COMMUTATORS AND NORMAL OPERATORS

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Let X be a Banach space and $L(X)$ the Banach algebra of bounded linear operators on X. An operator T in $L(X)$ is *hermitian* if $\| e^{itT} \| = 1$ $(t \in R)$, and is *normal* if $T = R + iJ$ where R and J are commuting normal operators; R and J are then determined uniquely by T, and we may write $T^* = R - iJ$. These definitions extend those for operators on Hilbert spaces. More details may be found in [**1**].

Given T in $L(X)$ we may define the left-multiplication operator $\lambda_T : L(X) \to L(X) : A \mapsto TA$ and the right-multiplication operator $\rho_T : L(X) \to L(X) : A \to AT$. It is easy to check (see [**2**], for instance) that λ_T and ρ_T are hermitian in $L(L(X))$ if T is hermitian in $L(X)$. It follows that $\lambda_{N_1} - \rho_{N_2}$ is normal in $L(L(X))$ if N_1 and N_2 are normal in $L(X)$.

Putnam [**4**] proved that if H is a Hilbert space, if $A, B \in L(H)$, and if A is normal and commutes with $AB - BA$, then A commutes with B. The following result extends Putnam's theorem to operators on Banach spaces.

PROPOSITION 1. *Suppose $T = N + Q$ where Q is quasinilpotent in $L(X)$, $N = R + iJ$ is normal in $L(X)$ and N commutes with Q. Suppose further that $T^2x = 0$ for some $x \in X$. Then $Rx = Jx = 0$.*

Proof. We first observe that Q commutes with both R and J ([**2**], Lemma 3).

Let $Y = \overline{\text{lin}}\,\{R^pJ^qQ^rx : p, q, r = 0, 1, 2, \ldots\}$. Then Y is invariant under R, J and Q; and $(T \mid Y)^2 = 0$. Thus $T \mid Y - Q \mid Y = R \mid Y + iJ \mid Y$ is both normal and quasinilpotent. Hence (see [**1**], §38) $R \mid Y = J \mid Y = 0$: that is, $Rx = Jx = 0$.

L. A. Harris has proved that if N is normal in $L(X)$ and $Nx = 0$, then $N^*x = 0$. This is an immediate corollary of the above proposition: for $Nx = 0$ implies $N^2x = 0$, from which $Rx = Jx = 0$. However, we do not always have $\| Nx \| = \| N^*x \|$. For the operator $\lambda_{N_1} - \rho_{N_2}$ is normal on $L(H)$ when $H = \mathbf{C}^2$, $N_1 = \text{diag}(1, -1)$ and $N_2 = \text{diag}(i, 0)$; but $\| N_1T - TN_2 \| = 4$ while $\| N_1^*T - TN_2^* \| = 2\sqrt{2}$ if $T = \begin{bmatrix} 1+i & 2 \\ -1+i & -2 \end{bmatrix}$.

We give a short proof of the following result due to Palmer ([**3**], Lemma 2.7); the notation is as above.

PROPOSITION 2. *Let $N = R + iJ$, where R and J commute and R^pJ^q is hermitian for $p, q = 0, 1, 2, \ldots$; then $\| Nx \| = \| N^*x \|$ $(x \in X)$.*

Proof. The closure of the set of polynomials in R and J forms a commutative C^*-algebra under the operator norm and the natural involution (by the Vidav-Palmer theorem; see [**1**], chapter 5). For $\varepsilon > 0$ the functional calculus gives

$$\| N - N^2(N^*N + \varepsilon I)^{-1}N^* \| = \| \varepsilon N(N^*N + \varepsilon I)^{-1} \| \leqq \sqrt{\varepsilon}/2,$$

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and $\| N^2(N^*N+\varepsilon I)^{-1} \| \leqq 1$. Hence, for $x \in X$,

$$\| Nx \| = \lim_{\varepsilon \to 0} \| N^2(N^*N+\varepsilon I)^{-1}N^*x \| \leqq \| N^*x \| .$$

This gives $\| N^*x \| \leqq \| N^{**}x \| = \| Nx \|$, which completes the proof.

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