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NEW ALGEBRAIC-TRIGONOMETRIC INEQUALITIES OF LAUB–ILANI TYPE

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Abstract

The Laub–Ilani inequality ['A subtle inequality', *Amer. Math. Monthly* **97** (1990), 65–67] states that $x^x + y^y \geqslant x^y + y^x$ for nonnegative real numbers x, y. We introduce and prove new trigonometric and algebraic-trigonometric inequalities of Laub–Ilani type and propose some conjectural algebraic-trigonometric inequalities of similar forms.

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1. Introduction

The origins of the Laub–Ilani inequality go back to at least 1985 when an intriguing power-exponential inequality appeared in the *Problems and Solutions* section of the *American Mathematical Monthly* as Problem E3116 by Laub [3]. Five years later, the solution to the problem by Ilani was published [4]. The Laub–Ilani inequality, originally entitled 'a subtle inequality', is the following assertion.

THEOREM 1.1 [3, 4]. (a) *If x and y are nonnegative real numbers, then*

$$x^x + y^y \geqslant x^y + y^x. \tag{1.1}$$

(b) *If $\{x_1, x_2, \ldots, x_n\}$ is a sequence of nonnegative real numbers and $\{y_1, y_2, \ldots, y_n\}$ is any permutation of this sequence, then*

$$x_1^{x_1} + x_2^{x_2} + \cdots + x_n^{x_n} \geqslant x_1^{y_1} + x_2^{y_2} + \cdots + x_n^{y_n}. \tag{1.2}$$

Inequality (1.1) was apparently rediscovered around 2006, when Zeikii [8] posted the inequality

$$a^a + b^b \geqslant a^b + b^a, \qquad 0 < a, b \leqslant 1,$$

and the conjecture

$$a^{2a} + b^{2b} \geqslant a^{2b} + b^{2a}, \qquad 0 < a, b \leqslant 1,$$

on the *Mathlinks Forum*. In 2009, Matejicka [5] discussed some inequalities and conjectures of similar type and Cirtoaje [1] proved that

$$a^{ra} + b^{rb} \geqslant a^{rb} + b^{ra}, \qquad a, b, r \in \mathbb{R}^+, \tag{1.3}$$

where $\mathbb{R}^+$ denotes the set of positive real numbers, provided that $0 \leqslant r < e$ and either $a \geqslant b \geqslant 1/e$ or $1/e \geqslant a \geqslant b > 0$. In the same paper, Cirtoaje also conjectured some further power-exponential inequalities such as

$$a^{ra} + b^{rb} + c^{rc} \geqslant a^{rb} + b^{rc} + c^{ra}, \tag{1.4}$$

for all positive real numbers $a, b, c, r \in \mathbb{R}^+$ with $a \leqslant b \leqslant c$, if and only if $r \leqslant e$. Recently, Miyagi and Nishizawa [7] proved another conjecture of Cirtoaje that, if a, b are nonnegative real numbers satisfying $a + b = 1$ and $k \geqslant 1$, then

$$a^{(2b)^k} + b^{(2a)^k} \leqslant 1.$$

Further results and the developments up to 2014 are described in [2].

To the best of author's knowledge, trigonometric or algebraic-trigonometric versions of the Laub–Ilani inequalities (1.1) and (1.2) have not been addressed in the literature. The aim of this paper is to prove some new results of this type and to pose some further conjectures.

2. New inequalities

We begin with two auxiliary results which will be used in the analysis. The first is an algebraic inequality, while the second is of algebraic-trigonometric type.

LEMMA 2.1. *Let x and y be real numbers such that $0 < x < y < 1$. Then*

$$(1 - x^{x+1})/(1 - x) < (1 - y^{y+1})/(1 - y).$$

PROOF. Since $0 < x < y < 1$, it is sufficient to prove that $f(t) = (1 - t^{t+1})/(1 - t)$ is an increasing function of t for $t \in (0, 1)$. Observe that

$$f'(t) = \frac{1 - t^{t+1} - t^t(1 - t)(t \ln t + t + 1)}{(1 - t)^2}.$$

Multiply both sides of the inequality $t \ln t + t + 1 < 1 + t$ by $t^t(1 - t)$ and rearrange:

$$1 - t^{t+1} - t^t(1 - t)(t \ln t + t + 1) > 1 - t^t(1 + t - t^2).$$

Clearly $1 + t - t^2 > 0$. If $t^t(1 + t - t^2) < 1$ or, equivalently, $t \ln t + \ln(1 + t - t^2) < 0$, then the result follows immediately.

Now let $g(t) = t \ln t + \ln(1 + t - t^2)$. Then $g'(t) = \ln k(t) - \ln h(t)$, where

$$k(t) = t \quad \text{and} \quad h(t) = \exp((t^2 + t - 2)/(1 + t - t^2)).$$

So $g'(t) = 0$ if $k(t) = h(t)$ for some $t \in (0, 1)$. Observe that

$$h''(t) = \frac{p(t)h(t)}{(1 + t - t^2)^4},$$

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where

$$p(t) = (14t^4 - 4t^5) + (28t^2 - 28t^3) + (8t^2 - 2t + 1).$$

Since $p(t) > 0$ and $h(t) > 0$, we see that $h''(t) > 0$ and so h is concave up on $(0, 1)$. Compare $\lim_{t\to 0^+} h(t) = e^{-2} > \lim_{t\to 0^+} k(t) = 0$, $h(1/2) = e^{-1} < k(1/2) = 1/2$ and $\lim_{t\to 1^-} h(t) = \lim_{t\to 1^-} k(t) = 1$. Since $k(t) = t$ is linear, the equation $k(t) - h(t) = 0$ has exactly one solution in $(0, 1)$ and hence $g'(t)$ changes sign on $(0, 1)$ only once, that is, $g(t)$ has exactly one extremum point, say, ξ on $(0, 1)$ at which $g'(\xi) = 0$. Since g is decreasing on $(0, \xi)$, increasing on $(\xi, 1)$ and $\lim_{t\to 0^+} g(t) = \lim_{t\to 1^-} g(t) = 0$, it follows that $g(t) < 0$ for all $t \in (0, 1)$ and so $t^t(1 + t - t^2) < 1$ for $0 < t < 1$. This completes the proof. □

Lemma 2.2. *Let* $x \in (0, 1)$. *Then*

(a) $(1 - x^{x+1}) \sin x > x^{x+1}(1 - x)$.

(b) $x(\sin x)^x < \sin x$.

Proof. (a) For $0 < x < 1$, we have $\sin x > x - x^3/3!$, which we can rearrange as $x^{x+1}/ \sin x < 6x^x/(6 - x^2)$. If $f(x) = 6x^x/(6 - x^2) < k(x) = (1 - x^{x+1})/(1 - x)$, $x \in (0, 1)$, the result follows immediately. Since

$$f'(x) = \frac{6x^x((1 + \ln x)(6 - x^2) + 2x)}{(6 - x^2)^2},$$

we have $f'(x) = 0$ if $(1 + \ln x)(6 - x^2) = -2x$ for some $x \in (0, 1)$. Consider the functions $g(x) = (1 + \ln x)(6 - x^2)$ and $h(x) = -2x$, $x \in (0, 1)$. It is easy to see that $g'(x) = ((6 - x^2)/x) - 2x(\ln x + 1) > 5 - 2 > 0$, that is, g is increasing on $(0, 1)$. On the other hand, h is decreasing on $(0, 1)$, $\lim_{x\to 0^+} g(x) = -\infty < \lim_{x\to 0^+} h(x) = 0$ and $\lim_{x\to 1^-} g(x) = 5 > \lim_{x\to 1^-} h(x) = -2$. So the equation $g(x) = h(x)$ has exactly one solution, say, $\eta \in (0, 1)$ such that $f'(x) < 0$ on $(0, \eta)$ and $f'(x) > 0$ on $(\eta, 1)$. Therefore f is decreasing on $(0, \eta)$ and increasing on $(\eta, 1)$ and, hence, has its absolute minimum value at $x = \eta$. Note also that $\lim_{x\to 0^+} f(x) = 1$ and $\lim_{x\to 1^-} f(x) = 6/5$. We can complete the proof of part (a) by comparing the values of the functions f and k. Consider the point $\zeta = 0.8 \in (0, 1)$. Then $f(\zeta) = f(0.8) \doteq 0.936\,39$. In addition, k is increasing, by Lemma 2.1, and $k(\zeta) = k(0.8) \doteq 1.653\,95$. So,

$$\begin{cases} f(x) < 1, x \in (0, 0.8) \\ k(x) > 1, \ x \in (0, 0.8) \end{cases} \quad \text{and} \quad \begin{cases} f(x) < 1.2, x \in (0.8, 1) \\ k(x) > 1.6, \ x \in (0.8, 1). \end{cases}$$

Thus $f(x) < k(x)$ for $x \in (0, 1)$, which completes the proof of part (a).

(b) From the Taylor series expansion of $\sin x$,

$$x(\sin x)^x = x^{x+1}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right)^x.$$

We now use Bernoulli's inequality, namely $(1 + a)^r \leqslant 1 + ra$, for $a > -1$, $0 \leqslant r \leqslant 1$. (The inequality is strict if $a \neq 0$ and $r \neq 0, 1$.) This gives

$$x(\sin x)^x < x^{x+1}\left(1 - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) = x^{x+1}(1 + \sin x - x).$$

So part (b) follows on noticing that $x^{x+1}(1 + \sin x - x) < \sin x$, from part (a). □

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We call F *symmetric* if $F(x,y) = F(y,x)$ for all x, y where $F(x,y)$ is defined. Similarly, we call the inequality $F(x,y) > G(x,y)$ *symmetric* if it implies $F(y,x) > G(y,x)$ for all x, y where $F(x,y)$ and $G(x,y)$ are defined. All our inequalities are symmetric in this sense.

We are now ready to state our main results.

Theorem 2.3. *If* $0 < x < y \leqslant 1$, *then*

$$\sin(x^x) + \sin(y^y) > \sin(y^x) + \sin(x^y).$$

Proof. Consider the function f defined by

$$f(t) = \sin(y^y) - \sin(y^t) - (\sin(t^y) - \sin(t^t)),$$

where y is fixed with $y \in (0,1]$ and $0 < t \leqslant y$. Clearly $f(y) = 0$. So if $f'(t) < 0$ for all $t \in (0,y)$ then $f(t) > 0$ and hence $f(x) > 0$.

To demonstrate this, write

$$f'(t) = g(t) + h(t),$$

where

$$g(t) = t^t \cos(t^t) \ln t - y^t \cos(y^t) \ln y \quad \text{and} \quad h(t) = t^t \cos(t^t) - yt^{y-1} \cos(t^y).$$

To show that $g(t) < 0$, consider the function $k(s) = s^t \cos(s^t) \ln s$, $t \leqslant s \leqslant y$. Then

$$k'(s) = s^{t-1}(\cos(s^t)(1 + t \ln s) - s^t t \sin(s^t) \ln s).$$

Note that $t \ln t \geqslant -e^{-1}$ for $t \in (0,1]$. Since $1 + t \ln s \geqslant 1 + t \ln t \geqslant (e-1)/e > 0$ and $s^t t \sin(s^t) \ln s \leqslant 0$, it follows that $k'(s) > 0$, that is, k is increasing on (t,y) and so $k(t) - k(y) < 0$, which proves that $g(t) < 0$. Moreover, from [4, page 66],

$$t^t - yt^{y-1} < 0, \quad 0 < t < y \leqslant 1. \tag{2.1}$$

Since $\cos(t^t) < \cos(t^y)$ for $0 < t < y$, it follows from (2.1) that $t^t \cos(t^t) < yt^{y-1} \cos(t^y)$, that is, $h(t) < 0$. Hence $f'(t) = g(t) + h(t) < 0$. □

We conjecture a similar trigonometric inequality for the *cosine* function.

Conjecture 2.4. If $0 < x < y \leqslant \pi/2$, then

$$\cos(x^x) + \cos(y^y) < \cos(y^x) + \cos(x^y).$$

The next theorem gives fully trigonometric analogues of the simple Laub–Ilani inequality (1.1).

Theorem 2.5. (a) *If* $0 < x < y \leqslant \pi/2$, *then*

$$(\sin x)^{\sin x} + (\sin y)^{\sin y} > (\sin x)^{\sin y} + (\sin y)^{\sin x}, \tag{2.2}$$
$$(\cos x)^{\cos x} + (\cos y)^{\cos y} > (\cos x)^{\cos y} + (\cos y)^{\cos x}. \tag{2.3}$$

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(b) *If* $0 < x < y \leqslant 1$, *then*

$$(\cos x)^{\sin x} + (\cos y)^{\sin y} < (\cos x)^{\sin y} + (\cos y)^{\sin x}.$$

PROOF. (a) Since $\sin t, \cos t \geqslant 0$ for $t \in (0, \pi/2]$, inequalities (2.2) and (2.3) are direct consequences of Theorem 1.1.

(b) Consider the function f defined by

$$f(t) = (\cos t)^{\sin t} + (\cos y)^{\sin y} - ((\cos t)^{\sin y} + (\cos y)^{\sin t}),$$

where y is fixed with $y \in (0, 1]$ and $0 < t \leqslant y$. Since $f(y) = 0$, the result follows if f is increasing. To see this, write

$$f'(t) = g(t)\cos t - \frac{\sin t}{\cos t} h(t),$$

where

$$g(t) = (\cos t)^{\sin t}\ln(\cos t) - (\cos y)^{\sin t}\ln(\cos y)$$

and

$$h(t) = (\cos t)^{\sin t}\sin t - (\cos t)^{\sin y}\sin y.$$

Observe that

$$\begin{aligned} g(t) &= -\int_t^y \frac{d}{ds}((\cos s)^{\sin t}\ln(\cos s))\,ds \\ &= \int_t^y ((\cos s)^{\sin t - 1}\sin s)(1 + \sin t\ln(\cos s))\,ds. \end{aligned}$$

Since $\cos s > \cos 1$,

$$1 > 1 + (\sin t)\ln(\cos s) > 1 + (\sin 1)\ln(\cos 1) \doteq 0.481\,97 > 0.$$

So $g(t) > 0$. Similarly,

$$h(t) = -\int_t^y \frac{d}{ds}((\cos t)^{\sin s}\sin s)\,ds = -\int_t^y (\cos s\cos^{\sin s} t)(1 + \sin s\ln(\cos t))\,ds.$$

Again, $1 > 1 + (\sin s)\ln(\cos t) > 1 + (\sin 1)\ln(\cos 1) \doteq 0.481\,97 > 0$ and this gives $h(t) < 0$. Combining $g(t) > 0$ and $h(t) < 0$ yields $f'(t) > 0$, that is, the function f is increasing. This completes the proof of (b). □

THEOREM 2.6. *If* $0 < x < y \leqslant 1$, *then*

$$(\cos x)^x + (\cos y)^y < (\cos x)^y + (\cos y)^x.$$

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Proof. Let $f(t) = (\cos t)^y - (\cos t)^t - ((\cos y)^y - (\cos y)^t)$, where y is fixed, $y \in (0, 1]$ and $0 < t \leqslant y$. Since $f(y) = 0$, it will be sufficient to show that f is decreasing. Clearly,

$$f'(t) = g(t) + h(t),$$

where

$$g(t) = (\cos y)^t \ln(\cos y) - (\cos t)^t \ln(\cos t)$$

and

$$h(t) = \frac{\sin t}{\cos t}(t(\cos t)^t - y(\cos t)^y).$$

First, consider the function $p(s) = s(\cos t)^s, t \leqslant s \leqslant y$. Then

$$p'(s) = (\cos t)^s(1 + s\ln(\cos t))$$

and $p'(s) > 0$ for $s \in (t, y)$, because $1 + s\ln(\cos t) \geqslant 1 + \ln(\cos 1) \doteq 0.384\,37 > 0$. Hence the function p is increasing, which implies that $t(\cos t)^t < y(\cos t)^y$, that is, $h(t) < 0$ for $0 < t < y$. Next, let $q(s) = (\cos s)^t \ln(\cos s), t \leqslant s \leqslant y$. Then

$$q'(s) = -((\cos s)^{t-1} \sin s)(1 + t\ln(\cos s))$$

and $q'(s) < 0$ for $s \in (t, y)$, because $1 + t\ln(\cos s) \geqslant 1 + \ln(\cos 1) \doteq 0.384\,37 > 0$. Hence the function q is decreasing. So $q(t) > q(y)$ for $t < y$, which implies that $g(t) < 0$ for $0 < t < y$. Therefore f is decreasing and the proof is complete. □

Before giving the *sine* version of the last result, we prove some auxiliary results that will be used in the proof.

Lemma 2.7. *Let* $y \in (0, \pi/2]$ *be fixed and let*

$$g(t) = (\sin t)^t \ln(\sin t) - (\sin y)^t \ln(\sin y), \quad 0 < t \leqslant y.$$

Then $g(t) < 0$ *for* $0 < t < y$ *and* g *is increasing on* $0 < t < y$.

Proof. Observe that

$$g(t) = -\int_t^y \frac{d}{ds}((\sin s)^t \ln(\sin s))\,ds = -\int_t^y \frac{\cos s}{\sin s}(\sin s)^t(1 + t\ln(\sin s))\,ds.$$

Since $s > t$ and $\ln(\sin s) < 0$, we have $1 + t\ln(\sin s) > 1 + s\ln(\sin s)$. We claim that $1 + s\ln(\sin s) > 0$ for $0 < s < \pi/2$ which will prove the first part of the lemma.

Jordan's inequality [6] states that

$$\frac{\sin x}{x} \geqslant \frac{2}{\pi}, \quad x \in (0, \pi/2], \tag{2.4}$$

where the equality holds if and only if $x = \pi/2$. For $s \in (0, 1)$, it follows that $(\sin s)^s > (2s/\pi)^s$. Now, $v(s) = (2s/\pi)^s$ is decreasing on $(0, \pi/2e)$ and increasing on $(\pi/2e, 1)$, while $\lim_{s\to 0^+} v(s) = 1$, $\lim_{s\to 1^-} v(s) = 2/\pi$ and $v(\pi/2e) = \exp(-\pi/2e) > 1/2$. So $1/2 < (2s/\pi)^s < 1$ and hence $1 + s\ln(\sin s) > 1 + \ln(1/2) > 0$ for $s \in (0, 1)$. On the

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other hand, if $1 \leqslant s < \pi/2$, then $1 + s\ln(\sin s) \geqslant 1 + s\ln(\sin 1) \geqslant 1 + \pi\ln(\sin 1)/2 > 0$. So $1 + s\ln(\sin s) > 0$ for $(0, \pi/2)$. Consequently, $1 + t\ln(\sin s) > 0$ for $t < s < y$ and, hence, $g(t) < 0$.

To prove that g is increasing on $0 < t < y$, consider g' which is given by

$$g'(t) = (\sin t)^t(\ln(\sin t))^2 - (\sin y)^t(\ln(\sin y))^2 + (1 + t\ln(\sin t))\frac{\cos t}{\sin t}(\sin t)^t. \tag{2.5}$$

Since $1 + t\ln(\sin t) > 0$ and $\cos t(\sin t)^{t-1} > 0$, the last term in (2.5) is positive. Moreover, since $g(t) < 0$ and $\ln(\sin t) < 0$,

$$(\sin t)^t(\ln(\sin t))^2 - (\sin y)^t\ln(\sin t)\ln(\sin y) > 0. \tag{2.6}$$

Since $-\ln(\sin t) > -\ln(\sin y)$ for $0 < t < y$, (2.6) yields

$$(\sin t)^t(\ln(\sin t))^2 - (\sin y)^t(\ln(\sin y))^2 > 0.$$

Therefore $g'(t) > 0$ for $0 < t < y \leqslant \pi/2$. □

Lemma 2.8. *If* $0 < t < y \leqslant 1$ *or* $3/5 \leqslant t < y \leqslant \pi/2$, *then*

$$t(\sin t)^t - y(\sin t)^y < 0.$$

Proof. First note that the regions $0 < t < y \leqslant 1$ and $3/5 \leqslant t < y \leqslant \pi/2$ are partially overlapping in the ty-plane.

(a) Suppose that $0 < t < y \leqslant 1$. From part (b) of Lemma 2.2, $t(\sin t)^t < \sin t$, that is,

$$\ln t < (1 - t)\ln(\sin t), \quad 0 < t < 1.$$

Let $p(s) = \ln(t/s)$ and $q(s) = (s - t)\ln(\sin t)$ for $t \leqslant s \leqslant y$. Then $p'(s) = -1/s < 0$ and $p''(s) = 1/s^2 > 0$ on $(0, 1)$, so p is decreasing and concave up on $(0, 1)$. On the other hand, q is linear and decreasing on $(0, 1)$. Since $p(t) = q(t) = 0$ and $\lim_{s\to 1^-} p(s) = \ln t < \lim_{s\to 1^-} q(s) = (1 - t)\ln(\sin t)$, we have $p(s) < q(s)$ and so

$$\frac{t}{s} < \frac{(\sin t)^s}{(\sin t)^t}, \quad t < s \leqslant y.$$

Then taking $s = y$ leads to the desired inequality for $0 < t < y \leqslant 1$.

(b) Suppose that $3/5 \leqslant t < y \leqslant \pi/2$. Let $w(s) = s(\sin t)^s$, $t \leqslant s \leqslant y$. Then

$$w'(s) = (\sin t)^s(1 + s\ln(\sin t)).$$

Since $1 + s\ln(\sin t) \geqslant 1 + s\ln(\sin(3/5)) > 1 + \pi\ln(\sin(3/5))/2 \doteq 0.102\,19 > 0$, it follows that $w'(s) > 0$ for $t < s < y$, that is, w is increasing, which implies that $w(t) = t(\sin t)^t < w(y) = y(\sin t)^y$ for $t < y$. □

We can now state the *sine* version of Theorem 2.6.

Theorem 2.9. *If* $0 < x < y \leqslant \pi/2$, *then*

$$(\sin x)^x + (\sin y)^y > (\sin y)^x + (\sin x)^y.$$

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PROOF. Consider the function f defined by

$$f(t) = (\sin y)^y - (\sin y)^t - ((\sin t)^y - (\sin t)^t),$$

where y is fixed with $0 < y \leqslant \pi/2$ and $0 < t \leqslant y$. Since $f(y) = 0$, it will follow that $f(t) > 0$ if $f'(t) < 0$ for all $t \in (0, y)$. To estimate f', write

$$f'(t) = g(t) + h(t),$$

where

$$g(t) = (\sin t)^t \ln(\sin t) - (\sin y)^t \ln(\sin y) \quad \text{and} \quad h(t) = (t(\sin t)^t - y(\sin t)^y)\frac{\cos t}{\sin t}.$$

To determine the sign of the function f', we examine three partially overlapping regions in the ty-plane.

(a) Over the region $0 < t < y \leqslant 1$, we have $g(t) < 0$ by Lemma 2.7 and $h(t) < 0$ by Lemma 2.8. Hence $f'(t) = g(t) + h(t) < 0$ for $0 < t < y \leqslant 1$.

(b) For the region $3/5 \leqslant t < y \leqslant \pi/2$, we have $g(t) < 0$ by Lemma 2.7 and $h(t) < 0$ by Lemma 2.8. Hence $f'(t) = g(t) + h(t) < 0$ for $3/5 \leqslant t < y \leqslant \pi/2$.

(c) Finally, consider the region $0 < t \leqslant 3/5$, $1 \leqslant y \leqslant \pi/2$. By part (b) of Lemma 2.2, $t(\sin t)^t < \sin t$. Moreover, $y(\sin t)^y > (\sin t)^{\pi/2}$. Thus,

$$h(t) = (t(\sin t)^t - y(\sin t)^y)\frac{\cos t}{\sin t} < (1 - (\sin t)^{(\pi-2)/2}) \cos t := k(t). \tag{2.7}$$

The function k is decreasing, since both the functions $p(t) = 1 - (\sin t)^{(\pi-2)/2} > 0$ and $q(t) = \cos t > 0$ are decreasing for $0 < t \leqslant 3/5$. On the other hand, if we set $u(s) = (\sin s)^t \ln(\sin s)$, $1 \leqslant s \leqslant y$, we find $u'(s) = \cos s(\sin s)^{t-1}(1 + t\ln(\sin s)) > 0$, which implies that u is increasing on $(1, y)$. So

$$(\sin y)^t \ln(\sin y) \geqslant (\sin 1)^t \ln(\sin 1)$$

and, hence,

$$\begin{aligned} g(t) &= (\sin t)^t \ln(\sin t) - (\sin y)^t \ln(\sin y) \\ &\leqslant (\sin t)^t \ln(\sin t) - (\sin 1)^t \ln(\sin 1) := z(t). \end{aligned} \tag{2.8}$$

From Lemma 2.7 it is easily seen that $z(t) < 0$ and $z(t)$ is increasing on $(0, 3/5)$.

Consequently, $f'(t) = g(t) + h(t) \leqslant z(t) + k(t)$. Now, using inequalities (2.7) and (2.8) and the properties of the functions k and z, we compute:

$f'(t) \leqslant -0.003$	for $t \in (0, 0.2]$,	$f'(t) \leqslant -0.091$	for $t \in (0.2, 0.3]$,
$f'(t) \leqslant -0.094$	for $t \in (0.3, 0.35]$,	$f'(t) \leqslant -0.056$	for $t \in (0.35, 0.40]$,
$f'(t) \leqslant -0.064$	for $t \in (0.4, 0.425]$,	$f'(t) \leqslant -0.050$	for $t \in (0.425, 0.450]$,
$f'(t) \leqslant -0.039$	for $t \in (0.450, 0.475]$,	$f'(t) \leqslant -0.030$	for $t \in (0.475, 0.500]$,
$f'(t) \leqslant -0.022$	for $t \in (0.500, 0.525]$,	$f'(t) \leqslant -0.015$	for $t \in (0.525, 0.550]$,
$f'(t) \leqslant -0.008$	for $t \in (0.550, 0.575]$,	$f'(t) \leqslant -0.003$	for $t \in (0.575, 0.600]$.

Hence, $f'(t) = g(t) + h(t) < 0$ for $0 < t \leqslant 3/5$, $1 \leqslant y \leqslant \pi/2$. This completes the proof. □

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Theorem 2.10. *If* $0 < x < y \leqslant \pi/2$ *then*

$$x^{\sin x} + y^{\sin y} > x^{\sin y} + y^{\sin x}.$$

Proof. For $0 < x < y = 1$, we have $x^{\sin x} > x^{\sin y} = x^{\sin 1}$ and, hence, the inequality holds. The inequality holds also for $1 = x < y \leqslant \pi/2$, since $y^{\sin y} > y^{\sin x} = y^{\sin 1}$. Further, $\sin x < \sin y$ for $0 < x < 1 < y \leqslant \pi/2$, so $x^{\sin x} > x^{\sin y}$ and $y^{\sin y} > y^{\sin x}$ and, again, the inequality holds. For the remaining values of x, y, that is, for $0 < x < y < 1$ and $1 < x < y \leqslant \pi/2$, consider the function f defined by

$$f(t) = y^{\sin y} - y^{\sin t} - (t^{\sin y} - t^{\sin t}),$$

where $y \in (0, \pi/2]$ is fixed and $0 < t \leqslant y$. Since $f(y) = 0$, the inequality follows if $f'(t) < 0$ for $0 < t < y < 1$ and $1 < t < y \leqslant \pi/2$. Write

$$f'(t) = g(t)\cos t + t^{-1}h(t),$$

where

$$g(t) = (t^{\sin t}\ln t - y^{\sin t}\ln y) \quad \text{and} \quad h(t) = t^{\sin t}\sin t - t^{\sin y}\sin y.$$

To determine the sign of $g(t)$ for $0 < t < y < 1$ and $1 < t < y \leqslant \pi/2$, rewrite g as

$$g(t) = -\int_t^y \frac{d}{ds}(s^{\sin t}\ln s)\,ds = -\int_t^y s^{\sin t - 1}(1 + (\sin t)\ln s)\,ds.$$

If $0 < t \leqslant s \leqslant y < 1$, then $1 + (\sin t)\ln s \geqslant 1 + (\sin s)\ln s > 1 + s\ln s$. Let $k(s) = s\ln s$ for $0 < s < 1$. The global minimum of k is $k(e^{-1}) = -e^{-1}$. Hence $1 + (\sin t)\ln s > 1 + s\ln s \geqslant 1 - e^{-1} > 0$ and $g(t) < 0$ for $0 < t < y < 1$. On the other hand, if $1 < t < s < y \leqslant \pi/2$, then $1 + (\sin t)\ln s > 0$ since $\ln s > 0$. Hence $g(t) < 0$ for $1 < t < y \leqslant \pi/2$.

To determine the sign of $h(t)$ for $0 < t < y < 1$, we use the inequality [4, page 66]

$$p^p - qp^{q-1} < 0, \quad 0 < p < q \leqslant 1.$$

Let $p = \sin t$ and $q = \sin y$. Then $\sin t^{\sin t} - \sin t^{\sin y - 1}\sin y < 0$, that is,

$$\frac{\sin t}{\sin y} < \sin t^{\sin y - \sin t}.$$

But since

$$\sin t^{\sin y - \sin t} < t^{\sin y - \sin t}, \quad 0 < t < y < 1,$$

we have

$$\frac{\sin t}{\sin y} < t^{\sin y - \sin t},$$

which shows that $h(t) < 0$ for $0 < t < y < 1$. Moreover, for $1 < t < y \leqslant \pi/2$, we have $\sin t < \sin y$ and hence $t^{\sin t} < t^{\sin y}$. Then $t^{\sin t}\sin t < t^{\sin y}\sin t < t^{\sin y}\sin y$, which shows that $h(t) < 0$ for $1 < t < y \leqslant \pi/2$.

Consequently, $f'(t) < 0$ for $0 < t < y < 1$ and $1 < t < y \leqslant \pi/2$, since $g(t) < 0$ and $h(t) < 0$ for the corresponding values of t. This completes the proof. □

The following result is the *cosine* version of Theorem 2.10.

THEOREM 2.11. *If* $0 < x < y$ *and* $1 \leqslant y \leqslant \pi/2$ *then*

$$x^{\cos x} + y^{\cos y} < x^{\cos y} + y^{\cos x}.$$

PROOF. The inequality holds for $x < y = 1$ and $x < y = \pi/2$, since $x^{\cos x} < x^{\cos 1}$ and $x^{\cos x} < (\pi/2)^{\cos x}$, respectively. For $x < y$, $1 < y < \pi/2$, we have $y^{\cos y} > x^{\cos y}$ and $\cos x > \cos y$. Then

$$\begin{aligned} y^{\cos x} - y^{\cos y} &= y^{\cos y}(y^{\cos x - \cos y} - 1) \\ &> x^{\cos y}(y^{\cos x - \cos y} - 1) > x^{\cos y}(x^{\cos x - \cos y} - 1) = x^{\cos x} - x^{\cos y}, \end{aligned}$$

which completes the proof. □

Finally, we give two conjectures based on numerous tests we have performed to check the validity of the trigonometric and algebraic-trigonometric versions of the n-term Laub–Ilani inequality given by (1.2).

CONJECTURE 2.12. If $x_1, x_2, \ldots, x_n \in (0, 1]$ and if $\{y_1, y_2, \ldots, y_n\}$ is any permutation of the finite sequence $\{x_1, x_2, \ldots, x_n\}$, then

(a) $\sin x_1^{x_1} + \sin x_2^{x_2} + \cdots + \sin x_n^{x_n} > \sin x_1^{y_1} + \sin x_2^{y_2} + \cdots + \sin x_n^{y_n}$,

(b) $\cos^{x_1} x_1 + \cos^{x_2} x_2 + \cdots + \cos^{x_n} x_n < \cos^{y_1} x_1 + \cos^{y_2} x_2 + \cdots + \cos^{y_n} x_n$,

(c) $\cos^{r_1} x_1 + \cos^{r_2} x_2 + \cdots + \cos^{r_n} x_n < \cos^{t_1} x_1 + \cos^{t_2} x_2 + \cdots + \cos^{t_n} x_n$, where $r_i = \sin x_i$ and $t_i = \sin y_i$ for $i = 1, 2, \ldots, n$.

CONJECTURE 2.13. If $x_1, x_2, \ldots, x_n \in (0, \pi/2]$ and if $\{y_1, y_2, \ldots, y_n\}$ is any permutation of the finite sequence $\{x_1, x_2, \ldots, x_n\}$, then

(a) $\cos x_1^{x_1} + \cos x_2^{x_2} + \cdots + \cos x_n^{x_n} < \cos x_1^{y_1} + \cos x_2^{y_2} + \cdots + \cos x_n^{y_n}$,

(b) $\sin^{x_1} x_1 + \sin^{x_2} x_2 + \cdots + \sin^{x_n} x_n > \sin^{y_1} x_1 + \sin^{y_2} x_2 + \cdots + \sin^{y_n} x_n$,

(c) $x_1^{\sin x_1} + x_2^{\sin x_2} + \cdots + x_n^{\sin x_n} > x_1^{\sin y_1} + x_2^{\sin y_2} + \cdots + x_n^{\sin y_n}$.

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