.
 \end{aligned}$$

Thus

$$|Su|_\omega \geq |S_{j_z} u|_\omega > |u|_\omega \tag{3.42}$$

and (3.40) is true.

Now Theorem 2.2 guarantees that S has a fixed point u with $u \in C \cap (\overline{\Omega_\alpha} \setminus \Omega_\beta)$ if $\beta < \alpha$, whereas $u \in C \cap (\overline{\Omega_\beta} \setminus \Omega_\alpha)$ if $\alpha < \beta$. Hence, equivalently (M1) has a constant-sign solution $u \in (A_\omega(\mathbb{R}))^n$ with $\min\{\alpha, \beta\} \leq |u|_\omega \leq \max\{\alpha, \beta\}$. Note that $|u|_\omega \neq \alpha$ and $|u|_\omega \neq \beta$. To see this, suppose $|u|_\omega = \alpha$ or $|u|_\omega = \beta$. Then, since $u = Su$ we have, noting (3.41) and (3.42),

$$\alpha = |u|_\omega = |Su|_\omega \leq \max_{1 \leq i \leq n} \psi_i(\alpha, \alpha, \dots, \alpha) K_{1,i} < \alpha = |u|_\omega$$

or $\beta = |u|_\omega = |Su|_\omega \geq |S_{j_i} u|_\omega > |u|_\omega = \beta$, which are contradictions. Hence

$$\min\{\alpha, \beta\} < |u|_\omega < \max\{\alpha, \beta\}.$$

Finally, $|u|_\omega = |u_i|_\omega$ for some $i \in \{1, 2, \dots, n\}$. Since $u \in C$, we have, for $t \in \mathbb{R}$, $\theta_i u_i(t) \geq M_i |u_i|_\omega = M_i |u|_\omega > M_i \min\{\alpha, \beta\}$. □

REMARK 3.2. In Theorem 3.1, it is possible to replace (3.3), (3.4), (3.6)–(3.8) and (3.11) with the following:

$$g'_i(s) \equiv g_i(t, s) \in L^1[0, \omega] \text{ for each } t \in [0, \omega]; \tag{3.43}$$

$$\text{the map } t \mapsto g'_i \text{ is continuous from } [0, \omega] \text{ to } L^1[0, \omega]; \tag{3.44}$$

$$\begin{cases} g_i(t, s) \geq 0 & \text{for all } t \in [0, \omega] \text{ and a.e. } s \in [t - \tau, t], \\ g_i(t + \omega, s + \omega) = g_i(t, s) & \text{for all } t \in \mathbb{R} \text{ and a.e. } s \in \mathbb{R}; \end{cases} \tag{3.45}$$

$$f_i : [0, \omega] \times \mathbb{R}^n \rightarrow \mathbb{R} \text{ is continuous.} \tag{3.46}$$

As in [7, Section 3], we need only notice that (3.43)–(3.46) imply that $S : C \rightarrow C$ is continuous and completely continuous.

Our next result gives the existence of *multiple* constant-sign periodic solutions.

THEOREM 3.3. *Let $1 \leq p \leq \infty, q$ be such that $1/p + 1/q = 1, 0 < \omega < \infty$ and let $\theta_i \in \{1, -1\}, 1 \leq i \leq n$, be fixed. Assume that (3.3)–(3.15) hold for each $1 \leq i \leq n$. Let (3.16) be satisfied for $\alpha = \alpha_\ell, \ell = 1, \dots, k$, and (3.17) be satisfied for $\beta = \beta_\ell, \ell = 1, \dots, m$.*

(a) *If $m = k + 1$ and $0 < \beta_1 < \alpha_1 < \dots < \beta_k < \alpha_k < \beta_{k+1}$, then (M1) has (at least) $2k$ constant-sign solutions $u^1, \dots, u^{2k} \in (A_\omega(\mathbb{R}))^n$ such that*

$$0 < \beta_1 < |u^1|_\omega < \alpha_1 < |u^2|_\omega < \beta_2 < \dots < \alpha_k < |u^{2k}|_\omega < \beta_{k+1}.$$

(b) *If $m = k$ and $0 < \beta_1 < \alpha_1 < \dots < \beta_k < \alpha_k$, then (M1) has (at least) $2k - 1$ constant-sign solutions $u^1, \dots, u^{2k-1} \in (A_\omega(\mathbb{R}))^n$ such that*

$$0 < \beta_1 < |u^1|_\omega < \alpha_1 < |u^2|_\omega < \beta_2 < \dots < \beta_k < |u^{2k-1}|_\omega < \alpha_k.$$

(c) If $k = m + 1$ and $0 < \alpha_1 < \beta_1 < \dots < \alpha_m < \beta_m < \alpha_{m+1}$, then (M1) has (at least) $2m$ constant-sign solutions $u^1, \dots, u^{2m} \in (A_\omega(\mathbb{R}))^n$ such that

$$0 < \alpha_1 < |u^1|_\omega < \beta_1 < |u^2|_\omega < \alpha_2 < \dots < \beta_m < |u^{2m}|_\omega < \alpha_{m+1}.$$

(d) If $k = m$ and $0 < \alpha_1 < \beta_1 < \dots < \alpha_k < \beta_k$, then (M1) has (at least) $2k - 1$ constant-sign solutions $u^1, \dots, u^{2k-1} \in (A_\omega(\mathbb{R}))^n$ such that

$$0 < \alpha_1 < |u^1|_\omega < \beta_1 < |u^2|_\omega < \alpha_2 < \dots < \alpha_k < |u^{2k-1}|_\omega < \beta_k.$$

EXAMPLE 1. Consider the nonlinear system of integral equations

$$u_i(t) = \int_{t-\tau}^t g_i(t, s)\theta_i h_i(s) [|u_1(s)|^\gamma + |u_2(s)|^\gamma + \dots + |u_n(s)|^\gamma] ds, \tag{3.47}$$

$t \in \mathbb{R}$, $1 \leq i \leq n$, where $0 < \gamma < 1$ and $\theta_i \in \{-1, 1\}$, $1 \leq i \leq n$, are fixed. For each $1 \leq i \leq n$, assume (3.3)–(3.7) hold,

$$\begin{cases} \text{there exists } \omega > 0 \text{ with } h_i(t + \omega) = h_i(t) \text{ for a.e. } t \in \mathbb{R}, \\ h_i \text{ is nonnegative and } h_i \in L^p[0, \omega] \end{cases} \tag{3.48}$$

and

$$\inf_{t \in [0, \omega]} \int_{t-\tau}^t g_i(t, s)h_i(s) ds > 0. \tag{3.49}$$

Then (3.47) has at least one constant-sign solution $u \in (A_\omega(\mathbb{R}))^n$ such that

$$\begin{cases} 0 < \min\{\alpha, \beta\} < |u|_\omega < \max\{\alpha, \beta\} \text{ and} \\ \theta_k u_k(t) > M_k \min\{\alpha, \beta\} \text{ for all } t \in \mathbb{R} \text{ and some } k \in \{1, \dots, n\}, \end{cases} \tag{3.50}$$

where

$$M_i = \frac{K_{2,i}}{K_{1,i}} \left(\min_{1 \leq j \leq n} \frac{K_{2,j}}{K_{1,j}} \right)^{\gamma/(1-\gamma)} \in (0, 1), \quad 1 \leq i \leq n \tag{3.51}$$

with

$$K_{2,i} = \inf_{t \in [0, \omega]} \int_{t-\tau}^t g_i(t, s)h_i(s) ds, \quad K_{1,i} = \sup_{t \in [0, \omega]} \int_{t-\tau}^t g_i(t, s)h_i(s) ds \tag{3.52}$$

and α, β are positive numbers satisfying

$$\alpha > \left(n \max_{1 \leq j \leq n} K_{1,j} \right)^{1/(1-\gamma)} \quad \text{and} \quad \beta < M_0^{\gamma/(1-\gamma)} \left(\min_{1 \leq j \leq n} K_{2,j} \right)^{1/(1-\gamma)}, \tag{3.53}$$

where $M_0 = \min_{1 \leq j \leq n} M_j \in (0, 1)$.

To see that the above is true, we shall apply Theorem 3.1 with

$$\begin{cases} f_i(t, u) = \theta_i h_i(t) [|u_1|^\gamma + |u_2|^\gamma + \dots + |u_n|^\gamma], \\ \rho_i = 1, \quad b_i = h_i, \quad \psi_i(u) = |u_1|^\gamma + |u_2|^\gamma + \dots + |u_n|^\gamma, \\ \phi_i(m) = m^\gamma \quad \text{and} \quad a_{j,k}(x) = x^\gamma, \quad 1 \leq i, j, k \leq n. \end{cases} \tag{3.54}$$

Note that (3.8)–(3.14) are clearly satisfied. Next, the inequality in condition (3.15) is reduced to $M_i \leq (K_{2,i}/K_{1,i})M_0^\gamma$, $1 \leq i \leq n$, which will be satisfied if we set

$$M_i = \frac{K_{2,i}}{K_{1,i}} M_0^\gamma, \quad 1 \leq i \leq n. \tag{3.55}$$

It follows immediately that $M_0 = \min_{1 \leq j \leq n} M_j = \min_{1 \leq j \leq n} (K_{2,j}/K_{1,j})M_0^\gamma$ or

$$M_0 = \left(\min_{1 \leq j \leq n} \frac{K_{2,j}}{K_{1,j}} \right)^{1/(1-\gamma)}. \tag{3.56}$$

Substituting (3.56) in (3.55) yields (3.51).

Further, (3.16) holds since noting (3.54) we find

$$\max_{1 \leq j \leq n} \psi_j(\alpha, \alpha, \dots, \alpha) K_{1,j} = n\alpha^\gamma \max_{1 \leq j \leq n} K_{1,j} < \alpha^\gamma \alpha^{1-\gamma} = \alpha,$$

where we have also used (3.53).

Finally, (3.17) is fulfilled since in view of (3.53) and (3.54) we have

$$\begin{aligned} \rho_j a_{j,z}(M_0\beta) K_{2,j} &= (M_0\beta)^\gamma K_{2,j} \geq (M_0\beta)^\gamma \min_{1 \leq j \leq n} K_{2,j} \\ &> (M_0\beta)^\gamma \beta^{1-\gamma} M_0^{-\gamma} = \beta. \end{aligned}$$

We now conclude from Theorem 3.1 that the system (3.47) has at least one constant-sign solution $u \in (A_\omega(\mathbb{R}))^n$ satisfying (3.50)–(3.53).

The next result uses the nonlinear alternative (Theorem 2.3) to show the existence of a periodic solution (which need *not* be of constant sign).

THEOREM 3.4. *Let $1 \leq p \leq \infty$, q be such that $1/p + 1/q = 1$ and $0 < \omega < \infty$. For each $1 \leq i \leq n$, assume that (3.3), (3.4), (3.6)–(3.8) and (3.11) hold and*

$$f_i(t + \omega, u) = f_i(t, u) \quad \text{for a.e. } t \in \mathbb{R} \text{ and } u \in \mathbb{R}^n. \tag{3.57}$$

Suppose there exists a constant c , independent of λ , such that

$$|u|_\omega \neq c \tag{3.58}$$

for any solution $u \in (A_\omega(\mathbb{R}))^n$ of the system

$$u_i(t) = \lambda \int_{t-\tau}^t g_i(t, s) f_i(s, u(s)) ds, \quad t \in \mathbb{R}, \quad 1 \leq i \leq n, \tag{3.59}_\lambda$$

where $\lambda \in (0, 1)$. Then, (M1) has at least one solution $u \in (A_\omega(\mathbb{R}))^n$ with $|u|_\omega \leq c$.

PROOF. We shall apply Theorem 2.3. Let $E = B = ((A_\omega(\mathbb{R}))^n, |\cdot|_\omega)$ and $U = \{u \in B \mid |u|_\omega < c\}$. Clearly, a solution of (3.59) $_\lambda$ is a fixed point of the equation $u = \lambda Su$. As seen in the proof of Theorem 3.1, (3.3), (3.4), (3.6)–(3.8), (3.11) and (3.57) guarantee that $S : B \rightarrow B$ is continuous and completely continuous. In view of (3.58), we cannot have conclusion (b) of Theorem 2.3, hence conclusion (a) of Theorem 2.3 must hold, that is, S has a fixed point in \bar{U} , or equivalently the system (M1) has a solution $u \in \bar{U}$ with $|u|_\omega \leq c$. \square

Using Theorem 3.4, we shall obtain the existence of a constant-sign periodic solution in the next result.

THEOREM 3.5. Let $1 \leq p \leq \infty, q$ be such that $1/p + 1/q = 1, 0 < \omega < \infty$ and let $\theta_i \in \{1, -1\}, 1 \leq i \leq n$, be fixed. For each $1 \leq i \leq n$, assume that (3.3)–(3.9), (3.11) and (3.57) hold and

$$\left\{ \begin{array}{l} \text{there exists a function } \psi_i : \prod_{j=1}^n [0, \infty)_j \rightarrow [0, \infty) \text{ continu-} \\ \text{ous and 'nondecreasing' in the sense that for each } 1 \leq j \leq \\ n, \text{ if } |u_j| \leq |v_j|, \text{ then } \psi_i(u_1, \dots, u_{j-1}, u_j, u_{j+1}, \dots, u_n) \leq \\ \psi_i(u_1, \dots, u_{j-1}, v_j, u_{j+1}, \dots, u_n), \text{ and a function } b_i : \mathbb{R} \rightarrow \\ [0, \infty), b_i \in L^p[0, \omega], b_i(t + \omega) = b_i(t) \text{ for a.e. } t \in \mathbb{R}, \text{ with} \\ \theta_i f_i(t, u) \leq b_i(t) \psi_i(u) \text{ for a.e. } t \in [0, \omega] \text{ and all } u \in \bar{K}. \end{array} \right. \tag{3.60}$$

Moreover, suppose

$$\left\{ \begin{array}{l} \text{there exists } \alpha > 0 \text{ with } \max_{1 \leq j \leq n} \psi_j(\alpha, \alpha, \dots, \alpha) K_{1,j} < \alpha, \text{ where} \\ K_{1,j} = \sup_{t \in [0, \omega]} \int_{t-\tau}^t g_j(t, s) b_j(s) ds. \end{array} \right. \tag{3.61}$$

Then, (M1) has at least one constant-sign solution $u \in (A_\omega(\mathbb{R}))^n$ with $|u|_\omega < \alpha$.

PROOF. We shall employ Theorem 3.4. To begin, we consider the system

$$u_i(t) = \int_{t-\tau}^t g_i(t, s) \hat{f}_i(s, u(s)) ds, \quad t \in \mathbb{R}, \quad 1 \leq i \leq n, \tag{3.62}$$

where $\hat{f}_i : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is defined by

$$\hat{f}_i(t, u_1, u_2, \dots, u_n) = f_i(t, \theta_1|u_1|, \theta_2|u_2|, \dots, \theta_n|u_n|), \quad 1 \leq i \leq n. \tag{3.63}$$

We shall prove that (3.62) has a solution. For this, we consider the system

$$u_i(t) = \lambda \int_{t-\tau}^t g_i(t, s) \hat{f}_i(s, u(s)) ds, \quad t \in \mathbb{R}, \quad 1 \leq i \leq n, \tag{3.64}_\lambda$$

where $\lambda \in (0, 1)$. Let $u \in (A_\omega(\mathbb{R}))^n$ be any solution of (3.64) $_\lambda$. If we can show that

$$|u|_\omega \neq \alpha, \tag{3.65}$$

then by Theorem 3.4 it follows that (3.62) has a solution.

Now, using (3.5), (3.9), the fact that $(\theta_1|u_1|, \theta_2|u_2|, \dots, \theta_n|u_n|) \in \tilde{K}$, (3.31), (3.32) and (3.63) we get for $t \in [0, \omega]$ and $1 \leq i \leq n$,

$$\begin{aligned} \theta_i u_i(t) &= \lambda \int_{t-\tau}^t g_i(t, s) \theta_i \hat{f}_i(s, u_1(s), u_2(s), \dots, u_n(s)) ds \\ &= \lambda \int_{t-\tau}^t g_i(t, s) \theta_i f_i(s, \theta_1|u_1(s)|, \theta_2|u_2(s)|, \dots, \theta_n|u_n(s)|) ds \geq 0, \end{aligned}$$

which means that

$$|u_i(t)| = \theta_i u_i(t), \quad t \in [0, \omega], \quad 1 \leq i \leq n. \tag{3.66}$$

An application of (3.33),(3.60) and (3.66) yields for $t \in [0, \omega]$ and $1 \leq i \leq n$,

$$\begin{aligned} |u_i(t)| = \theta_i u_i(t) &\leq \int_{t-\tau}^t g_i(t, s) \theta_i f_i(s, \theta_1|u_1(s)|, \dots, \theta_n|u_n(s)|) ds \\ &\leq \int_{t-\tau}^t g_i(t, s) b_i(s) \psi_i(\theta_1|u_1(s)|, \dots, \theta_n|u_n(s)|) ds \\ &\leq \psi_i(|u|_\omega, \dots, |u|_\omega) \int_{t-\tau}^t g_i(t, s) b_i(s) ds \\ &\leq \psi_i(|u|_\omega, \dots, |u|_\omega) K_{1,i}. \end{aligned}$$

This immediately leads to

$$|u_i|_\omega \leq \psi_i(|u|_\omega, \dots, |u|_\omega) K_{1,i}, \quad 1 \leq i \leq n. \tag{3.67}$$

Now $|u|_\omega = |u_z|_\omega$ for some $z \in \{1, 2, \dots, n\}$. Then it follows from (3.67) that

$$|u|_\omega \leq \psi_z(|u|_\omega, \dots, |u|_\omega) K_{1,z}. \tag{3.68}$$

Noting (3.61) and (3.68), we conclude that $|u|_\omega \neq \alpha$. Hence (3.65) is proved.

It now follows from Theorem 3.4 that the system (3.62) has a solution $u^* = (u_1^*, u_2^*, \dots, u_n^*) \in (A_\omega(\mathbb{R}))^n$ with $\|u^*\| \leq \alpha$, and

$$u_i^*(t) = \int_{t-\tau}^t g_i(t, s) \hat{f}_i(s, u^*(s)) ds, \quad t \in \mathbb{R}, \quad 1 \leq i \leq n.$$

Using a similar argument as above, together with $u^* \in (A_\omega(\mathbb{R}))^n$, it can be easily seen that

$$|u_i^*(t)| = \theta_i u_i^*(t), \quad t \in \mathbb{R}, \quad 1 \leq i \leq n, \tag{3.69}$$

and

$$|u^*|_\omega \neq \alpha. \tag{3.70}$$

Therefore u^* is of constant sign and $|u^*|_\omega < \alpha$. Further, using (3.63) and (3.69), we have for $t \in \mathbb{R}$ and $1 \leq i \leq n$,

$$\begin{aligned} u_i^*(t) &= \int_{t-\tau}^t g_i(t, s) \hat{f}_i(s, u^*(s)) ds = \int_{t-\tau}^t g_i(t, s) f_i(s, \theta_1 |u_1^*(s)|, \dots, \theta_n |u_n^*(s)|) ds \\ &= \int_{t-\tau}^t g_i(t, s) f_i(s, \theta_1^2 u_1^*(s), \dots, \theta_n^2 u_n^*(s)) ds \\ &= \int_{t-\tau}^t g_i(t, s) f_i(s, u_1^*(s), \dots, u_n^*(s)) ds. \end{aligned}$$

Hence u^* is in fact a solution of (M1). The proof is now complete. □

In Theorem 3.5, it is possible for $|u|_\omega$ to be zero. However, we can combine Theorem 3.5 with Theorem 3.1 to obtain the existence of multiple *nontrivial* constant-sign periodic solutions, stated as the next result.

THEOREM 3.6. *Let $1 \leq p \leq \infty, q$ be such that $1/p + 1/q = 1, 0 < \omega < \infty$ and let $\theta_i \in \{1, -1\}, 1 \leq i \leq n$, be fixed. Assume that (3.3)–(3.15) and (3.57) hold for each $1 \leq i \leq n$. Let (3.16) be satisfied for $\alpha = \alpha_\ell, \ell = 1, 2, \dots, k$, and (3.17) be satisfied for $\beta = \beta_\ell, \ell = 1, 2, \dots, m$.*

(a) *If $m = k + 1$ and $0 < \beta_1 < \alpha_1 < \dots < \beta_k < \alpha_k < \beta_{k+1}$, then (M1) has (at least) $2k$ constant-sign solutions $u^1, \dots, u^{2k} \in (A_\omega(\mathbb{R}))^n$ such that*

$$0 < \beta_1 < |u^1|_\omega < \alpha_1 < |u^2|_\omega < \beta_2 < \dots < \alpha_k < |u^{2k}|_\omega < \beta_{k+1}.$$

(b) *If $m = k$ and $0 < \beta_1 < \alpha_1 < \dots < \beta_k < \alpha_k$, then (M1) has (at least) $2k - 1$ constant-sign solutions $u^1, \dots, u^{2k-1} \in (A_\omega(\mathbb{R}))^n$ such that*

$$0 < \beta_1 < |u^1|_\omega < \alpha_1 < |u^2|_\omega < \beta_2 < \dots < \beta_k < |u^{2k-1}|_\omega < \alpha_k.$$

(c) *If $k = m + 1$ and $0 < \alpha_1 < \beta_1 < \dots < \alpha_m < \beta_m < \alpha_{m+1}$, then (M1) has (at least) $2m + 1$ constant-sign solutions $u^0, \dots, u^{2m} \in (A_\omega(\mathbb{R}))^n$ such that*

$$0 \leq |u^0|_\omega < \alpha_1 < |u^1|_\omega < \beta_1 < |u^2|_\omega < \alpha_2 < \dots < \beta_m < |u^{2m}|_\omega < \alpha_{m+1}.$$

(d) If $k = m$ and $0 < \alpha_1 < \beta_1 < \dots < \alpha_k < \beta_k$, then (M1) has (at least) $2k$ constant-sign solutions $u^0, \dots, u^{2k-1} \in (A_\omega(\mathbb{R}))^n$ such that

$$0 \leq |u^0|_\omega < \alpha_1 < |u^1|_\omega < \beta_1 < |u^2|_\omega < \alpha_2 < \dots < \alpha_k < |u^{2k-1}|_\omega < \beta_k.$$

PROOF. In (a) and (b), we just apply Theorem 3.1 repeatedly. In (c) and (d), Theorem 3.5 is used to obtain the existence of $u^0 \in (A_\omega(\mathbb{R}))^n$ with $0 \leq \|u^0\| < \alpha_1$; the results then follow by repeated use of Theorem 3.1. \square

Our next result makes use of Leggett and Williams' fixed point theorem (Theorem 2.4).

THEOREM 3.7. Let $1 \leq p \leq \infty$, q be such that $1/p + 1/q = 1$, $0 < \omega < \infty$ and let $\theta_i \in \{1, -1\}$, $1 \leq i \leq n$, be fixed. Assume that (3.3)–(3.11) and (3.60) hold for each $1 \leq i \leq n$. Moreover, suppose

$$\exists j \in \{1, \dots, n\} \text{ such that } K_{2,j} = \inf_{t \in [0, \omega]} \int_{t-\tau}^t g_j(t, s) b_j(s) ds > 0; \tag{3.71}$$

$$\left\{ \begin{array}{l} \text{for the same } j \text{ as in (3.71), there exists } r > 0 \text{ with } r < \\ \phi_j(r, \dots, r) K_{2,j}, \text{ where } \phi_j : \prod_{i=1}^n [0, \infty)_i \rightarrow [0, \infty) \text{ is continu-} \\ \text{ous, } \phi_j(u_1, \dots, u_n)/|u_j| \text{ is 'nonincreasing' in the sense that for each} \\ 1 \leq k \leq n, \text{ if } 0 < |u_k| \leq |v_k| \leq r, \text{ then } \phi_j(u_1, \dots, u_n)/|u_j| \geq \\ \phi_j(v_1, \dots, v_n)/|v_j|, \text{ and } b_j(t)\phi_j(u) \leq \theta_j f_j(t, u) \text{ for a.e. } t \in [0, \omega] \\ \text{and all } u \in K \end{array} \right. \tag{3.72}$$

and

$$\left\{ \begin{array}{l} \text{there exists } R (\neq r) \text{ with } R > \max_{1 \leq i \leq n} \psi_i(R, \dots, R) K_{1,i}, \text{ where} \\ K_{1,i} = \sup_{t \in [0, \omega]} \int_{t-\tau}^t g_i(t, s) b_i(s) ds. \end{array} \right. \tag{3.73}$$

Then (M1) has at least one constant-sign solution $u \in (A_\omega(\mathbb{R}))^n$ with

$$\min\{r, R\} \leq |u|_\omega \leq \max\{r, R\} \quad \text{and} \quad |u|_\omega \neq R. \tag{3.74}$$

PROOF. Let $B = ((A_\omega(\mathbb{R}))^n, |\cdot|_\omega)$ and

$$\begin{aligned} C &= \{u \in B \mid \text{for each } 1 \leq i \leq n, \theta_i u_i(t) \geq 0 \text{ for } t \in [0, \omega]\} \\ &= \{u \in B \mid \text{for each } 1 \leq i \leq n, \theta_i u_i(t) \geq 0 \text{ for } t \in \mathbb{R}\}. \end{aligned}$$

Also, let $u_0 \equiv (\theta_1, \dots, \theta_n)$. Then

$$\begin{aligned} C(u_0) &= \{u \in C \mid \text{there exists } \lambda > 0 \text{ with } u(t) - \lambda u_0 \in C \text{ for } t \in [0, \omega]\} \\ &= \{u \in C \mid \text{for each } 1 \leq i \leq n, \theta_i u_i(t) > 0 \text{ for } t \in [0, \omega]\}. \end{aligned}$$

As seen in the proof of Theorem 3.1, conditions (3.3)–(3.11) guarantee that $S : C \rightarrow C$ is continuous and completely continuous. To apply Theorem 2.4, we shall first show that

$$|Su|_\omega \leq |u|_\omega \text{ for } u \in \partial \bar{C}_R. \tag{3.75}$$

Let $u \in \partial \bar{C}_R$. Then $|u|_\omega = R$. Using (3.33) and (3.60), we find for $t \in [0, \omega]$ and $1 \leq i \leq n$,

$$\begin{aligned} |S_i u(t)| &= \theta_i S_i u(t) \leq \int_{t-\tau}^t g_i(t, s) b_i(s) \psi_i(u(s)) ds \\ &\leq \psi_i(|u_1|_\omega, \dots, |u_n|_\omega) \int_{t-\tau}^t g_i(t, s) b_i(s) ds \leq \psi_i(R, \dots, R) K_{1,i}. \end{aligned}$$

This yields, together with (3.73),

$$|Su|_\omega = \max_{1 \leq i \leq n} |S_i u|_\omega \leq \max_{1 \leq i \leq n} \psi_i(R, \dots, R) K_{1,i} < R = |u|_\omega \tag{3.76}$$

and hence (3.75) is proved.

Next, we shall verify that

$$Su \not\leq u, \text{ that is, } u - Su \notin C, \text{ for } u \in \partial \bar{C}_r \cap C(u_0). \tag{3.77}$$

Let $u \in \partial \bar{C}_r \cap C(u_0)$. Then

$$|u|_\omega = r \text{ and } r \geq \theta_i u_i(t) > 0, \quad t \in [0, \omega], \quad 1 \leq i \leq n. \tag{3.78}$$

By a similar argument as in getting (3.31)–(3.34) from (3.72) we have the following for $u \in C, t \in [0, \omega]$, a.e. $s \in [t - \tau, t]$ and some $j \in \{1, 2, \dots, n\}$ (the same j as in (3.71)):

$$\theta_j f_j(s, u(s)) = \theta_j f_j(s_\omega, u(s)) \geq b_j(s_\omega) \phi_j(u(s)) = b_j(s) \phi_j(u(s)). \tag{3.79}$$

Thus, for $t \in [0, \omega]$ and the same j as in (3.71), using (3.78) and (3.79) we get

$$\begin{aligned} \theta_j S_j u(t) &\geq \int_{t-\tau}^t g_j(t, s) b_j(s) \phi_j(u(s)), ds \\ &= \int_{t-\tau}^t g_j(t, s) b_j(s) \frac{\phi_j(u(s))}{\theta_j u_j(s)} \theta_j u_j(s) ds \\ &\geq \frac{\phi_j(r, \dots, r)}{r} \int_{t-\tau}^t g_j(t, s) b_j(s) \theta_j u_j(s) ds. \end{aligned} \tag{3.80}$$

Let $t_{0,j} \in [0, \omega]$ be such that $\inf_{t \in [0, \omega]} \theta_j u_j(t) = \theta_j u_j(t_{0,j}) > 0$. Then it follows from (3.72) and (3.80) that for $t \in [0, \omega]$,

$$\begin{aligned} \theta_j S_j u(t) &\geq \frac{\phi_j(r, \dots, r)}{r} \theta_j u_j(t_{0,j}) \int_{t-\tau}^t g_j(t, s) b_j(s) ds \\ &\geq \left[\frac{\phi_j(r, \dots, r)}{r} K_{2,j} \right] \theta_j u_j(t_{0,j}) > \theta_j u_j(t_{0,j}). \end{aligned}$$

Thus in particular we have $\theta_j S_j u(t_{0,j}) > \theta_j u_j(t_{0,j})$, and so (3.77) is proved.

It now follows from Theorem 2.4 that system (M1) has a constant-sign solution $u \in (A_\omega(\mathbb{R}))^n$ with $\min\{r, R\} \leq |u|_\omega \leq \max\{r, R\}$. Note that $|u|_\omega \neq R$. In fact, if $|u|_\omega = R$, then from (3.76) we have $|u|_\omega = |Su|_\omega < R = |u|_\omega$ which is a contradiction. This completes the proof. \square

REMARK 3.8. If the inequality in (3.73) is changed to

$$R \geq \max_{1 \leq i \leq n} \psi_i(R, \dots, R) K_{1,i},$$

then (3.74) is correspondingly changed to $\min\{r, R\} \leq |u|_\omega \leq \max\{r, R\}$.

REMARK 3.9. Theorem 3.7 improves the results obtained in [9].

A repeated application of Theorem 3.7 yields the existence of *multiple* solutions as follows.

THEOREM 3.10. *Let $1 \leq p \leq \infty$, q be such that $1/p + 1/q = 1$, $0 < \omega < \infty$ and let $\theta_i \in \{1, -1\}$, $1 \leq i \leq n$, be fixed. Assume that (3.3)–(3.11) and (3.60) hold for each $1 \leq i \leq n$, and (3.71) holds. Let (3.72) be satisfied for $r = r_\ell$, $\ell = 1, 2, \dots, k$, and (3.73) be satisfied for $R = R_\ell$, $\ell = 1, 2, \dots, m$.*

(a) *If $m = k + 1$ and $0 < R_1 < r_1 < \dots < R_k < r_k < R_{k+1}$, then (M1) has (at least) $2k$ constant-sign solutions $u^1, \dots, u^{2k} \in (A_\omega(\mathbb{R}))^n$ such that*

$$0 < R_1 < |u^1|_\omega \leq r_1 \leq |u^2|_\omega < R_2 < \dots \leq r_k \leq |u^{2k}|_\omega < R_{k+1}.$$

(b) *If $m = k$ and $0 < R_1 < r_1 < \dots < R_k < r_k$, then (M1) has (at least) $2k - 1$ constant-sign solutions $u^1, \dots, u^{2k-1} \in (A_\omega(\mathbb{R}))^n$ such that*

$$0 < R_1 < |u^1|_\omega \leq r_1 \leq |u^2|_\omega < R_2 < \dots < R_k < |u^{2k-1}|_\omega \leq r_k.$$

(c) *If $k = m + 1$ and $0 < r_1 < R_1 < \dots < r_m < R_m < r_{m+1}$, then (M1) has (at least) $2m$ constant-sign solutions $u^1, \dots, u^{2m} \in (A_\omega(\mathbb{R}))^n$ such that*

$$0 < r_1 \leq |u^1|_\omega < R_1 < |u^2|_\omega \leq r_2 \leq \dots < R_m < |u^{2m}|_\omega \leq r_{m+1}.$$

(d) If $k = m$ and $0 < r_1 < R_1 < \dots < r_k < R_k$, then (M1) has (at least) $2k - 1$ constant-sign solutions $u^1, \dots, u^{2k-1} \in (A_\omega(\mathbb{R}))^n$ such that

$$0 < r_1 \leq |u^1|_\omega < R_1 < |u^2|_\omega \leq r_2 \leq \dots \leq r_k \leq |u^{2k-1}|_\omega < R_k.$$

EXAMPLE 2. Consider the nonlinear system of integral equations

$$u_i(t) = \int_{t-\tau}^t g_i(t, s)\theta_i h_i(s) [|u_1(s)|^\gamma + \dots + |u_n(s)|^\gamma + |u_j(s)|^\delta] ds, \tag{3.81}$$

$t \in \mathbb{R}$, $1 \leq i \leq n$; where $\gamma > 0$, $0 < \delta < 1$, $j \in \{1, 2, \dots, n\}$ and $\theta_i \in \{-1, 1\}$, $1 \leq i \leq n$, are fixed. For each $1 \leq i \leq n$, assume (3.3)–(3.7) and (3.48) hold, and also

$$\inf_{t \in [0, \omega]} \int_{t-\tau}^t g_j(t, s)h_j(s) ds > 0. \tag{3.82}$$

Then (3.81) has at least one constant-sign solution $u \in (A_\omega(\mathbb{R}))^n$ such that

$$\min\{r, R\} \leq |u|_\omega \leq \max\{r, R\} \quad \text{and} \quad |u|_\omega \neq R, \tag{3.83}$$

where r and R are positive numbers satisfying

$$r < \left\{ \inf_{t \in [0, \omega]} \int_{t-\tau}^t g_j(t, s)h_j(s) ds \right\}^{1/(1-\delta)} \tag{3.84}$$

and

$$nR^{\gamma-1} + R^{\delta-1} < \left(\max_{1 \leq i \leq n} K_{1,i} \right)^{-1} \tag{3.85}$$

with

$$K_{1,i} = \sup_{t \in [0, \omega]} \int_{t-\tau}^t g_i(t, s)h_i(s) ds, \quad 1 \leq i \leq n. \tag{3.86}$$

To see that the above is true, we shall apply Theorem 3.7 with

$$\begin{cases} f_i(t, u) = \theta_i h_i(t) [|u_1|^\gamma + \dots + |u_n|^\gamma + |u_j|^\delta], \\ b_i = h_i, \quad \psi_i(u) = |u_1|^\gamma + \dots + |u_n|^\gamma + |u_j|^\delta, \quad 1 \leq i \leq n \\ \phi_j(u) = |u_j|^\delta. \end{cases} \tag{3.87}$$

Note that (3.8)–(3.11) and (3.60) are clearly satisfied. Next, in view of (3.82) and (3.87), condition (3.71) is satisfied with the fixed j in (3.81). For this fixed j , it is obvious that $\phi_j(u)/|u_j| = 1/|u_j|^{1-\delta}$ is nonincreasing. Moreover, the inequality $r < \phi_j(r, \dots, r)K_{2,j}$ in (3.72) reduces to

$$\frac{r}{\phi_j(r, r, \dots, r)} = r^{1-\delta} < K_{2,j} = \inf_{t \in [0, \omega]} \int_{t-\tau}^t g_j(t, s)h_j(s) ds$$

which is equivalent to (3.84). Hence condition (3.72) is fulfilled. Finally, the inequality in (3.73) is reduced to $R > (nR^\gamma + R^\delta)(\max_{1 \leq i \leq n} K_{1,i})$ which leads to (3.85).

It now follows from Theorem 3.7 that system (3.81) has at least one constant-sign solution $u \in (A_\omega(\mathbb{R}))^n$ satisfying (3.83)–(3.86).

Our next result also employs Leggett and Williams’ fixed point theorem (Theorem 2.4).

THEOREM 3.11. *Let $1 \leq p \leq \infty$, q be such that $1/p + 1/q = 1$, $0 < \omega < \infty$ and let $\theta_i \in \{1, -1\}$, $1 \leq i \leq n$, be fixed. Assume that (3.3)–(3.11) and (3.60) hold for each $1 \leq i \leq n$. Moreover, suppose*

$$\left\{ \begin{array}{l} \text{there exists some } j \in \{1, 2, \dots, n\} \text{ such that } g_j(t, s) \geq a_j(s) \text{ for all} \\ t \in [0, \omega] \text{ and a.e. } s \in [0, \omega], \text{ where } a_j \in L^q[0, \omega], a_j \text{ is nonnegative,} \\ \text{and } a_j(t + \omega) = a_j(t) \text{ for a.e. } t \in \mathbb{R}, \end{array} \right. \quad (3.88)$$

$$\left\{ \begin{array}{l} \text{for the same } j \text{ as in (3.88), there exists } r > 0 \text{ and a continuous} \\ \text{function } \phi_j : \prod_{i=1}^n [0, \infty)_i \rightarrow [0, \infty), \text{ where } \phi_j(u_1, \dots, u_n)/|u_j| \\ \text{is ‘nonincreasing’ in the sense that for each } 1 \leq k \leq n, \text{ if} \\ 0 < |u_k| \leq |v_k| \leq r, \text{ then } \phi_j(u_1, \dots, u_n)/|u_j| \geq \phi_j(v_1, \dots, v_n)/|v_j|, \\ \text{with } b_j(t)\phi_j(u) \leq \theta_j f_j(t, u) \text{ for a.e. } t \in [0, \omega] \text{ and all } u \in K, \end{array} \right. \quad (3.89)$$

$$\left\{ \begin{array}{l} \text{for the same } j \text{ and } r \text{ as in (3.89), we have} \\ \left(\frac{\phi_j(r, \dots, r)}{r} \right)^N \left(\prod_{i=1}^N \int_{I_i} a_j(s)b_j(s) ds \right) > 1, \\ \text{where } N \text{ is the smallest positive integer such that } \omega/N \leq \tau/2 \text{ and} \\ I_i = [(i - 1)\omega/N, i\omega/N], i = 0, 1, \dots, N, \end{array} \right. \quad (3.90)$$

and also (3.73) holds. Then (M1) has at least one constant-sign solution $u \in (A_\omega(\mathbb{R}))^n$ with $\min\{r, R\} \leq |u|_\omega \leq \max\{r, R\}$ and $|u|_\omega \neq R$.

PROOF. Let B, C and u_0 be defined as in the proof of Theorem 3.7. Then the same arguments give (3.75).

Next, we shall show that (3.77) is true. Let $u \in \partial \bar{C}_r \cap C(u_0)$. Then (3.78) follows. Further, from (3.89) we obtain (3.79) for $u \in C$, $t \in [0, \omega]$, a.e. $s \in [t - \tau, t]$ and some $j \in \{1, \dots, n\}$ (the same j as in (3.88)). Thus for $t \in [0, \omega]$ and the same j as in (3.88), using (3.78), (3.79), (3.88) and (3.89) we find

$$\begin{aligned} \theta_j S_j u(t) &\geq \int_{t-\tau}^t g_j(t, s) b_j(s) \phi_j(u(s)) ds \\ &\geq \int_{t-\tau}^t a_j(s) b_j(s) \frac{\phi_j(u(s))}{\theta_j u_j(s)} \theta_j u_j(s) ds \end{aligned}$$

$$\geq \frac{\phi_j(r, \dots, r)}{r} \int_{t-\tau}^t a_j(s)b_j(s)\theta_j u_j(s) ds. \tag{3.91}$$

We claim that there exists a $t_{0,j} \in [0, \omega]$ with

$$\frac{\phi_j(r, \dots, r)}{r} \int_{t_{0,j}-\tau}^{t_{0,j}} a_j(s)b_j(s)\theta_j u_j(s) ds > \theta_j u(t_{0,j}). \tag{3.92}$$

If our claim is true, then it follows from (3.91) that

$$\theta_j S_j u(t_{0,j}) \geq \frac{\phi_j(r, \dots, r)}{r} \int_{t_{0,j}-\tau}^{t_{0,j}} a_j(s)b_j(s)\theta_j u_j(s) ds > \theta_j u(t_{0,j})$$

and therefore (3.77) holds. Using a similar argument as in the proof of Theorem 3.7, we then apply Theorem 2.4 to obtain the result.

It remains to prove our claim. Suppose (3.92) is false. Then

$$\theta_j u_j(t) \geq \frac{\phi_j(r, r, \dots, r)}{r} \int_{t-\tau}^t a_j(s)b_j(s)\theta_j u_j(s) ds \quad \text{for all } t \in [0, \omega]. \tag{3.93}$$

Note that if $t \in I_i$ for some $i \in \{1, \dots, N\}$, then $I_{i-1} \subseteq [t - \tau, t]$ since $\omega/N \leq \tau/2$. This together with (3.93) gives for $i \in \{1, \dots, N\}$,

$$\begin{aligned} & \int_{I_i} a_j(t)b_j(t)\theta_j u_j(t) dt \\ & \geq \frac{\phi_j(r, \dots, r)}{r} \int_{I_i} a_j(t)b_j(t) \int_{t-\tau}^t a_j(s)b_j(s)\theta_j u_j(s) ds dt \\ & \geq \frac{\phi_j(r, \dots, r)}{r} \int_{I_i} a_j(t)b_j(t) \int_{I_{i-1}} a_j(s)b_j(s)\theta_j u_j(s) ds dt \\ & = \frac{\phi_j(r, \dots, r)}{r} \left(\int_{I_i} a_j(t)b_j(t) dt \right) \left(\int_{I_{i-1}} a_j(s)b_j(s)\theta_j u_j(s) ds \right). \end{aligned}$$

Applying the above repeatedly yields

$$\begin{aligned} & \int_{I_N} a_j(t)b_j(t)\theta_j u_j(t) dt \\ & \geq \left(\frac{\phi_j(r, \dots, r)}{r} \right)^N \left(\prod_{i=1}^N \int_{I_i} a_j(t)b_j(t) dt \right) \left(\int_{I_0} a_j(s)b_j(s)\theta_j u_j(s) ds \right) \\ & = \left(\frac{\phi_j(r, \dots, r)}{r} \right)^N \left(\prod_{i=1}^N \int_{I_i} a_j(t)b_j(t) dt \right) \left(\int_{I_N} a_j(s)b_j(s)\theta_j u_j(s) ds \right). \tag{3.94} \end{aligned}$$

If $\int_{I_N} a_j(t)b_j(t)\theta_j u_j(t) dt > 0$, then (3.94) leads to

$$\left(\frac{\phi_j(r, \dots, r)}{r}\right)^N \left(\prod_{i=1}^N \int_{I_i} a_j(t)b_j(t) dt\right) \leq 1,$$

which contradicts (3.90). On the other hand, if $\int_{I_N} a_j(t)b_j(t)\theta_j u_j(t) dt = 0$, then since $\theta_j u_j(t) > 0$ for $t \in [0, \omega]$, we must have $a_j(t)b_j(t) = 0$ for a.e. $t \in I_N$. This again contradicts (3.90). Hence we have shown that our claim (3.92) is true. The proof is now complete. \square

REMARK 3.12. Remark 3.8 also holds for Theorem 3.11.

REMARK 3.13. Theorem 3.11 extends the results obtained in [15].

Our final result generalises Theorem 3.11 to give the existence of *multiple* constant-sign periodic solutions.

THEOREM 3.14. *Let $1 \leq p \leq \infty$, q be such that $1/p + 1/q = 1$, $0 < \omega < \infty$ and let $\theta_i \in \{1, -1\}$, $1 \leq i \leq n$, be fixed. Assume that (3.3)–(3.11) and (3.60) hold for each $1 \leq i \leq n$, and (3.88) holds. Let (3.89) and (3.90) be satisfied for $r = r_\ell$, $\ell = 1, 2, \dots, k$, and (3.73) be satisfied for $R = R_\ell$, $\ell = 1, 2, \dots, m$. Then conclusions (a)–(d) of Theorem 3.10 hold.*

REMARK 3.15. Similar to Remark 3.2, in Theorems 3.3–3.14 we can replace conditions (3.3), (3.4), (3.6)–(3.8) and (3.11) with (3.43)–(3.46).

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