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Cercles de remplissage for the Riemann Zeta Function

P. M. Gauthier

Abstract. The celebrated theorem of Picard asserts that each non-constant entire function assumes every value infinitely often, with at most one exception. The Riemann zeta function has this Picard behaviour in a sequence of discs lying in the critical band and whose diameters tend to zero. According to the Riemann hypothesis, the value zero would be this (unique) exceptional value.

S. M. Voronin has shown that the Riemann zeta function has a remarkable universality property. The following is an improved version due to B. Bagchi (see [4, p. 309]).

***Theorem 1* (Universality)** *For each compact set K having connected complement and lying in the critical strip $1/2 < \sigma < 1$, for each function f continuous and zero-free on K and holomorphic on the interior of K, and for each $\epsilon > 0$, there exists a t_ϵ such that,*

$$|\zeta(s + it_\epsilon) - f(s)| < \epsilon \quad \textit{for all } s \in K.$$

Notice that without the assumption that f be zero-free this would contradict the Riemann hypothesis. This universality theorem will be exploited to study the distribution of values of the Riemann zeta function.

Let f be a function meromorphic in a domain Ω of the complex plane $\mathbf{C}$. We shall call a sequence $\{D_n\}$ of (euclidean) discs in Ω a sequence of *disques de remplissage* for the function f if the diameters of the discs D_n tend to zero and if, respectively, the images $f(D_n)$ cover all of the Riemann sphere $\bar{\mathbf{C}}$ except possibly for two sets E_n and F_n, whose (spherical) diameters tend to zero. Of course, since any subsequence of a sequence of disques de remplissage is again a sequence of disques de remplissage, once we have the existence of such a sequence, it follows that given an arbitrary sequence of positive numbers $\{\epsilon_n\}$, there is a sequence of disques de remplissage whose respective radii are smaller than ϵ_n. In the literature, one speaks of *cercles* rather than *disques* de remplissage. We shall call the sequence of centers of such discs a sequence of *centres de remplissage*.

If the function f assumes some value of the Riemann sphere at most finitely often, then, for large n, one of the exceptional sets, say E_n, contains this value. In particular, if f is holomorphic, then we may assume that E_n contains the point at infinity. If moreover, f assumes some finite value a at most finitely often, then we may assume

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that the sets F_n contain a. Since the spherical diameters of E_n and F_n shrink to zero, it follows that f assumes every finite value infinitely often in the union of the disques de remplissage, with the exception of the value a. One can rephrase this by saying that in a sequence of disques de remplissage, a meromorphic function assumes every value of $\bar{\mathbf{C}}$ (with at most two exceptions) infinitely often. For a holomorphic function, one of the exceptional values is, of course, the value infinity, so there can only be one exceptional finite value. The Riemann hypothesis implies that if the Riemann zeta function $\zeta(s)$ has a sequence of disques de remplissage disjoint from the trivial zeros $-2, -4, -6, \dots$ and from the critical line $\Re s = 1/2$, then the value 0 is the (unique) associated exceptional value.

Let $|A, B|$ denote the euclidean distance between two subsets A and B of $\mathbf{C}$. If A is a singleton $\{a\}$, then we write $|a, B|$ rather than $|\{a\}, B|$ and similarly if B is a singleton. Let d denote a distance on $\bar{\mathbf{C}}$.

The following lemma is implicit in the fundamental paper [3] by A. Ostrowski (see also [2]),

Lemma 1 *A sequence $\{z_n\}$ is a sequence of centres de remplissage for a function f meromorphic in Ω if there is a sequence $\{z_n'\}$ such that*

$$|z_n - z_n'| = o(|z_n, \partial\Omega|) \quad \textit{while} \quad d\big(f(z_n), f(z_n')\big) \not\to 0.$$

Theorem 2 *For each σ_0 in the interval $1/2 \le \sigma \le 1$, the function $\zeta(s)$ has a sequence of centres de remplissage on the line $\Re s = \sigma_0$.*

Remarks

1. The case $\sigma_0 = 1$ generalizes Theorem 11.1 in [4].
2. The result fails for $\sigma_0 > 1$ since, for each $\delta > 1$, the function $\zeta(s)$ is bounded in the half-plane $\Re s \ge \delta$.

Proof Suppose first that $1/2 < \sigma_0 < 1$. Let $\{\sigma_n\}$ be a sequence of points in $1/2 < \sigma < 1$ distinct from σ_0 and converging to σ_0. Let p_n be a polynomial with $p_n(\sigma_0) = 0$ and $p_n(\sigma_n) = 1$. By the Universality Theorem, there exists a sequence of positive numbers t_n such that, setting $s_n = \sigma_0 + it_n$ and $s_n' = \sigma_n + it_n$, we have

$$|\zeta(s_n) - p_n(\sigma_0)| < 1/n \quad \text{and} \quad |\zeta(s_n') - p_n(\sigma_n)| < 1/n.$$

Thus, $\zeta(s_n) \to 0$ and $\zeta(s_n') \to 1$. Since $|s_n - s_n'| \to 0$, it follows from the lemma that the sequence $\{s_n\}$ are centres de remplissage for the function $\zeta(s)$. By choosing a subsequence, if necessary, we may assume that the associated disques de remplissage lie inside the critical strip $1/2 < \sigma < 1$ and thus, if the Riemann hypothesis is correct, the associated possible exceptional value exists and is in fact zero.

For the boundary lines $\sigma_0 = 1/2$ and $\sigma_0 = 1$, the existence of a sequence of centres de remplissage on the line $\Re s = \sigma_0$ is a consequence of the previous case and the lemma. Indeed, by taking a diagonal sequence from the critical strip $1/2 < \sigma < 1$, we may construct sequences $s_n' = \sigma_n' + it_n$ and $s_n'' = \sigma_n'' + it_n$ such that $\sigma_n' \to 1/2$,

$\sigma_n'' \rightarrow 1/2$ and $d\big(f(s_n'), f(s_n'')\big) \not\rightarrow 0$. It follows from the lemma that for some subsequence of t_n for which we retain the same notation, $s_n = 1/2 + it_n$ is a sequence of centres de remplissage. The argument for the case $\sigma_0 = 1$ is the same. This completes the proof of the theorem.

H. Bohr and R. Courant have shown [1] that, for fixed σ in the interval $1/2 < \sigma \leq 1$, the curve $\zeta(\sigma + it)$ is everywhere dense. For σ strictly inside the critical strip $1/2 < \sigma < 1$, this also follows from the Universality Theorem. For $\sigma = 1$ it does not. We do not know (see [4, p. 310]) whether the result of Bohr and Courant remains true for the critical line $\sigma = 1/2$. However, we do know [4, p. 209] that the zeta function is unbounded on the critical line.

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Département de mathématiques et de statistique et
Centre de rechèrches mathématiques
Université de Montréal
CP 6128 Centre Ville
Montréal, Québec
H3C 3J7
email: gauthier@dms.umontreal.ca

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