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Sharp Bertini Theorem for Plane Curves over Finite Fields

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Abstract. We prove that if C is a reflexive smooth plane curve of degree d defined over a finite field $\mathbb{F}_q$ with $d \le q+1$, then there is an $\mathbb{F}_q$-line L that intersects C transversely. We also prove the same result for non-reflexive curves of degree $p+1$ and $2p+1$ when $q = p^r$.

1 Introduction

A classical theorem of Bertini states that if X is a smooth quasi-projective variety in $\mathbb{P}^n$ defined over an infinite field k, then a general hyperplane section of X is smooth. Specializing to the case when $C \subseteq \mathbb{P}^2$ is a smooth plane curve, it follows that there exists a line L (defined over k) such that L intersects C transversely, meaning that $C \cap L$ consists of d distinct geometric points where $d = \deg(C)$. But when $k = \mathbb{F}_q$ is a finite field, it is possible to have a smooth plane curve $C \subseteq \mathbb{P}^2$ such that every line L defined over $\mathbb{F}_q$ is tangent to the curve C (see Example 2.2). Moreover, Poonen's Bertini Theorem [8, Theorem 1.2] guarantees that such smooth curves, where all the $\mathbb{F}_q$-lines are tangent, do exist in every sufficiently large degree (see Example 2.3). With a view toward an effective version of Poonen's theorem, one can ask the following question.

Question 1.1 *Suppose $C \subseteq \mathbb{P}^2$ is a smooth plane curve defined over $\mathbb{F}_q$. Let $d = \deg(C)$. What conditions on q and d will ensure that there is a line $L \subseteq \mathbb{P}^2$ defined over $\mathbb{F}_q$ such that L meets C transversely?*

Let us call L a *good line* if L meets C transversely. We expect that if q is large with respect to d, then good lines will exist. Indeed, if $q \ge d(d-1)$, then the dual curve C^* cannot be space-filling, *i.e.*, $C^*(\mathbb{F}_q) \ne (\mathbb{P}^2)^*(\mathbb{F}_q)$. This is because $\deg(C^*) \le d(d-1) \le q$ and a curve of degree of at most q cannot go through all the points of $(\mathbb{P}^2)^*(\mathbb{F}_q)$. Any point in $(\mathbb{P}^2)^*(\mathbb{F}_q) \smallsetminus C^*(\mathbb{F}_q)$ represents a good line $L \subseteq \mathbb{P}^2$ defined over $\mathbb{F}_q$. A generalization of this observation to higher dimensions is proved by Ballico [1, Theorem 1].

In this paper, we improve the quadratic bound $q \ge d(d-1)$ to the linear bound $q \ge d-1$.

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Theorem 1.2 *If C is a smooth reflexive plane curve defined over $\mathbb{F}_q$ with* $\deg(C) \leq q+1$, *then there is an $\mathbb{F}_q$-line L such that L intersects C transversely.*

The theorem is sharp in a sense that the statement cannot be improved to $q \geq d-2$. There is a counter-example when $q = d-2$ (see Example 2.2). The "reflexive" assumption on C is same as saying that C has finitely many flex points (see Section 2). As a natural follow-up, we can ask the following question.

Question 1.3 *Does Theorem 1.2 hold when C is non-reflexive?*

In Section 3, we prove a partial result in this direction.

Theorem 1.4 *Let C be a smooth non-reflexive plane curve of degree $p+1$ or $2p+1$ defined over $\mathbb{F}_q$ where $q = p^r$ with $r \geq 2$. Then there is an $\mathbb{F}_q$-line L such that L intersects C transversely.*

Finally, in the last section of the paper (Section 4), we focus exclusively on Frobenius non-classical curves, which are non-reflexive curves of a special kind. As we will see, Question 1.3 in this case is equivalent to a statement about collinear $\mathbb{F}_q$-points on the curve.

Conventions In order to avoid various pathologies, we will assume throughout the paper that the characteristic of the field is $p > 2$.

2 Reflexive Curves

In this section we review the theory of reflexive plane curves and prove Theorem 1.2. If C is a plane curve defined over a field k, we can consider the Gauss map $\varphi: C \to (\mathbb{P}^2)^*$ that associates with each smooth point p of C its tangent line. The dual curve C^* is defined to be the closure of $\varphi(C)$ inside $(\mathbb{P}^2)^*$. By looking at the Gauss map for the dual curve, we get $\varphi': C^* \to C^{**}$. In what follows, we will identify $\mathbb{P}^2$ and $(\mathbb{P}^2)^{**}$.

Definition 2.1 The curve C is called *reflexive* if $C = C^{**}$ and $\varphi' \circ \varphi: C \to C^{**}$ is the identity map.

A theorem of Wallace [9] asserts that C is reflexive if and only if φ is separable. As a result, all smooth plane curves in characteristic zero are reflexive. Recall that a point P of C is called a *flex point* if the tangent line at P meets the curve C at P with multiplicity at least 3. When $\operatorname{char}(k) = p > 2$, we have the following characterization: C is reflexive if and only if C has finitely many flex points [7, Proposition 1.5].

Before we prove Theorem 1.2, here are some counter-examples of smooth curves C where all the lines defined over $\mathbb{F}_q$ are tangent to C (so that no good line exists).

Example 2.2 Let C be a smooth plane curve with $\deg(C) = q+2$ such that $\#C(\mathbb{F}_q) = \#\mathbb{P}^2(\mathbb{F}_q)$. Such curves exist, and have been extensively studied by Homma and Kim [6]. For such a curve C, every $\mathbb{F}_q$-line L intersects C at $q+2$ points (counted with multiplicity). But $q+1$ of these points are already accounted by the points of $L(\mathbb{F}_q) =$

$\mathbb{P}^1(\mathbb{F}_q)$. Thus, the residual intersection multiplicity results from L being tangent to C at one of the $\mathbb{F}_q$-points.

Example 2.3 Fix a finite field $\mathbb{F}_q$. Let $\{L_1, \dots, L_{q^2+q+1}\}$ be all the $\mathbb{F}_q$-lines in the plane. Pick distinct (geometric) points $P_i \in L_i$ for each i. The condition that C is tangent to L_i at P_i is a statement about vanishing of the first few coefficients in the Taylor expansion at these finitely many points. By applying Poonen's Bertini theorem with Taylor conditions [8, Theorem 1.2], there exists some d_0 such that for every $d \geq d_0$, there exists a smooth plane curve $C \subseteq \mathbb{P}^2$ of degree d such that L_i is tangent to C at P_i. In particular, all $\mathbb{F}_q$-lines $L \subseteq \mathbb{P}^2$ are tangent to C. A closer inspection of the proof reveals that the integer d_0 is in the order of q^2 (essentially because we imposed $q^2 + q + 1$ local conditions).

We will now prove the main theorem of this paper.

Theorem 1.2 *If C is a smooth reflexive plane curve defined over $\mathbb{F}_q$ with* $\deg(C) \leq q + 1$*, then there is an $\mathbb{F}_q$-line L such that L intersects C transversely.*

Proof Let Φ be the Frobenius map, defined on points by

$$\Phi\big([X:Y:Z]\big) = [X^q : Y^q : Z^q].$$

We will write $T_P(C)$ for the tangent line to C at a (geometric) point P. Set

$$N = \#\big\{P \in C(\overline{\mathbb{F}_q}) : \Phi(P) \in T_P(C)\big\},$$

which is finite, because C is reflexive [4]. Let $d = \deg(C)$. The following inequality is proved in [5, Theorem 8.41]:

$$(*) \qquad 2 \cdot \#C(\mathbb{F}_q) + N \leq d(q + d - 1)$$

under the assumption that C has finitely many flex points and that characteristic of the field is $p > 2$. This is the step where we use the hypothesis that C is reflexive.

Assume, to the contrary, that every $\mathbb{F}_q$-line is tangent to the curve C at some (geometric) point. Let us divide these lines into two groups: if L is tangent to C at an $\mathbb{F}_q$-rational point, we will call L a *rational tangent*. Otherwise, we will call L a *special tangent*. Since every $\mathbb{F}_q$-line is tangent to C, and there are $q^2 + q + 1$ lines defined over $\mathbb{F}_q$, we get

$$\#\{\text{rational tangents}\} + \#\{\text{special tangents}\} = q^2 + q + 1$$

and

$$\#\{\text{rational tangents}\} \leq \#C(\mathbb{F}_q)$$

Now, if L is a special tangent, it is tangent to the curve C at a non-$\mathbb{F}_q$-point P. Then L is also tangent to C at $P, \Phi(P), \Phi^2(P), \dots, \Phi^{e-1}(P)$ where $e = [k(P):\mathbb{F}_q]$ is the degree of the point P. Since $e \geq 2$, the line L contributes at least 2 elements to N. As a result,

$$2 \cdot \#\{\text{special tangents}\} \leq N.$$

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Combining all the inequalities above, we obtain that

$$\begin{aligned} q^2+q+1 &= \#\{\text{rational tangents}\} + \#\{\text{special tangents}\} \\ &\leq \#C(\mathbb{F}_q) + \frac{N}{2} \leq \frac{1}{2}d(q+d-1) \qquad \text{(using } (*)) \\ &\leq \frac{1}{2}(q+1)\big(q+(q+1)-1\big) = \frac{1}{2}(q+1)(2q) = q^2+q, \end{aligned}$$

which is a contradiction. ∎

When $q = p$ is a prime, every smooth curve of degree at most p is reflexive. Moreover, Pardini [7, Proposition] has shown that every smooth non-reflexive curve of degree $p+1$ (over any field of characteristic p) is projectively equivalent to the curve given by the equation $xy^p + yz^p + zx^p = 0$. For this curve, many good lines exist. For instance, take two $\mathbb{F}_p$-points on the curve, and join them with a line L. Then L will intersect C transversely.

Consequently, we deduce the result for all smooth plane curves over $\mathbb{F}_p$ where p is prime.

Corollary 2.4 *If C is a smooth plane curve defined over $\mathbb{F}_p$ with $\deg(C) \leq p+1$, where p is a prime, then there is an $\mathbb{F}_p$-line L such that L intersects C transversely.*

3 Non-reflexive Curves

In this section, we will restrict attention to non-reflexive curves and prove Theorem 1.4.

Let $C \subseteq \mathbb{P}^2$ be a smooth non-reflexive curve defined over $\mathbb{F}_q$ with $q = p^r$ where $r \geq 2$. Pardini [7, Corollary 2.4] has shown that C is defined by an equation of the form:

$$a^p x + b^p y + c^p z = 0$$

where $a, b, c \in \mathbb{F}_q[x, y, z]$ are homogeneous polynomials of degree $t \geq 1$. In particular, $\deg(C) = tp + 1$.

We establish a Bertini-type theorem for the case $t = 1$ and $t = 2$.

Theorem 1.4 *Let C be a smooth non-reflexive plane curve of degree $p+1$ or $2p+1$ defined over $\mathbb{F}_q$ where $q = p^r$ with $r \geq 2$. Then there is an $\mathbb{F}_q$-line L such that L intersects C transversely.*

Proof If $\deg(C) = p+1$, then C is projectively equivalent to the curve given by the equation $xy^p + yz^p + zx^p = 0$, for which many good lines L exist (see the discussion before Corollary 2.4). For the rest of the proof, we will assume that $\deg(C) = 2p+1$. Since C is non-reflexive, by [7, Corollary 4.3] the degree of the dual curve is

$$\deg(C^*) = \frac{d(d-1)}{p} = \frac{(2p+1)(2p)}{p} = 4p+2.$$

For $p \geq 5$, we observe that $\deg(C^*) = 4p + 2 \leq p^2 \leq q$, so C^* cannot contain all of $(\mathbb{P}^2)^*(\mathbb{F}_q)$, and hence any point $L \in (\mathbb{P}^2)^*(\mathbb{F}_q) \smallsetminus C^*(\mathbb{F}_q)$ will be a desired line that intersects C transversely.

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When $p = 3$, the inequality $\deg(C^*) = 4p + 2 = 14 \le p^r = q$ still holds for $r \ge 3$. The only case that requires a separate analysis is $(p, r) = (3, 2)$, which corresponds to degree $2 \cdot 3 + 1 = 7$ curve defined over $\mathbb{F}_{3^2} = \mathbb{F}_9$. The rest of the proof is devoted to studying this remaining case.

Let C be a smooth non-reflexive curve of degree 7 defined over $\mathbb{F}_9$. Assume, to the contrary, that all the lines defined over $\mathbb{F}_9$ are tangent to C. Following the same terminology used in the proof of Theorem 1.2, we call L a *rational tangent* if L is tangent to C at some $\mathbb{F}_9$-point. Otherwise, L is called a *special tangent*. Since C is non-reflexive, each tangent line L must intersect the curve at the tangency point with multiplicity at least 3 ([7, Proposition 1.5]). Then the following hold.

(i) If L is a rational tangent, then $L \cap C$ contains at most five $\mathbb{F}_9$-points.
(ii) If L is a special tangent, then $L \cap C$ contains a conjugate pair of $\mathbb{F}_{81}$-points and a single $\mathbb{F}_9$-point. In symbols, $L \cap C = \{Q, Q^\sigma, P\}$, where $Q \in \mathbb{P}^2(\mathbb{F}_{81}) \setminus \mathbb{P}^2(\mathbb{F}_9)$ and $P \in \mathbb{P}^2(\mathbb{F}_9)$.

Consider the following incidence correspondence of points and lines:

$$\mathcal{I} = \left\{ (P, L) : L \in (\mathbb{P}^2)^*(\mathbb{F}_9) \text{ and } P \in (C \cap L)(\mathbb{F}_9) \right\}.$$

Each $P \in C(\mathbb{F}_9)$ is contained in $q + 1 = 10$ different $\mathbb{F}_9$-lines. Therefore, $\#\mathcal{I} = \#C(\mathbb{F}_9) \cdot 10$. On the other hand, using (i) and (ii) above, each special tangent L contributes 1 point, while each rational tangent L contributes at most 5 points to $\#\mathcal{I}$. Thus, $\#\mathcal{I} \le S + 5R$ where S and R are the number of special and rational tangents, respectively. We deduce that $\#C(\mathbb{F}_9) \cdot 10 \le S + 5R$. Since $\#C(\mathbb{F}_9) \ge R$, we get $10R \le S + 5R$, which implies $5R \le S$. Since $S + R = 9^2 + 9 + 1 = 91$, we have $5(91 - S) \le S$, so that $S \ge \frac{5 \cdot 91}{6} = 75.8333\ldots$. Thus, $S \ge 76$.

Next, take any rational tangent L_0. Every special tangent line intersects L_0 in one of its ten $\mathbb{F}_9$-points. Since $\frac{S}{10} \ge \frac{76}{10} > 7$, there exists $P_0 \in L_0(\mathbb{F}_q)$ such that there are at least 8 special tangent lines that pass through P_0. By looking at the ten $\mathbb{F}_9$-lines passing through P_0, we can estimate $\#C(\mathbb{F}_9)$ as follows. Each of the 8 special tangents will contribute at most 1 rational point, while the remaining (at most 2) rational tangents will contribute at most 5 rational points. Thus, one gets $\#C(\mathbb{F}_9) \le 8 + 2 \cdot 5 = 18$. Consider the incidence correspondence

$$\mathcal{J} = \left\{ (P, L) : L \text{ is a special tangent and } P \in (C \cap L)(\mathbb{F}_9) \right\}.$$

By (i) above, every special tangent contains exactly one $\mathbb{F}_9$-point of C, so that $\#\mathcal{J} = S$. As a result,

$$S = \#\mathcal{J} = \sum_{P \in C(\mathbb{F}_9)} \#\{\text{special tangents passing through } P\}.$$

Since

$$\frac{S}{\#C(\mathbb{F}_9)} \ge \frac{76}{18} > 4,$$

there exists a point $P \in C(\mathbb{F}_9)$ such that at least 5 special tangents pass through P. Consider the corresponding line P^* in the dual space $(\mathbb{P}^2)^*$, which consists of all lines passing through P. Let us look at the intersection of the line P^* and the dual curve C^* inside $(\mathbb{P}^2)^*$. The intersection has all the ten $\mathbb{F}_9$-points of P^*, since all the $\mathbb{F}_9$-lines are tangent to C. However, each of the special tangents is bitangent to C, so it

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is a node in C^*, and hence will contribute 2 to the intersection. It follows that $P^* \cap C^*$ has at least $5 \cdot 2 + 5 = 15$ intersections, contradicting the fact that $\deg(C^*) = 14$. ■

Remark 3.1 As we saw above, the hardest part of the proof is the case $p = 3$. This answers a question of Felipe Voloch, who asked, in a private communication, whether or not there exists a transverse line for a degree 7 smooth non-reflexive curve defined over $\mathbb{F}_9$. The small primes still persist when we try to extend Theorem 1.3 to non-reflexive curves of degree $3p + 1$. Indeed, if C is a smooth non-reflexive curve of degree $3p + 1$, then

$$\deg(C^*) = \frac{(3p+1)(3p)}{p} = 9p + 3 \leq p^2 \leq q$$

for $p \geq 11$; the usual argument shows that $(C^*)(\mathbb{F}_q) \neq (\mathbb{P}^2)^*(\mathbb{F}_q)$, implying that good lines exist for $p \geq 11$. However, the main difficulty lies with the primes $p = 3, 5, 7$.

4 Connection to Frobenius Non-classical Curves

In this section, we observe the implications of a Bertini-type theorem for a special class of non-reflexive curves, known as Frobenius non-classical curves.

Definition 4.1 Let $C \subseteq \mathbb{P}^2$ be a smooth plane curve defined over $\mathbb{F}_q$. Then C is called *Frobenius non-classical* if $\Phi(P) \in T_P(C)$ for every P, where $T_P(C)$ is the tangent line to C at the point P, and $\Phi: \mathbb{P}^2 \to \mathbb{P}^2$ is the q-th power Frobenius map.

We should remark that the usual definition of Frobenius non-classical is stated differently (by looking at the order sequence of C), but the definition given above is equivalent in the case of smooth plane curves [4, Proposition 1].

Example Let C be the curve defined over $\mathbb{F}_{q^2}$ by the equation

$$x^{q+1} + y^{q+1} + z^{q+1} = 0.$$

It can be checked that C is a smooth Frobenius non-classical curve for $\mathbb{F}_{q^2}$.

If C is a smooth Frobenius non-classical plane curve of degree d defined over $\mathbb{F}_q$ where $q = p^r$, then it is known that C is non-reflexive [4, Proposition 1] and $\sqrt{q} + 1 \leq d \leq \frac{q-1}{q'-1}$, where q' is the generic order of contact of the curve with a tangent line [4, Propositions 5 and 6]. In particular, $\deg(C) \leq q - 1$ always holds. So Question 1.3 is equivalent to the following.

Question 4.2 *If C is a smooth Frobenius non-classical plane curve defined over $\mathbb{F}_q$, does there exist an $\mathbb{F}_q$-line L such that L intersects C transversely?*

The existence of such a line L can be verified for the curve $x^{q+1} + y^{q+1} + z^{q+1} = 0$, and more generally, for the curve given by the equation

$$x^{q^{n-1}+\cdots+q+1} + y^{q^{n-1}+\cdots+q+1} + z^{q^{n-1}+\cdots+q+1} = 0,$$

where $n \geq 2$. These curves are indeed smooth and Frobenius non-classical with respect to the field $\mathbb{F}_{q^n}$ [4, Theorem 2].

If the Question 4.2 has an affirmative answer, then it implies that there is a line L defined over $\mathbb{F}_q$ such that $L \cap C$ consists of $d = \deg(C)$ distinct $\mathbb{F}_q$-rational points. Indeed, if L contains a non-$\mathbb{F}_q$-point Q, then we observe that $Q, \Phi(Q) \in T_Q(C)$ (since C is Frobenius non-classical) and $Q, \Phi(Q) \in L$ (as L is defined over $\mathbb{F}_q$), implying that $L = T_Q(C)$ is a tangent line. Thus, any good (transverse) line L intersects C at $\deg(C)$ distinct $\mathbb{F}_q$-points. This allows us to reformulate Question 4.2 as follows.

Question 4.3 *If C is a smooth Frobenius non-classical plane curve defined over $\mathbb{F}_q$, then does C have $d = \deg(C)$ many $\mathbb{F}_q$-rational points on a line?*

Question 4.3 is motivated by the fact that Frobenius non-classical curves have many $\mathbb{F}_q$-points. In fact, the $\mathbb{F}_q$-points on these curves have been used in [2, 3] to construct certain complete arcs in the plane. Moreover, the following theorem due to Hefez and Voloch [4, Theorem 1] gives the exact the number of $\mathbb{F}_q$-points on *any* smooth Frobenius non-classical plane curve.

Theorem 4.4 (Hefez–Voloch) *If $C \subseteq \mathbb{P}^2$ is a smooth Frobenius non-classical curve of degree d defined over $\mathbb{F}_q$, then $\#C(\mathbb{F}_q) = d(q - d + 2)$.*

We can apply Theorem 4.4 directly to get an estimate on the number of collinear points of C. Consider the incidence correspondence $\{(P, L) : L \in (\mathbb{P}^2)^*(\mathbb{F}_q) \text{ and } P \in (L \cap C)(\mathbb{F}_q)\}$. Since each $\mathbb{F}_q$-point P is contained in $q + 1$ lines,

$$\#C(\mathbb{F}_q)(q+1) = \sum_{P \in C(\mathbb{F}_q)} (q+1) = \sum_L \#(L \cap C)(\mathbb{F}_q).$$

The sum on the right runs over all $q^2 + q + 1$ lines. Thus, an $\mathbb{F}_q$-line on average contains

$$\frac{\#C(\mathbb{F}_q)(q+1)}{q^2+q+1} = \frac{d(q-d+2)(q+1)}{q^2+q+1} > \frac{d(q-d+2)}{q+1} > d\Big(1 - \frac{d}{q+1}\Big)$$

$\mathbb{F}_q$-points of C. As q gets larger, this number approaches d. This heuristic suggests that Question 4.3 may have an affirmative answer.

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