J. Austral. Math. Soc. (Series A) 54 (1993), 334–351

THE SPECTRAL THEOREM FOR WELL-BOUNDED OPERATORS

IAN DOUST and QIU BOZHOU

(Received 13 December 1990; revised 26 March 1991)

Communicated by P. G. Dodds

Abstract

Well-bounded operators are those which possess a bounded functional calculus for the absolutely continuous functions on some compact interval. Depending on the weak compactness of this functional calculus, one obtains one of two types of spectral theorem for these operators. A method is given which enables one to obtain both spectral theorems by simply changing the topology used. Even for the case of well-bounded operators of type (B), the proof given is more elementary than that previously in the literature.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 47 B 40, secondary 46 A 50, 46 H 30, 47 A 60.

1. Introduction

It is now 30 years since Smart introduced the class of well-bounded operators to provide a spectral theory for operators whose eigenvalue expansions may only converge conditionally. The original papers of Smart and Ringrose [11, 8] considered operators on a reflexive Banach space which possessed a functional calculus smaller than that for self-adjoint operators on a Hilbert space; namely ones with a functional calculus for the absolutely continuous functions on some compact interval of the real line. That is, they studied operators T for which there exist constants $a, b, K \in \mathbb{R}$ such that

$$\|p(T)\| \le K \left\{ |p(a)| + \int_a^b |p'(t)|\, dt \right\}$$

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https://doi.org/10.1017/S1446788700031827 Published online by Cambridge University Press

for all polynomials p. Operators which satisfied this condition were called *well-bounded.* They showed that such a functional calculus gave rise to a type of spectral resolution of the identity for the operator which we now call a spectral family. Later, Ringrose [9] studied operators with this type of functional calculus on general Banach spaces. The situation here is rather less satisfactory as, although one obtains some type of spectral resolution (known in this case as a decomposition of the identity), it consists of projections which act on the dual of the Banach space and in general it is not uniquely determined by the well-bounded operator. Furthermore, the exact conditions for a family of projections to be a decomposition of the identity are somewhat cumbersome. An alternative approach to that of Smart and Ringrose was introduced by Sills [10] who used a technique that involved first extending the AC functional calculus to the second dual, a space which contains idempotent elements.

In 1971 Berkson and Dowson [3] considered classes of well-bounded operators whose decomposition of the identity $\{E(\lambda)\}$ consists of projections which are the adjoints of projections $\{F(\lambda)\}$ on the original Banach space. Those well-bounded operators for which the family $\{F(\lambda)\}$ has certain continuity properties were said to be of type (B), and it is these operators which have become most important in applications of the theory. The well-bounded operators of type (B) were later characterised as being those for which the functional calculus for T is "weakly compact" (see [3, Theorem 4.2; 12]). This means that, for all x in the Banach space X, the map $AC \to X$, $f \mapsto f(T)x$ is compact in the weak topology; or equivalently, that the algebra homomorphism $f \mapsto f(T)$ is compact in the weak operator topology. Subsequently, the possession of a weakly compact AC functional calculus has often been taken as the definition of well-bounded operators of type (B). In recent years several proofs of the spectral theorem for these operators have been given which show that the possession of such a weakly compact functional calculus allows one to define a unique spectral family $\{E(\lambda)\}$ on X for which $T = \int_J^{\oplus} \lambda \, dE(\lambda)$. A consequence of this theorem is that one can extend the AC functional calculus for T to one for all functions g of bounded variation by $g(T) = \int_J^{\oplus} g(\lambda) \, dE(\lambda)$. A good survey of the more recent work in this area may be found in [2].

The aim of this paper is to give new proofs of the spectral theorems for both well-bounded operators of type (B) and general well-bounded operators, proofs which stress the role of the different types of compactness one has in the two cases. We believe that these proofs are somewhat simpler than those in the literature, at least to the extent that our methods employ a unified approach to the two situations and do so without requiring large amounts of mathematical machinery.

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The methods in this paper have their origins in the proofs of the spectral theorem for well-bounded operators of type (B) (in particular those of Berkson and Gillespie [4, 2]). By considering suitable algebra homomorphisms with certain compactness properties, these methods will allow us to construct either a spectral family or a decomposition of the identity as appropriate.

We shall also comment on the definition of decomposition of the identity which was introduced by Ringrose and the integration theory that such a decomposition of the identity permits. It is not hard to check that one of Ringrose's conditions is redundant in the sense that it is not needed to ensure that the equation

$$\langle Tx, x^*\rangle = b\langle x, x^*\rangle - \int_a^b \langle x, E(\lambda)x^*\rangle\, d\lambda, \qquad (x \in X,\ x^* \in X^*),$$

defines a well-bounded operator. The extra condition, a weak type of right continuity, does however allow one to ensure uniqueness of the decomposition of the identity in certain situations, such as when the Banach space is reflexive. Without such a condition even the identity operator fails to have a unique decomposition of the identity. We propose here an alternative continuity condition which is perhaps more natural in the present setting and which allows one to retain the uniqueness theorems in these important cases.

2. Preliminaries

We shall present here a brief outline of some of the details of the weak topologies and integration theories that we shall need. The main point in this paper is that the difference between well-bounded operators of type (B) and general well-bounded operators lies in the type of compactness that one has for the functional calculus.

Throughout X and Y will denote complex Banach spaces. Real spaces can be covered by changing $\mathbb{C}$ to $\mathbb{R}$ in appropriate places in the proofs. The dual of X will be denoted by X^*, the algebra of all bounded operators on X by $B(X)$ and the set of projections (or idempotent operators) on X by $\mathrm{Proj}(X)$. The Banach algebra of all absolutely continuous functions on the interval $[a, b] \subset \mathbb{R}$ will be denoted by $AC[a, b]$ (or just AC). This forms a subalgebra of the algebra $BV[a, b]$ of all functions of bounded variation on $[a, b]$ equipped with the norm

$$\|f\|_{BV} = |f(b)| + \operatorname*{var}_{[a,b]} f.$$

The most usual weak topology on $B(X)$ is the weak operator topology, which is generated by the linear functionals $\varphi(A) = \langle Ax, x^*\rangle$ for $x \in X$

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and $x^* \in X^*$. It is well-known that if X is reflexive then the closed unit ball of $B(X)$ is compact in the weak operator topology, and so any Banach algebra homomorphism mapping in $B(X)$ is compact in this topology. On the space $B(X^*)$ one can also consider the weak-$*$ operator topology which is the locally convex topology generated by the functionals $\varphi(A) = \langle x, Ax^* \rangle$ for $x \in X$ and $x^* \in X^*$. The following lemma can be proved by adapting your favourite proof that for any Hilbert space $\mathscr{H}$ the unit ball of $B(\mathscr{H})$ is weak operator topology compact.

LEMMA 2.1. *Let X be any Banach space. Then the closed unit ball of $B(X^*)$ is compact in the weak-$*$ operator topology.*

We shall need the following corollary of a theorem of Barry in order to show that certain nets of operators converge in the strong operator topology on $B(X)$. A net $\{T_\alpha\}_{\alpha \in A}$ of operators on X is said to be *naturally ordered* if $T_\alpha T_\beta = T_\beta T_\alpha = T_\alpha$ whenever $\beta > \alpha$.

LEMMA 2.2. *Let $\{T_\alpha\}_{\alpha \in A}$ be a naturally ordered net of operators on X and suppose that there exists a weak operator topology compact subset $S \subset B(X)$ such that $T_\alpha \in S$ for all $\alpha \in A$. Then $\{T_\alpha\}_{\alpha \in A}$ converges in the strong operator topology.*

PROOF. Suppose that $x \in X$ and $\alpha \in A$. Then as S is weak operator topology compact, there exists a subnet $\{T_\gamma\}_{\gamma \in \Gamma}$ of $\{T_\alpha\}_{\alpha \in A}$ with weak operator limit T say. Thus, for all $x^* \in X^*$, $\langle T_\gamma x, x^* \rangle \to \langle Tx, x^* \rangle$, and so, by the properties of subnets, Tx lies in the weak closure of $\{T_\beta x : \beta \geq \alpha\}$. In other words

$$Tx \in \bigcap_\alpha \text{wk-cl}\{T_\beta x : \beta \geq \alpha\}.$$

A theorem of Barry [1] shows that the non-emptiness of this set is a sufficient condition for the net $\{T_\alpha\}_{\alpha \in A}$ to converge in the strong operator topology.

We shall show that when the AC functional calculus for T is weak operator topology compact, then T will possess a type of diagonalization called a spectral family.

DEFINITION 2.3. A *spectral family of projections* on a Banach space X is a projection-valued function $E : \mathbb{R} \to \text{Proj}(X)$ such that

(i) E is right continuous in the strong operator topology and has a strong left hand limit at each point in $\mathbb{R}$;
(ii) E is uniformly bounded, that is there exists $K < \infty$ such that $\|E(\lambda)\| < K$ for all $\lambda \in \mathbb{R}$;
(iii) $E(\lambda)E(\mu) = E(\mu)E(\lambda) = E(\min\{\lambda, \mu\})$ for all λ, $\mu \in \mathbb{R}$;

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(iv) $E(\lambda) \to 0$ (respectively $E(\lambda) \to I$) in the strong operator topology as $\lambda \to -\infty$ (respectively $\lambda \to \infty$).

If $E(\lambda) = 0$ for all $\lambda < a \in \mathbb{R}$ and $E(\lambda) = I$ for all $\lambda \geq b \in \mathbb{R}$, then we say that E is *concentrated on* $[a, b]$.

Spectral families possess a well-developed integration theory. Suppose that $\{E(\lambda)\}_{\lambda \in \mathbf{R}}$ is a spectral family concentrated on the compact interval $J = [a, b]$. Let $\mathscr{P}$ denote the directed set of all partitions of $[a, b]$ (partially ordered by inclusion). If $g \in BV[a, b]$, the algebra of functions of bounded variation on $[a, b]$, then the sums

$$\mathscr{S}(g, \Lambda) = g(a)E(a) + \sum_{\lambda_j \in \Lambda} g(\lambda_j)(E(\lambda_j) - E(\lambda_{j-1}))$$

converge as the partition $\Lambda \in \mathscr{P}$ gets finer. This limit is denoted by $\int_J^{\oplus} g(\lambda)\, dE(\lambda)$. The map $g \mapsto \int_J^{\oplus} g(\lambda)\, dE(\lambda)$ is a continuous Banach algebra homomorphism from $BV[a, b]$ to $B(X)$ (and this homomorphism turns out to be compact). The details of this theory may be found in [6] or [5].

General well-bounded operators have an integration theory with respect to a family of projections known as a decomposition of the identity.

DEFINITION 2.4. A *decomposition of the identity* (*for* X) is a family of projections $\{E(\lambda)\}_{\lambda \in \mathbf{R}} \subset \mathrm{Proj}(X^*)$ such that

(i) E is concentrated on some compact interval $J = [a, b] \subset \mathbb{R}$;
(ii) $E(\lambda)E(\mu) = E(\mu)E(\lambda) = E(\min\{\lambda, \mu\})$ for all $\lambda, \mu \in \mathbb{R}$;
(iii) E is uniformly bounded;
(iv) for all $x \in X$ and $x^* \in X^*$, the function $\lambda \mapsto \langle x, E(\lambda)x^* \rangle$ is Lebesgue measurable;
(v) for all $x \in X$, the map $\gamma_x : X^* \to L^\infty[a, b]$, $x^* \mapsto \langle x, E(\cdot)x^* \rangle$ is continuous when X^* and $L^\infty[a, b]$ are given their weak-$*$ topologies as duals of X and $L^1[a, b]$ respectively;
(vi) for all $s \in \mathbb{R}$, if the map $\lambda \mapsto E(\lambda)$ has right weak-$*$ operator topology limit at s, then this limit is $E(s)$.

Our definition is slightly different from that originally introduced by Ringrose [9]. He used the following right continuity condition instead of the above condition (vi).

(vi)$'$ for all $x \in X$, $x^* \in X^*$ and $s \in [a, b)$, if the map

$$t \mapsto \int_a^t \langle x, E(\lambda)x^* \rangle\, d\lambda$$

is right differentiable at s, then the value of its right derivative at that point is $\langle x, E(s)x^* \rangle$.

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It is easy to see that condition (vi)$'$ is a stronger condition: suppose that $\{E(\lambda)\}$ satisfies conditions (i)–(v) and condition (vi)$'$, and that $E(\lambda)$ has a weak-$*$ operator topology limit E as $\lambda \to s^+$. Fix $x \in X$ and $x^* \in X^*$. Then $\langle x, (E(\lambda) - E)x^*\rangle \to 0$ as $\lambda \to s^+$. A calculation along the lines of the fundamental theorem of calculus shows that the map $t \mapsto \int_a^t \langle x, (E(\lambda) - E)x^*\rangle\, d\lambda$ is right differentiable at s with right derivative zero. It follows from (vi)$'$ that the value of this right derivative is $\langle x, (E(s) - E)x^*\rangle$. Hence $E(s) = E$ as required. Condition (vi) and Lemma; 2.2 imply that a decomposition of the identity on a reflexive space is always formed from the adjoints of a spectral family.

Given a decomposition of the identity, one may define an operator on X by

$$\langle Tx, x^*\rangle = b\langle x, x^*\rangle - \int_a^b \langle x, E(\lambda)x^*\rangle\, d\lambda, \qquad (x \in X, x^* \in X^*).$$

An induction proof shows that such an operator is necessarily well-bounded. Details of this integration theory and the functional calculus (for T^*) which it produces are given in Section 4.

3. The spectral theorem

Suppose that $[a, b]$ is a compact interval of the real line and that ψ: $AC[a, b] \to B(Y)$ is an algebra homomorphism taking values in the bounded operators on a Banach space Y. For $\lambda \in [a, b)$ and $\delta \in (0, b - \lambda)$, let $\mathscr{F}_{\lambda,\delta}$ be the set of all real-valued functions $f \in AC[a, b]$ such that $f = 1$ on $[a, \lambda]$, $f = 0$ on $[\lambda + \delta, b]$ and f is decreasing on $[\lambda, \lambda + \delta]$. Let $g_{\lambda,\delta}$ denote the element of $\mathscr{F}_{\lambda,\delta}$ which is linear on $[\lambda, \lambda + \delta]$. Heuristically the idea is to produce projections in $B(Y)$ using the "nearly idempotent" functions $g_{\lambda,\delta}$ in $AC(J)$.

Let τ be some topology on $B(Y)$. Define

$$\mathscr{K}_{\lambda,\delta} = \tau\text{-cl}\{\psi(f) : f \in \mathscr{F}_{\lambda,\delta}\} \subset B(Y),$$

$$\mathscr{K}_\lambda = \bigcap_{\delta>0} \mathscr{K}_{\lambda,\delta},$$

$$\mathscr{C}_\lambda = \{S \in B(Y) : S \text{ is a } \tau\text{-cluster point of } \{\psi(g_{\lambda,\delta})\}_{\delta>0}\} \subset \mathscr{K}_\lambda.$$

For $\lambda < a$, let $\mathscr{K}_\lambda = \mathscr{C}_\lambda = \{0\}$ and for $\lambda \geq b$, let $\mathscr{K}_\lambda = \mathscr{C}_\lambda = \{I\}$. The sets $\{\mathscr{K}_\lambda\}_{\lambda \in \mathbf{R}}$ are called the *K-sets* for ψ (relative to τ). In general, some of the K-sets for such an algebra homomorphism may be empty, but this cannot happen if we have some compactness. In what follows, τ will be either the weak operator topology or the weak-$*$ operator topology.

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LEMMA 3.1. *Let Y be a complex Banach space, let Γ be a total linear subspace of Y^*, and let τ denote the weak topology on $B(Y)$ given by the linear functionals $\varphi(A) = \langle Ay, y^*\rangle$ and $y \in Y$ and $y^* \in \Gamma$. Suppose that $\psi: AC[a, b] \to B(Y)$ is a τ-compact algebra homomorphism such that for all $f \in AC[a, b]$, $\psi(f)^*\Gamma \subset \Gamma$. Then the K-sets for ψ relative to τ satisfy the following properties*:

(i) $\varnothing \neq \mathscr{C}_\lambda \subset \mathscr{K}_\lambda \subset \mathrm{Proj}(Y)$ *for all* $\lambda \in \mathbb{R}$;
(ii) *if* $\lambda < \mu$, $E \in \mathscr{K}_\lambda$ *and* $F \in \mathscr{K}_\mu$ *then* $EF = FE = E$;
(iii) *the sets* $\mathscr{K}_\lambda$ *are uniformly bounded*; *that is, there exists a constant* K *such that* $\|E\| \leq K$ *for all* $E \in \bigcup_{\lambda\in\mathbb{R}} \mathscr{K}_\lambda$;
(iv) *for each* $\lambda \in [a, b)$, *all the elements of* $\mathscr{K}_\lambda$ *have the same range.*

PROOF. As each $\mathscr{F}_{\lambda,\delta}$ is non-empty and norm bounded ($\|f\|_{BV} \leq 1$ for all $f \in \mathscr{F}_{\lambda,\delta}$), and ψ is τ-compact, it follows that each set $\mathscr{K}_{\lambda,\delta}$ is a non-empty, τ-compact subset of $B(Y)$. Also, since $\delta_1 < \delta_2$ implies $\mathscr{K}_{\lambda,\delta_1} \subset \mathscr{K}_{\lambda,\delta_2}$, the sets $\{\mathscr{K}_{\lambda,\delta}\}_{\delta>0}$ have the finite intersection property and so by compactness $\mathscr{K}_\lambda$ is non-empty and τ-compact. Let

$$\mathscr{M}_\lambda = \{y \in Y : \psi(f)y = 0 \quad \text{for all } f \in \cup_{\delta>0}(1 - \mathscr{F}_{\lambda,\delta})\}.$$

We shall show that $\mathscr{M}_\lambda$ is the range of every element of $\mathscr{K}_\lambda$. Suppose that $y \in \mathscr{M}_\lambda$ and $E \in \mathscr{K}_\lambda$. If we fix $\delta > 0$, then there exists a net $\{g_\alpha\}_{\alpha\in A}$ in $\mathscr{F}_{\lambda,\delta}$ such that, for all $y^* \in \Gamma$

$$\langle Ey, y^*\rangle = \lim_{\alpha\in A}\langle \psi(g_\alpha)y, y^*\rangle = \lim_{\alpha\in A}\langle (I - \psi(f_\alpha))y, y^*\rangle,$$

where $f_\alpha = 1 - g_\alpha \in 1 - \mathscr{F}_{\lambda,\delta}$. Thus $\langle Ey, y^*\rangle = \langle y, y^*\rangle$. It follows (since Γ is total) that $Ey = y$ and so $y \in \mathrm{Ran}\, E$. Thus $\mathscr{M}_\lambda \subset \mathrm{Ran}\, E$.

Suppose now that $Ey = x$ and that for some $\delta > 0$, $f \in 1 - \mathscr{F}_{\lambda,\delta}$. To show that $x \in \mathscr{M}_\lambda$ we must show that $\psi(f)x = 0$. Fix $\varepsilon > 0$. Then, as f is a continuous increasing function, there exists $\delta_0 > 0$ such that $0 \leq f(t) \leq \varepsilon/2$ for $t \in [\lambda, \lambda + \delta_0]$. Thus, as $E \in \mathscr{K}_{\lambda,\delta_0}$, there exists a net $\{g_\alpha\}_{\alpha\in A}$ in $\mathscr{F}_{\lambda,\delta_0}$ such that $E = \tau\text{-}\lim_{\alpha\in A} \psi(g_\alpha)$. For all $\alpha \in A$,

$$\begin{aligned}\|fg_\alpha\|_{BV} &= \int_a^b |(fg_\alpha)'| = \int_a^b |f'g_\alpha + fg_\alpha'| \\ &\leq \int_\lambda^{\lambda+\delta_0} |f'g_\alpha| + \int_\lambda^{\lambda+\delta_0} |fg_\alpha'| \leq \int_\lambda^{\lambda+\delta_0} |f'| + \frac{\varepsilon}{2}\int_\lambda^{\lambda+\delta_0} |g_\alpha'| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

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as $\int_\lambda^{\lambda+\delta_0}|f'| = \int_\lambda^{\lambda+\delta_0} f' = f(\lambda+\delta_0) \le \varepsilon/2$. Thus, for all $y^* \in \Gamma$,

$$\begin{aligned}
|\langle \psi(f)x, y^*\rangle| &= \langle \psi(f)Ey, y^*\rangle| = |\langle Ey, \psi(f)^*y^*\rangle| \\
&= \left|\lim_{\alpha\in A}\langle \psi(g_\alpha)y, \psi(f)^*y^*\rangle\right| \quad (\text{since } \psi(f)^*y^* \in \Gamma) \\
&= \left|\lim_{\alpha\in A}\langle \psi(fg_\alpha)y, y^*\rangle\right| \le \sup_{\alpha\in A}\|\psi(fg_\alpha)\|\,\|y\|\,\|y^*\| \\
&\le \|y\|\,\|y^*\|\,\|\psi\|\varepsilon.
\end{aligned}$$

Since Γ is total and ε is arbitrary, $\psi(f)x = 0$ and so $\operatorname{Ran} E \subset \mathscr{M}_\lambda$. This completes the proof that for $E \in \mathscr{K}_\lambda$, $\mathscr{M}_\lambda = \operatorname{Ran} E$. Note that the first part of this proof shows that $Ey = y$ for all $y \in \operatorname{Ran} E$, so $E \in \operatorname{Proj}(Y)$.

Suppose now that $a \le \lambda < \mu < b$, that $E_\lambda \in \mathscr{K}_\lambda$, and that $E_\mu \in \mathscr{K}_\mu$. We shall show that $E_\lambda E_\mu = E_\lambda$. Let $\delta = \mu - \lambda$ and $\rho = b - \mu$. Then, as $E_\lambda \in \mathscr{K}_\lambda \subset \mathscr{K}_{\lambda,\delta}$, there exists a net $\{g_\alpha\}_{\alpha\in A}$ of functions in $\mathscr{F}_{\lambda,\delta}$ such that $E_\lambda = \tau\text{-}\lim_{\alpha\in A}\psi(g_\alpha)$. Similarly, there exists a net $\{h_\beta\}_{\beta\in B}$ of functions in $\mathscr{F}_{\mu,\rho}$ such that $E_\mu = \tau\text{-}\lim_{\beta\in B}\psi(h_\beta)$. Thus, for $y \in Y$ and $y^* \in \Gamma$,

$$\langle E_\lambda E_\mu y, y^*\rangle = \lim_{\alpha\in A}\langle \psi(g_\alpha)E_\mu y, y^*\rangle = \lim_{\alpha\in A}\langle E_\mu y, \psi(g_\alpha)^*y^*\rangle.$$

By the hypothesis, $\psi(g_\alpha)^*y^* \in \Gamma$ for all $\alpha \in A$, so

$$\begin{aligned}
\langle E_\lambda E_\mu y, y^*\rangle &= \lim_{\alpha\in A}\left\{\lim_{\beta\in B}\langle \psi(h_\beta)y,\ \psi(g_\alpha)^*y^*\rangle\right\} \\
&= \lim_{\alpha\in A}\left\{\lim_{\beta\in B}\langle \psi(g_\alpha h_\beta)y, y^*\rangle\right\} \\
&= \lim_{\alpha\in A}\left\{\lim_{\beta\in B}\langle \psi(g_\alpha)y, y^*\rangle\right\}
\end{aligned}$$

as $g_\alpha h_\beta = g_\alpha$ for all $\alpha \in A$ and $\beta \in B$. Thus

$$\langle E_\lambda E_\mu y, y^*\rangle = \langle E_\lambda y, y^*\rangle$$

and so $E_\lambda E_\mu = E_\lambda$. An almost identical proof shows that $E_\mu E_\lambda = E_\lambda$.

To see that the sets are uniformly bounded, it suffices to note that for $\lambda \in [a, b)$ and $\delta \in (0, b-\lambda)$,

$$\mathscr{K}_{\lambda,\delta} \subset \tau\text{-cl}\{\psi(f) : \|f\|_{BV} \le 1\} \subset \{U \in B(Y) : \|U\| \le \|\psi\|\}.$$

This is because the unit ball in $B(Y)$ is closed in the τ topology.

It remains to show that $\varnothing \ne \mathscr{C}_\lambda \subset \mathscr{K}_\lambda$. Fix $\lambda \in [a, b)$. Note that the net $\{\psi(g_{\lambda,\delta})\}_{\delta>0}$ is eventually in each of the τ-compact sets $\mathscr{K}_{\lambda,\delta}$. It must therefore have a τ-cluster point and all such points must lie in the intersection, $\mathscr{K}_\lambda$, of these sets.

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Under the conditions of Lemma 3.1 then, the K-sets for ψ are always nonempty. In general the nets $\{g_{\lambda,\delta}\}_{\delta>0}$ need not have a limit point however, so $\mathscr{K}_\lambda$ will contain more than one projection. However, if τ is the weak operator topology, then the K-sets each contain a unique projection.

LEMMA 3.2. *Under the conditions of Lemma* 3.1, *if* $\Gamma = Y^*$ *(that is,* τ *is the weak operator topology on* $B(Y)$*), then the* K*-sets* $\mathscr{K}_\lambda$ *for* ψ *are all singleton sets.*

PROOF. That $\mathscr{K}_\lambda$ is a singleton if $\lambda < a$ or if $\lambda \geq b$ is true by definition. We shall show that if $\lambda \in [a, b)$, and $\delta \in (0, b-\lambda)$ then $\mathscr{K}_{\lambda,\delta}$ is commutative. This will imply that $\mathscr{K}_\lambda$ is also commutative. Since two commuting projections with the same range must be identical, Lemma 3.1 will imply that each of the K-sets contains just one element.

Suppose that E_1 and E_2 are elements of $\mathscr{K}_{\lambda,\delta}$. Then $E_1 = \tau\text{-}\lim_{\alpha\in A}\psi(g_\alpha)$ and $E_2 = \tau\text{-}\lim_{\beta\in B}\psi(h_\beta)$ for some nets $\{g_\alpha\}_{\alpha\in A}$ and $\{h_\beta\}_{\beta\in B}$ in $\mathscr{F}_{\lambda,\delta}$. Thus if $y \in Y$ and $y^* \in Y^*$,

$$\begin{aligned}
\langle E_1E_2y, y^*\rangle &= \lim_{\alpha\in A}\langle\psi(g_\alpha)E_2y, y^*\rangle = \lim_{\alpha\in A}\langle E_2y, \psi(g_\alpha)^*y^*\rangle \\
&= \lim_{\alpha\in A}\left\{\lim_{\beta\in B}\langle\psi(h_\beta)y, \psi(g_\alpha)^*y^*\rangle\right\} = \lim_{\alpha\in A}\left\{\lim_{\beta\in B}\langle\psi(g_\alpha h_\beta)y, y^*\rangle\right\} \\
&= \lim_{\alpha\in A}\left\{\lim_{\beta\in B}\langle\psi(h_\beta)\psi(g_\alpha)y, y^*\rangle\right\} = \lim_{\alpha\in A}\langle E_2\psi(g_\alpha)y, y^*\rangle \\
&= \lim_{\alpha\in A}\langle\psi(g_\alpha)y, E_2^*y^*\rangle = \langle E_1y, E_2^*y^*\rangle \\
&= \langle E_2E_1y, y^*\rangle.
\end{aligned}$$

Thus $E_1E_2 = E_2E_1$ and so $\mathscr{K}_{\lambda,\delta}$ is commutative.

Actually, the same proof can be used to show a little more.

LEMMA 3.3. *Let the* K*-sets for* ψ *be as in Lemma* 3.1 *and suppose that* $\lambda \in \mathbb{R}$. *Then if there exists* $E \in \mathscr{K}_\lambda$ *such that* $E^*\Gamma \subset \Gamma$ *then* $\mathscr{K}_\lambda$ *is a singleton.*

LEMMA 3.4. *Let* ψ, τ *and the sets* $\mathscr{C}_\lambda$ *be as in Lemma* 3.1. *For all* $\lambda \in [a, b)$, *let* $E(\lambda) \in \mathscr{C}_\lambda$ *be a* τ*-cluster point of* $\{\psi(g_{\lambda,\delta})\}_{\delta>0}$. *Then for all* $y \in Y$ *and* $y^* \in \Gamma$, *and all* $s \in [a, b)$,

(i) *the map* $\lambda \mapsto \langle E(\lambda)y, y^*\rangle$ *is a bounded measurable function*;

(ii) *for all* $f \in AC[a, b]$,

$$\langle\psi(f)y, y^*\rangle = f(b)\langle y, y^*\rangle - \int_a^b f'(\lambda)\langle E(\lambda)y, y^*\rangle\, d\lambda;$$

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(iii) *if* $E(\lambda)$ *has right* τ*-limit as* $\lambda \to s^+$, *then this limit is* $E(s)$;
(iv) *if the map* $t \mapsto \int_a^t \langle E(\lambda)y, y^*\rangle\, d\lambda$ *is right differentiable at* s, *then the value of this right derivative is* $\langle E(s)y, y^*\rangle$.

PROOF. For all $y \in Y$ and $y^* \in \Gamma$, the mapping $f \mapsto \langle \psi(f)y, y^*\rangle$ is a continuous linear functional on $AC(J)$. Since $AC(J)$ is linearly isometric to $L^1(J) \oplus \mathbb{C}$, there must be some $\varphi_{y,y^*} \in L^\infty(J)$ and $c_{y,y^*} \in \mathbb{C}$ such that

$$\langle \psi(f)y, y^*\rangle = \int_J f'(t)\varphi_{y,y^*}(t)\, dt + c_{y,y^*} f(b)$$

for all $f \in AC(J)$. Taking $f \equiv 1$ shows that $c_{y,y^*} = \langle y, y^*\rangle$. Fix $\lambda \in [a, b)$. Since $E(\lambda) \in \mathscr{E}_\lambda$, there exists a net $\{\delta_\nu\}_{\nu \in A}$ converging to zero such that $E(\lambda) = \tau\text{-}\lim_{\nu \in A} \psi(g_{\lambda,\delta_\nu})$. It is easy to see that

$$\langle \psi(g_{\lambda,\delta_\nu})y, y^*\rangle = -1/\delta_\nu \int_\lambda^{\lambda+\delta_\nu} \varphi_{y,y^*}(t)\, dt.$$

The Lebesgue differentiation theorem implies that the right hand side of the above equation converges to $-\varphi_{y,y^*}(\lambda)$ almost everywhere as $\delta_\nu \to 0^+$. Thus for almost all $\lambda \in [a, b)$, $\varphi_{y,y^*}(\lambda) = -\langle E(\lambda)y, y^*\rangle$. Since φ_{y,y^*} is essentially bounded and measurable, it is clear that the map $\lambda \mapsto \langle E(\lambda)y, y^*\rangle$ also has this property (and the uniform boundedness of the K-sets for ψ guarantees that this map is actually bounded and not just essentially bounded). We have then that

$$\langle \psi(f)y, y^*\rangle = f(b)\langle y, y^*\rangle - \int_a^b f'(\lambda)\langle E(\lambda)y, y^*\rangle\, d\lambda$$

for all $f \in AC(J)$.

Suppose now that $E(\lambda)$ has right τ-limit E at s. Fix $y \in Y$ and $y^* \in \Gamma$ and define $h(\lambda) = \langle (E(\lambda) - E)y, y^*\rangle$. Clearly $h(\lambda) \to 0$ as $\lambda \to s^+$. For $\delta > 0$, let

$$m_\delta = \inf_{\lambda \in (s, s+\delta)} h(\lambda) \quad \text{and} \quad M_\delta = \sup_{\lambda \in (s, s+\delta)} h(\lambda).$$

Since

$$m_\delta \le \frac{1}{\delta} \int_s^{s+\delta} h(\lambda)\, d\lambda \le M_\delta$$

and m_δ and M_δ both approach zero as $\delta \to 0^+$, the map

$$t \mapsto \int_a^t \langle E(\lambda)y, y^*\rangle\, d\lambda$$

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is right differentiable at s with right derivative $\langle Ey, y^*\rangle$. However, by the definition of $E(s)$, there is (as above), a net $\{\delta_\nu\}_{\nu\in A}$ converging to zero such that

$$\frac{1}{\delta_\nu}\int_s^{s+\delta_\nu}\langle E(\lambda)y, y^*\rangle\, d\lambda = \langle\psi(g_{s,\delta_\nu})y, y^*\rangle \to \langle E(s)y, y^*\rangle$$

and so $E = E(s)$ as required.

The proof of property (iv) is similar.

These lemmas contain most of the proof of the spectral theorem for well-bounded operators of type (B).

Theorem 3.5. *Suppose that $T \in B(X)$. Then T is well-bounded of type* (B) *if and only if there exists a spectral family $\{E(\lambda)\}_{\lambda\in\mathbf{R}}$ of projections on X concentrated on some compact interval $J \subset \mathbb{R}$ such that*

$$T = \int_J^{\oplus} \lambda\, dE(\lambda).$$

If this is the case then the spectral family is uniquely determined.

Proof. (*Necessity*) Suppose that T is well-bounded of type (B), that is that T has a weakly compact $AC(J)$ functional calculus for some compact interval $J = [a, b] \subset \mathbb{R}$. Let $\psi: AC(J) \to B(X)$ denote the algebra homomorphism $\psi(f) = f(T)$. It is easy to check that if we take $Y = X$ and $\Gamma = X^*$, then ψ satisfies the conditions of Lemmas 3.1 and 3.2. These lemmas imply then that the K-sets for ψ each contain a unique projection which we shall denote by $E(\lambda)$. We shall show that $\{E(\lambda)\}_{\lambda\in\mathbf{R}}$ is a spectral family. Most of the required properties come immediately from Lemma 3.1. It only remains to show that $\{E(\lambda)\}$ has a strong left limit and is strongly right continuous everywhere. The existence of strong left and right limits follows from Lemma 2.2. For $\lambda \in [a, b)$, denote the strong right limit of $E(t)$ as $t \to \lambda^+$ by $E(\lambda^+)$. It follows immediately that $E(\lambda^+)$ is the weak operator topology limit of $E(t)$ as $t \to \lambda^+$. Combining Lemmas 3.2 and 3.4 shows that $E(\lambda^+) = E(\lambda)$.

The necessity proof will be completed by showing that $T = \int_J^{\oplus} \lambda\, dE(\lambda)$. Fix $x \in X$ and $x^* \in X^*$. By Lemma 3.4

$$\langle\psi(f)x, x^*\rangle = f(b)\langle x, x^*\rangle - \int_a^b f'(\lambda)\langle E(\lambda)x, x^*\rangle\, d\lambda$$

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for all $f \in AC(J)$. Now

$$\begin{aligned}
\left\langle \left(\int_J^{\oplus} f\, dE \right) x, x^* \right\rangle
&= \lim_{\Lambda \in \mathscr{P}} \left\{ \langle f(b)E(b)x, x^* \rangle - \left\langle \sum_{\Lambda} (f(\lambda_j) - f(\lambda_{j-1}))E(\lambda_j)x, x^* \right\rangle \right\} \\
&= f(b)\langle x, x^* \rangle - \lim_{\Lambda \in \mathscr{P}} \left\{ \sum_{\Lambda} (f(\lambda_j) - f(\lambda_{j-1}))\langle E(\lambda_j)x, x^* \rangle \right\} \\
&= f(b)\langle x, x^* \rangle - \int_a^b f'(\lambda)\langle E(\lambda)x, x^* \rangle\, d\lambda .
\end{aligned}$$

Thus

$$\langle \psi(f)x, x^* \rangle = \left\langle \left(\int_J^{\oplus} f\, dE \right) x, x^* \right\rangle .$$

Since x and x^* are arbitrary, this equation, with $f(\lambda) = \lambda$, shows that $T = \int_J^{\oplus} \lambda\, dE(\lambda)$.

The proofs of sufficiency and uniqueness are quite standard and we refer the reader to [6] or [2].

We can now prove the spectral theorem for general well-bounded operators.

THEOREM 3.6. *Suppose that $T \in B(X)$. Then T is well-bounded if and only if there exists a decomposition of the identity for X, $\{E(\lambda)\}_{\lambda \in \mathbf{R}}$, concentrated on a compact interval $J = [a, b] \subset \mathbb{R}$, such that*

$$\langle Tx, x^* \rangle = b\langle x, x^* \rangle - \int_a^b \langle x, E(\lambda)x^* \rangle\, d\lambda$$

for all $x \in X$ and $x^* \in X^*$.

PROOF. (*Necessity*) Suppose that T is well-bounded, that is, it has an $AC(J)$ functional calculus $\xi \colon AC(J) \to B(X)$, where $J = [a, b]$ is a compact interval of $\mathbb{R}$. Define $\psi \colon AC(J) \to B(X^*)$ by $\psi(f) = \xi(f)^* = f(T)^*$.

If we take $Y = X^*$ and $\Gamma = X \subset X^{**}$ in Lemma 3.1, then τ is the weak-$*$ operator topology on $B(X^*)$. The weak-$*$ operator topology compactness of the unit ball in $B(X^*)$ ensures that the algebra homomorphism must be τ-compact. Also $\psi(f)^*\Gamma \subset \Gamma$ for all $f \in AC(J)$ since this is just saying that $\xi(f) \in B(X)$ for all such f. Thus we may apply Lemmas 3.1 and 3.4 to produce a family of projections $\{E(\lambda)\}_{\lambda \in \mathbf{R}} \subset \operatorname{Proj}(X^*)$ which satisfies conditions (i)–(iv) for a decomposition of the identity.

It thus remains to check that conditions (v) and (vi) hold. Given $u \in L^1(J)$, define $f_u(\lambda) = \int_\lambda^b u(t)\, dt$. Clearly $f_u \in AC(J)$ and $f'_u(\lambda) = -u(\lambda)$

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(a.e.). Fix $x \in X$ and define $A_x \colon L^1(J) \to X$ by $A_x(u) = \xi(f_u)x$. The operator A_x is then clearly continuous and satisfies

$$\langle A_x u, x^* \rangle = \langle \xi(f_u)x, x^* \rangle = \int_a^b u(\lambda)\langle x, E(\lambda)x^* \rangle \, d\lambda.$$

Thus $A_x^* \colon X^* \to L^\infty(J)$ satisfies

$$\langle u, A_x^* x^* \rangle = \int_a^b u(\lambda)\langle x, E(\lambda)x^* \rangle \, d\lambda.$$

Consequently, A_x^* is the map γ_x in the definition of condition (v), and, as it is the adjoint of a continuous linear map from $L^1(J)$ into X, it has the required continuity property.

That we can arrange for the family of projections $\{E(\lambda)\}$ to have the limited right continuity property of condition (vi) (or indeed of condition (vi)$'$) follows from Lemma 3.4.

(*Sufficiency*) Again this is quite standard (see Section 4).

In non-reflexive spaces it is well-known that a well-bounded operator need not have a unique decomposition of the identity. For such an example see [9, Section 6]. This non-uniqueness means that the extended functional calculus for T^* which will be presented in Section 4 is not uniquely determined.

4. An extended functional calculus for well-bounded operators

In this section we shall describe the functional calculus which is obtained from a decomposition of the identity. Functions in $BV[a, b]$ correspond in a natural way to measures on $[a, b]$. A consequence of the Lebesgue decomposition theorem [7, pp. 134, 182] is that such a measure μ decomposes uniquely as $\mu = \nu_1 + \nu_2 + \nu_3$, where ν_1 is absolutely continuous with respect to Lebesgue measure, ν_2 is purely atomic, and ν_3 is the continuous singular part of μ. Let $\mathcal{N}$ denote the set of all left-continuous functions in $BV[a, b]$ for which the continuous singular part of the corresponding measure vanishes. Then each function in $\mathcal{N}$ may be written as the sum of an absolutely continuous function plus a left-continuous "saltus" function. We shall use the decomposition of the identity for T to develop a functional calculus for T^* for the functions in $\mathcal{N}$.

Ringrose's proof of the spectral theorem for general well-bounded operators proceeds by using the AC-functional calculus for T to produce this extended functional calculus for T^*. This extended functional calculus is

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then used to define the decomposition of the identity for T. It has not been shown however how to define the $\mathcal{N}$ functional calculus which is naturally associated with a decomposition of the identity. If one proceeds by using the decomposition of the identity to define a well-bounded operator T, Ringrose's proof of the spectral theorem will produce an $\mathcal{N}$ functional calculus for T^*. However, this is not necessarily consistent with the original decomposition of the identity since a choice of ultrafilters has to be made to produce the extension. In other words, it could be that

$$\chi_{(-\infty,\lambda]}(T^*) \neq E(\lambda).$$

The extended functional calculus for a particular well-bounded operator need not be uniquely determined, whereas the functional calculus associated with a decomposition of the identity is. Our aim here is to produce the extended functional calculus directly from a given decomposition of the identity. To do this we consider the absolutely continuous and singular parts of the functions separately.

Suppose that $\{E(\lambda)\}_{\lambda\in\mathbf{R}} \subset B(X^*)$ is a decomposition of the identity, concentrated on some compact interval $[a, b]$. Conditions (iii), (iv) and (v) for a decomposition of the identity ensure that there exists a unique operator $T \in B(X)$ such that

$$\langle Tx, x^*\rangle = b\langle x, x^*\rangle - \int_a^b \langle x, E(\lambda)x^*\rangle\, d\lambda, \qquad (x \in X,\ x^* \in X^*).$$

This operator is known as the *decomposable operator associated with* T. A proof may be found in [6, Theorem 15.6] or [9, Theorem 1]. That T is well-bounded and that for $f \in AC[a, b]$

$$\langle x, f(T^*)x^*\rangle = f(b)\langle x, x^*\rangle - \int_a^b f'(\lambda)\langle x, E(\lambda)x^*\rangle\, d\lambda$$

is a standard induction argument using Fubini's Theorem (see [9, Theorem 2]). The fact that T is well-bounded implies that this defines a Banach algebra homomorphism Φ: $AC[a, b] \to B(X^*)$, $\Phi(f) = f(T^*)$. Next we shall extend this to a larger class of functions which contains the characteristic functions of intervals of the form $(\alpha, \beta]$.

Let LCS_0 be the set of all left continuous step functions on $[a, b]$ which are zero at a. Every $g \in LCS_0$ may be written either as

$$g = \sum_{j=1}^{n} \alpha_j \chi_{(\lambda_{j-1}, \lambda_j]},$$

where $\{a = \lambda_0, \lambda_1, \dots, \lambda_n = b\}$ is some partition of $[a, b]$, or as

$$g(t) = \sum_{\lambda_j < t} \beta_j,$$

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where $\beta_0 = \alpha_1$, $\beta_j = \alpha_{j+1} - \alpha_j$ $(1 \le j < n)$ and $\beta_n = -\alpha_n$. If we let $\Phi(\chi_{(\lambda_{j-1}, \lambda_j]}) = E(\lambda_j) - E(\lambda_{j-1})$, then we can extend Φ to all of LCS_0 by linearity:

$$\Phi(g) = \sum_{j=1}^{n} \alpha_j (E(\lambda_j) - E(\lambda_{j-1})) = \sum_{j=0}^{n} \beta_j (I - E(\lambda_j)) = -\sum_{j=0}^{n} \beta_j E(\lambda_j),$$

since $\sum_{j=0}^{n} \beta_j = 0$. Clearly then

$$\|\Phi(g)\| \le \sum_{j=0}^{n} |\beta_j| \, \|E(\lambda_j)\| \le M \|g\|_{BV},$$

where $M = \sup\{\|E(\lambda)\| : \lambda \in \mathbb{R}\}$. Since Φ is thus bounded on LCS_0, we can extend it by continuity to the closure in $BV[a, b]$ of LCS_0 which we shall denote by $\overline{LCS}$.

Suppose then that $f \in \mathcal{N}$. Then f has a unique decomposition

$$f = f_A + f_S,$$

where $f_A \in AC$ and $f_S \in \overline{LCS}$. Furthermore

$$\|f\|_{BV} = \|f_A\|_{BV} + \|f_S\|_{BV}.$$

For such f define

$$\Phi(f) = \Phi(f_A) + \Phi(f_S).$$

This produces a well-defined linear map from $\mathcal{N}$ to $B(X^*)$. Also

$$\|\Phi(f)\| \le \|\Phi(f_A)\| + \|\Phi(f_S)\| \le M\|f_A\|_{BV} + M\|f_S\|_{BV} \le M\|f\|_{BV},$$

so Φ is continuous.

The hardest part is to show that $\Phi: \mathcal{N} \to B(X^*)$ is an algebra homomorphism. Suppose that $f, g \in AC \oplus LCS_0$. Then f and g can be written as

$$f = f_0 + \sum_{i=1}^{n} f_i, \qquad g = g_0 + \sum_{j=1}^{m} g_j,$$

where $f_0, g_0 \in AC$, $f_i = \tilde{\alpha}_i \chi_{(\mu_{i-1}, \mu_i]}$ and $g_j = \alpha_j \chi_{(\lambda_{j-1}, \lambda_j]}$. Then

$$\begin{aligned}\Phi(fg) &= \Phi\left(\left(f_0 + \sum f_i\right)\left(g_0 + \sum g_j\right)\right) \\ &= \Phi(f_0 g_0) + \sum_j \Phi(f_0 g_j) + \sum_i \Phi(f_i g_0) + \sum_i \sum_j \Phi(f_i g_j).\end{aligned}$$

That $\Phi(f_0 g_0) = \Phi(f_0)\Phi(g_0)$ is a consequence of the fact that Φ is an algebra homomorphism on $AC[a, b]$. It is also not hard to check that $\Phi(f_i g_j) = \Phi(f_i)\Phi(g_j)$. The middle terms present more difficulties, however. Consider

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the term $\Phi(f_0 g_j)$. The splitting $f_0 g_j = h + k$ with $h \in AC$ and $k \in LCS_0$ is given by

$$h(t) = \begin{cases} 0, & a \le t \le \lambda_{j-1}, \\ \alpha_j(f_0(t) - f_0(\lambda_{j-1})), & \lambda_{j-1} < t \le \lambda_j, \\ \alpha_j(f_0(\lambda_j) - f_0(\lambda_{j-1})), & \lambda_j < t \le b, \end{cases}$$

and

$$k(t) = \begin{cases} 0, & a \le t \le \lambda_{j-1}, \\ \alpha_j f_0(\lambda_{j-1}), & \lambda_{j-1} < t \le \lambda_j, \\ \alpha_j(f_0(\lambda_{j-1}) - f_0(\lambda_j)), & \lambda_j < t \le b. \end{cases}$$

Fix $x \in X$ and $x^* \in X^*$. Then

$$\begin{aligned}
\langle x, \Phi(f_0 g_j)x^*\rangle &= \langle x, (\Phi(h) + \Phi(k))x^*\rangle \\
&= h(b)\langle x, x^*\rangle - \int_a^b h'(\lambda)\langle x, E(\lambda)x^*\rangle \, d\lambda \\
&\quad + \langle x, \alpha_j f_0(\lambda_{j-1})(E(\lambda_j) - E(\lambda_{j-1}))x^*\rangle \\
&\quad + \langle x, \alpha_j(f_0(\lambda_{j-1}) - f_0(\lambda_j))(E(b) - E(\lambda_j))x^*\rangle \\
&= \alpha_j(f_0(\lambda_j) - f_0(\lambda_{j-1}))\langle x, x^*\rangle - \alpha_j \int_{\lambda_{j-1}}^{\lambda_j} f'_0(\lambda)\langle x, E(\lambda)x^*\rangle \, d\lambda \\
&\quad + \alpha_j f_0(\lambda_{j-1})\langle x, (E(\lambda_j) - E(\lambda_{j-1}))x^*\rangle \\
&\quad + \alpha_j(f_0(\lambda_{j-1}) - f_0(\lambda_j))\langle x, (I - E(\lambda_j))x^*\rangle \\
&= -\alpha_j \int_{\lambda_{j-1}}^{\lambda_j} f'_0(\lambda)\langle x, E(\lambda)x^*\rangle \, d\lambda \\
&\quad - \alpha_j f_0(\lambda_{j-1})\langle x, E(\lambda_{j-1})x^*\rangle + \alpha_j f_0(\lambda_j)\langle x, E(\lambda_j)x^*\rangle \\
&= -\alpha_j \int_{\lambda_{j-1}}^{\lambda_j} f'_0(\lambda)\langle x, E(\lambda)x^*\rangle \, d\lambda \\
&\quad + \alpha_j f_0(\lambda_j)\langle x, E(\lambda_{j-1})x^*\rangle - \alpha_j f_0(\lambda_{j-1})\langle x, E(\lambda_{j-1})x^*\rangle \\
&\quad + \alpha_j f_0(\lambda_j)\langle x, E(\lambda_j)x^*\rangle - \alpha_j f_0(\lambda_j)\langle x, E(\lambda_{j-1})x^*\rangle \\
&= \alpha_j f_0(b)\langle x, (E(\lambda_j) - E(\lambda_{j-1}))x^*\rangle - \alpha_j \int_{\lambda_{j-1}}^{\lambda_j} f'_0(\lambda)\langle x, E(\lambda)x^*\rangle \, d\lambda \\
&\quad + \alpha_j(f_0(\lambda_j) - f_0(\lambda_{j-1}))\langle x, E(\lambda_{j-1}))x^*\rangle \\
&\quad - \alpha_j(f_0(b) - f_0(\lambda_j))\langle x, (E(\lambda_j) - E(\lambda_{j-1}))x^*\rangle \\
&= \alpha_j f_0(b)\langle x, (E(\lambda_j) - E(\lambda_{j-1}))x^*\rangle - \alpha_j \int_{\lambda_{j-1}}^{\lambda_j} f'_0(\lambda)\langle x, E(\lambda)x^*\rangle \, d\lambda
\end{aligned}$$

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$$
\begin{aligned}
&+\alpha_j \int_{\lambda_{j-1}}^{\lambda_j} f'_0(\lambda)\langle x, E(\lambda_{j-1})x^*\rangle\, d\lambda \\
&-\alpha_j \int_{\lambda_j}^{b} f'_0(\lambda)\langle x, (E(\lambda_j) - E(\lambda_{j-1}))x^*\rangle\, d\lambda \\
&= \alpha_j f_0(b)\langle x, (E(\lambda_j) - E(\lambda_{j-1}))x^*\rangle \\
&\quad -\alpha_j \int_a^b f'_0(\lambda)\langle x, E(\lambda)(E(\lambda_j) - E(\lambda_{j-1}))x^*\rangle\, d\lambda \\
&= f_0(b)\langle x, \Phi(g_j)x^*\rangle - \int_a^b f'_0(\lambda)\langle x, E(\lambda)\Phi(g_j)x^*\rangle\, d\lambda \\
&= \langle x, \Phi(f_0)\Phi(g_j)x^*\rangle .
\end{aligned}
$$

Thus $\Phi(f_0 g_j) = \Phi(f_0)\Phi(g_j)$. A similar proof shows that $\Phi(f_i g_0) = \Phi(f_i)\Phi(g_0)$. Consequently

$$
\begin{aligned}
\Phi(fg) &= \Phi(f_0)\Phi(g_0) + \sum_j \Phi(f_0)\Phi(g_j) + \sum_i \Phi(f_i)\Phi(g_0) + \sum_i \sum_j \Phi(f_i)\Phi(g_j) \\
&= \Phi(f)\Phi(g).
\end{aligned}
$$

The result extends to $f, g \in \mathscr{N}$ by the continuity of Φ.

Finally we remark that the above functional calculus may be obtained by using the Lebesgue-Stieltjes integral of the function $\langle x, E(\cdot)x^*\rangle$ with respect to the function $f \in \mathscr{N}$; more specifically

$$
\langle x, f(T^*)x^*\rangle = f(b)\langle x, x^*\rangle - \int_a^b \langle x, E(\lambda)x^*\rangle\, df(\lambda), \qquad (x \in X,\ x^* \in X^*).
$$

It is easy to check that the finiteness of the variation of f, and the boundedness of $\{E(\lambda)\}$ ensure that this integral exists. Again the major difficulty with this approach is showing that the map $f \mapsto f(T^*)$ is multiplicative.

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School of Mathematics
University of New South Wales
Kensington, NSW, 2033
Australia

Department of Applied Mathematics
Tongji University
Shanghai
People's Republic of China

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