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The product of vector-valued measures

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Let M (N) be a σ-algebra of subsets of a set S (T) and let X, Y be Banach spaces with $\langle\ ,\ \rangle$ a continuous bilinear map from $X \times Y$ into the scalar field. If $\mu : M \to X$ $(\nu : N \to Y)$ is a vector measure and λ is the scalar measure defined on the measurable rectangles $A \times B$, $A \in M$, $B \in N$, by $\lambda(A\times B) = \langle \mu(A), \nu(B) \rangle$, it is known that λ is generally not countably additive on the algebra generated by the measurable rectangles and therefore has no countably additive extension to the σ-algebra generated by the measurable rectangles. If μ (ν) is an indefinite Pettis integral $\mu = \int f d\alpha$ $\left(\nu = \int g d\beta\right)$, it is shown that a necessary and sufficient condition that λ have a countable additive extension to the σ-algebra generated by the measurable rectangles is that the function $F : (s, t) \to \langle f(s), g(t) \rangle$ is integrable with respect to $\alpha \times \beta$.

Let (S, M) and (T, N) be measurable spaces and let H be a Hilbert space. If $\mu : M \to H$ and $\nu : N \to H$ are vector measures ([6], IV.10), then for a measurable rectangle $A \times B$, $A \in M$, $B \in N$, we may define a scalar set function λ by

(1) $$\lambda(A\times B) = \big(\mu(A), \nu(B)\big) ,$$

where $(\ ,\ )$ is the inner product in H. If A denotes the algebra of subsets of $S \times T$ generated by the measurable rectangles, then λ can be extended to a finitely additive set function on A in the usual fashion. P. Masani posed the question as to whether λ is countably additive on A. Of course, if λ is countably additive on A, one would like to know

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if λ has a countably additive extension to the σ-algebra, Σ, generated by A. In [5] and [10], examples are given that show λ may fail to be countably additive on A. In this note we consider the question as to when λ has a (finite) countably additive extension to the σ-algebra Σ. In Theorems 1 and 3 we present necessary and sufficient conditions for an inner product of certain types of vector measures (indefinite Pettis integrals) to have such a countably additive extension. Our methods are applicable to a more general situation than described above so the results are considered in the more general setting.

Let X, Y be B-spaces and let $\langle\ ,\ \rangle$ be a continuous bilinear map from $X \times Y$ into the scalar field, which we assume to be $\mathbb{R}$ for convenience. (Strictly speaking, the inner product above does not fit into this setting if the Hilbert space is complex, but by introducing conjugates in the appropriate places this can be taken care of with no difficulty.) If $\mu : M \to X$ and $\nu : N \to Y$ are vector measures, we define a scalar set function λ on measurable rectangles $A \times B$, $A \in M$, $B \in N$, by

$$\lambda(A \times B) = \langle \mu(A), \nu(B) \rangle . \tag{2}$$

If A is again the algebra generated by the measurable rectangles, λ has a finitely additive extension to A. We consider the question as to when λ has a finite countably additive extension to Σ, the σ-algebra generated by A. The vector measures which we consider are vector measures which are indefinite Pettis integrals (see [8] for a discussion of the Pettis integral; in particular Theorem 3.7.2). A Hilbert space valued measure can be realized as an indefinite Pettis integral if (and only if) the measure has σ-finite variation ([9], 4.3) so Theorems 1 and 3 are applicable to a large class of Hilbert space valued measures (as well as Banach space valued measures). It should be noted that Theorems 1 and 2 of [9] give necessary and sufficient conditions for a vector measure to be an indefinite Pettis integral.

Henceforth, α (β) will denote a σ-finite positive measure on M (N).

THEOREM 1. *Let* $f : S \to X$ $(g : T \to Y)$ *be strongly measurable and Pettis integrable with respect to* α (β) *and set* $\mu(E) = \int_E f d\alpha$, $E \in M$

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$\left(\nu(E) = \int_E g d\beta \ , \quad E \in N\right)$. *If* λ *defined by* (2) *has a (finite) countably additive extension to* Σ *, then the function* $F : S\times T \to \mathbb{R}$ *, defined by* $F(s, t) = \langle f(s), g(t)\rangle$ *, is in* $L^1(\alpha\times\beta)$ *and* $\lambda(E) = \int_E F d\alpha\times\beta$ *,* $E \in \Sigma$.

Proof. First we show $\lambda \ll \alpha\times\beta$. For this it is enough to show $\lambda \ll \alpha\times\beta$ on A ([6], IV.9.13 and III.4.13). Suppose $E = \bigcup_{i=1}^{n} A_i\times B_i \in A$, with $\{A_i\times B_i\}$ pairwise disjoint, and $\alpha\times\beta(E) = 0$. Then $\alpha(A_i) = 0$ or $\beta(B_i) = 0$ for each i so that $\lambda(E) = 0$.

Let h be the Radon-Nikodým derivative of λ with respect to $\alpha\times\beta$. Since F is clearly measurable, it suffices to show $h = F$ $\alpha\times\beta$-almost everywhere. First consider the case where both α and β are finite. In this case recall that f (g) can be realized as an improper Bochner integral, that is, there is a sequence $\{A_n\}$ $(\{B_n\})$ of M-measurable sets (N-measurable sets) such that $A_n \uparrow S$ $(B_n \uparrow T)$ and $\int_A f d\alpha = \lim_n \text{Bochner} \int_{A\cap A_n} f d\alpha$ $\left(\int_B g d\beta = \lim \int_{B\cap B_n} g d\beta\right)$ ([9], 4.1). Now F is $\alpha\times\beta$ integrable in each $A_n \times B_n$ since f and g are Bochner integrable in A_n and B_n , respectively. Also if $A \times B$ is a measurable rectangle contained in $A_n \times B_n$,

$$(3) \qquad \lambda(A\times B) = \int_{A\times B} h d\alpha\times\beta = \left\langle \int_A f d\alpha, \int_B g d\beta \right\rangle = \int_A \left\langle f(s), \int_B g(t) d\beta(t) \right\rangle d\alpha(s)$$

$$= \int_A \int_B \langle f(s), g(t) \rangle d\beta(t) d\alpha(s) = \int_{A\times B} F d\alpha\times\beta$$

so that $h = F$ $\alpha\times\beta$-almost everywhere in $A_n \times B_n$. Since $A_n \uparrow S$ and $B_n \uparrow T$, $h = F$ $\alpha\times\beta$-almost everywhere in $S \times T$.

If α and β are σ-finite, write $S = \bigcup_{n=1}^{\infty} S_n$ with $\alpha(S_n) < \infty$ and $T = \bigcup_{n=1}^{\infty} T_n$ with $\beta(T_n) < \infty$ and apply the result above in each $S_n \times T_n$

to obtain $F = h$ almost everywhere in $S \times T$.

REMARK 2. Theorem 1 can be considered to be a vector analogue of Theorem 21.29 of [7], that is, the Radon-Nikodým derivative of the product measure λ is the "product" of the Radon-Nikodým derivatives of μ and ν , where "product" is interpreted to be given by the bilinear map $\langle\ ,\ \rangle$. One obvious case when F is integrable is when both f and g are Bochner integrable since $|F(s, t)| \leq c\|f(s)\|\|g(t)\|$.

THEOREM 3. *Let the notation be as in Theorem* 1. *If the function* F *is* $\alpha\times\beta$-*integrable, then* λ *has a finite countably additive extension to* Σ .

Proof. Define a measure λ' on Σ by $\lambda'(E) = \int_E F d\alpha\times\beta$. By the computation in (3), λ' and λ agree on any measurable rectangle and therefore agree on A . Hence λ' gives a countably additive extension of λ to Σ .

The example presented in [5] can be considered in the context of Theorems 1 and 3. For convenience let the measure space be the positive integers N with every subset of N measurable. Let $\{\phi_n\}$ be an orthonormal sequence in l^2 and set $\mu(A) = \sum_{m\in A} m^{-5/8}\phi_m$ for $A \subseteq N$. Then μ is a vector measure which is the indefinite Pettis integral of the function $f(m) = m^{-5/8}\phi_m$ with respect to the counting measure on N . Similarly, let $\{\chi_m\} = \{\psi_n/\|\psi_m\|\}$ be the normalized orthogonal sequence in Proposition 2 of [5] and set $\nu(A) = \sum_{m\in A} 2^{-5m/4}\chi_m$ so that ν is the indefinite Pettis integral of the function $g(m) = 2^{-5m/4}\chi_m$ with respect to the counting measure. Then the computation on page 845 of [5] shows that the function $F = (f, g)$ is not integrable with respect to the counting measure on $N \times N$.

Also with the aid of Theorems 1 and 3 a Fubini-type theorem for products of vector measures can be formulated. In particular, we can easily obtain

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PROPOSITION 4. *Let $H : S \times T \to R$ be bounded and Σ measurable. If F (as in Theorem 1) is $\alpha\times\beta$-integrable, then*

$$(4) \qquad \int_{S\times T} H d\lambda = \int_{S\times T} HF d\alpha\times\beta = \int_T \left\langle \int_S H(s, t) f(s) d\alpha(s), g(t) \right\rangle d\beta(t) .$$

Proof. For each t, $\int_S H(s, t) f(s) d\alpha(s)$ exists as a Pettis integral ([3], Theorem 4), and by Fubini's Theorem the function $t \to \int_S H(s, t) F(s, t) d\alpha(s) = \left\langle \int_S H(s, t) f(s) d\alpha(s), g(t) \right\rangle$ is β-integrable with (4) holding.

REMARK 5. Of course, the restriction that H be bounded in Proposition 4 is very undesirable. However, it is difficult to give conditions that insure that the integral $\int_S H(s, t) f(s) d\alpha(s)$ exists as a Pettis integral (see the remark preceding Theorem 4 in [3]).

REMARK 6. The iterated integral in (4) can also be considered in another way. The "inside" integral $\int_S H(s, t) f(s) d\alpha(s)$ is equal to $\int_S H(s, t) d\mu(s)$, where this integral is understood to be the vector-measure integral as in [6], IV.10. Also if H is bounded by $M > 0$, $\left\| \int_S H(s, t) d\mu(s) \right\| \le M \text{semi-var}(\mu)(S)$, ([6], IV.10.8), and therefore if the function $\psi : t \to \int_S H(s, t) d\mu(s)$ is measurable, the integral $\int_T \left\langle \int_S H(s, t) d\mu(s), d\nu(t) \right\rangle$ exists in the sense of Bartle ([2], Theorem 4) and is equal to $\int_{S\times T} H d\lambda$ in (4). Concerning the measurability of the function ψ it is clear by Fubini's Theorem that for each $x' \in X'$ the function $x'\psi$ is measurable. Therefore it suffices to observe that ψ has almost separable range ([6], III.6.11); for this, it is enough to observe that the measure μ is almost separably valued. There exists a sequence of M-simple X-valued functions $\{S_n\}$ such that $S_n \to f$ α-

almost everywhere and $\|S_n(\cdot)\| \le \|f(\cdot)\|$. For convenience, assume that $S_n \to f$ everywhere, the α-null set being ignored. Then $\left\{\int_A S_n d\alpha\right\}$ converges weakly to $\int_A f d\alpha = \mu(A)$ for each $A \in M$. Therefore if X_0 is the closed subspace spanned by the range of the $\{S_n\}$, $\mu(A) \in X_0$ for each $A \in M$ ([6], V.2.14). Thus μ has almost separable range and the measurability of ψ is established.

REMARK 7. Of course, the integrals in (4) and Remark 6 can also be evaluated as the iterated integrals,

$$(5)\quad \int_{S\times T} H d\lambda = \int_S \left\langle f(s), \int_T H(s, t) g(t) d\beta(t) \right\rangle d\alpha(s)$$

$$= \int_S \left\langle d\mu(s), \int_T H(s, t) d\nu(t) \right\rangle .$$

Under the additional restriction that the functions f and g in Theorem 1 are Bochner integrable, the requirement that the function H in Proposition 4 be bounded can be somewhat relaxed. That is, we have

PROPOSITION 8. *Suppose that the measures* μ *and* ν *in Theorem* 1 *are of bounded variation with variations* $v(\mu)$ *and* $v(\nu)$ *(that is,* f *and* g *are Bochner integrable). If* H *is integrable with respect to* $v(\mu) \times v(\nu)$ *, then* H *is integrable with respect to* λ *and equations* (4) *and* (5) *hold.*

Proof. For $A \in M$ and $B \in N$,

$$v(\lambda)(A\times B) = \int_{A\times B} |F| d\lambda \le \int_A \|f\| d\alpha \int_B \|g\| d\beta = v(\mu) \times v(\nu)(A\times B)$$

([4], II.10.5). Thus $v(\lambda) \le v(\mu) \times v(\nu)$ on A and hence on Σ , and H is integrable with respect to λ . For each $t \in T$, $H(\cdot, t)$ is integrable with respect to $v(\mu)$ and f is Bochner integrable with respect to α so that $H(\cdot, t)f$ is α-integrable with

$$\int_S H(s, t) f(s) d\alpha(s) = \int_S H(s, t) d\mu(s)$$

([4], II.10.4). Since

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$$\left\| \int_S H(s,\ t) f(s) d\alpha(s) \right\| \leq \int_S |H(s,\ t)| \|f(s)\| d\mu(s) \ ,$$

the function $t \to \int_S |H(s,\ t)| \|f(s)\| d\mu(s)$ is $v(\nu)$-integrable, and the function $\phi : t \to \int_S H(s,\ t) f(s) d\alpha(s)$ is measurable (by an argument as in Remark 6), the function ϕ is also $v(\nu)$-integrable. By [4], II.10.4 the the function $t \to \langle \phi(t),\ g(t) \rangle$ is β-integrable and

$$\int_T \left\langle \int_S H(s,\ t) f(s) d\alpha(s),\ g(t) \right\rangle d\beta(t) =$$

$$= \int_T \left\langle \int_S H(s,\ t) d\mu(s),\ d\nu(t) \right\rangle = \int_{S\times T} H d\lambda \ .$$

Similarly, (5) can be established.

In closing it should also be remarked that [1] contains a version of the Fubini Theorem for vector measures. The hypotheses of this theorem are so lengthy that it may be difficult to use the theorem in practice.

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