SERIES OF PRODUCTS OF BESSEL POLYNOMIALS

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1. Introduction. The Bessel polynomials, which arise as solution of the classical wave equation in spherical co-ordinates, are defined by Krall and Frink **(3)** by the equation

$$(1) \qquad \gamma_n(x, a, b) = {}_2F_0\left(-n, a+n-1; -\frac{x}{b}\right).$$

The purpose of this paper is to present some series of products of these polynomials when the two arguments are different as in the case of Legendre and Hermite polynomials. Such an explanation was given by Brafman **(2)**, namely:

$$(2) \quad \sum_{r=0}^{n} \binom{n}{r} \frac{\Gamma(2-a-n)}{\Gamma(2-a-n-r)} \left(-\frac{x}{b}-\frac{y}{b}\right)^{2r} \gamma_r\left(\frac{xy}{x+y}, a, b\right) = \gamma_n(x, a, b)\, \gamma_n(y, a, b).$$

These series will be stated and proved in § 2. The following formulae are required in the proofs:

$$(3) \quad {}_4F_3\left(\begin{matrix}\alpha, 1+\frac{1}{2}\alpha, d, e\\ \frac{1}{2}\alpha, 1+\alpha-d, 1+\alpha-e\end{matrix}; -1\right) = \frac{\Gamma(1+\alpha-d)\,\Gamma(1+\alpha-e)}{\Gamma(1+\alpha)\,\Gamma(1+\alpha-d-e)}$$

(1, p. 28, formula 3**)**. Gauss's theorem, namely:
If $R(\gamma) > 0$, $R(\gamma-\alpha-\beta) > 0$,

$$(4) \qquad F(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}$$

(4, p. 144, Example 2**)** and Saalschutz's theorem **(5)**, namely:
If $\rho+\sigma = \alpha+\beta+\gamma+1$ and if α, β, or γ is a negative integer,

$$(5) \quad F\left(\begin{matrix}\alpha, \beta, \gamma;\\ \rho, \sigma\end{matrix}\, 1\right) = \frac{\Gamma(\rho)\,\Gamma(1+\alpha-\sigma)\,\Gamma(1+\beta-\sigma)\,\Gamma(1+\gamma-\sigma)}{\Gamma(1-\sigma)\,\Gamma(\rho-\alpha)\,\Gamma(\rho-\beta)\,\Gamma(\rho-\gamma)}.$$

Frequent use will also be made of the factorial notation:

$$(\alpha; n) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \alpha(\alpha+1)\ldots(\alpha+n-1) \text{ for } n = 1, 2, 3, \ldots,$$

$$(\alpha, 0) = 1 \qquad \text{for } \alpha \neq 0.$$

2. Formulae and proofs: The expansions to be proved are:

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$$(6)\quad \sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{\Gamma(1-a-n-r)}{\Gamma(2-a-r)} (1-a-2r)\, \gamma_r(x,a,b)\, \gamma_r(y,a,b) = \left(\frac{x}{b}+\frac{y}{b}\right)^n \gamma_n\left(\frac{xy}{x+y}, a, b\right);$$

$$(7)\quad \sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{\Gamma(2-a-n-r)}{\Gamma(4-2a-2n-2r)} \left(\frac{x}{b}\right)^r \gamma_r(x,a,b) = \frac{\Gamma(2-a-n)}{\Gamma(4-a-2n)} \gamma_{2n}(x, 2a-2, b);$$

$$(8)\quad \sum_{r=0}^{n} \binom{n}{r} \frac{\Gamma(2-a)}{\Gamma(2-a-r)} \left(-\frac{2x}{b}\right)^r \gamma_r(\tfrac{1}{2}x, a, b) = \gamma_{2n}(x, a-2n, b);$$

$$(9)\quad \sum_{r=0}^{n} (-1)^r \frac{(1-a-2n+2r)\,\Gamma(1-a-2n+r)}{r!\{(n-r)!\,\Gamma(2-a-n+r)\}^2} \{\gamma_{n-r}(x,a,b)\}^2 = \{(n!)^2\,\Gamma(2-a)\}^{-1} \gamma_{2n}(x, a-2n, b);$$

$$(10)\quad \sum_{r=0}^{n} \binom{n}{r} \frac{(1+\alpha;r)(1-a-\alpha-n-r;r)(1-a-2r)}{(1-a-n-r;n+1)} \gamma_r(x,a,b). = \gamma_n(x, 1+a+\alpha, b).$$

Proof of (6). If on the left of (6) we replace $\gamma_r(x,a,b)\,\gamma_r(y,a,b)$ by the series in (3), then after changing the order of summation it becomes

$$\sum_{m=0}^{n} (-1)^m \left(\frac{x}{b}+\frac{y}{b}\right)^m \gamma_m\left(\frac{xy}{x+y}, a, b\right) \sum_{r=m}^{n} (-1)^r \binom{n}{r}\binom{r}{m} \frac{\Gamma(1-a-n-r)}{\Gamma(2-a-m-r)} (1-a-2r).$$

If we apply the formula

$$\sum_{r=m}^{n} (-1)^r \binom{n}{r}\binom{r}{m} \frac{\Gamma(1-a-n-r)}{\Gamma(2-a-m-r)} (1-a-2r) = (-1)^n \delta_{mn},$$

then the right-hand side of (6) is obtained. If $y = x$, (6) becomes

$$(11)\quad \sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{\Gamma(1-a-n-r)}{\Gamma(2-a-r)} (1-a-2r) \{\gamma_r(x,a,b)\}^2 = \left(\frac{2x}{b}\right)^n \gamma_n(\tfrac{1}{2}x, a, b).$$

Proof of (7). The following lemma is required in the proofs of (7) and (8):

LEMMA: *If n, N, r are positive integers, then*

$$(12)\quad (-n; N-n+2r) = (-1)^n\{(2n)!\}^{-1}(n!)2^{2r}(-2n; N) (\tfrac{1}{2}N - n; r)\,(\tfrac{1}{2}+\tfrac{1}{2}N-n; r).$$

Proof. We have

$$(-n; N-n+2r) = 2^{N-n-+2r}(-\tfrac{1}{2}n; \tfrac{1}{2}N - \tfrac{1}{2}n + r).$$

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But

$$(-\tfrac{1}{2}n;\tfrac{1}{2}N-\tfrac{1}{2}n+r)=(-\tfrac{1}{2}n;\tfrac{1}{2}N-\tfrac{1}{2}n)(\tfrac{1}{2}N-n;r)$$

and

$$(\tfrac{1}{2}-\tfrac{1}{2}n;\tfrac{1}{2}N-\tfrac{1}{2}n+r)=(\tfrac{1}{2}-\tfrac{1}{2}n,\tfrac{1}{2}N-\tfrac{1}{2}n)\,(\tfrac{1}{2}+\tfrac{1}{2}N-n;r),$$

so that

$$\begin{aligned}&\frac{2^{n-N-2r}(-n;N-n+2r)}{(\tfrac{1}{2}N-n;r)(\tfrac{1}{2}+\tfrac{1}{2}N-n;r)}\\&=(-\tfrac{1}{2}n;\tfrac{1}{2}N-\tfrac{1}{2}n)(\tfrac{1}{2}-\tfrac{1}{2}n;\tfrac{1}{2}N-\tfrac{1}{2}n)\\&=2^{n-N}(-n;N-n)\\&=\frac{\{(2n)(2n-1)\ldots(n+1)\}\{(-n;N-n)\}\{(n!)\}}{2^{N-n}(2n)!}\\&=\frac{(-1)^n\,n!}{2^{N-n}\,(2n)!}\{(-2n)(-2n+1)\ldots(-2n+N-1)\}\\&=(-1)^n2^{\,n-N}(n!)\,\{(2n)!\}^{-1}\,\{(-2n;N)\},\end{aligned}$$

from which the lemma follows.

Proof of (7). In the left-hand side of (7) write $n-r$ for r, substitute for $\gamma_{n-r}(x,a,b)$ from (1), then it becomes:

$$\sum_{r=0}^{n}\sum_{s=0}^{n-r}(-1)^{n-r+s}\frac{\Gamma(2-a-2n+r)(-n+r;s)}{r!\,s!\,(n-r)!}\;\frac{(a+n-r-1;s)}{\Gamma(4-2a-4n+2r)}\left(\frac{b}{x}\right)^{r-s}.$$

Here we put $s=r+N-n$, and the last expression becomes:

$$\begin{aligned}&\frac{1}{n!\,\Gamma(a+n-1)}\sum_{N=0}^{2n}\sum_{r=0}^{[n-\frac{1}{2}N]}(-1)^N\frac{(-n;N-n+2r)\,\Gamma(2-a-2n+r)}{r!\,\Gamma(4-2a-4n+2r)}\\&\qquad\frac{\Gamma(a+N-1)(2-a-n;r)}{\Gamma(1+N-n+r)}\left(\frac{b}{x}\right)^{n-N}\\&=(-1)^n\frac{2^{3n+2a-2}\;\pi^{\frac{1}{2}}}{(2n)!\,\Gamma(\frac{5}{2}-a-2n)\,\Gamma(a+n-1)}\sum_{N=0}^{2n}(-2)^N\\&\qquad\Gamma(a+N-1)(-2n;N)\left(\frac{b}{x}\right)^{n-N}\{\Gamma(1+N-n)\}^{-1}\\&\qquad\times{}_3F_2\left[\begin{matrix}\tfrac{1}{2}N-n,\tfrac{1}{2}+\tfrac{1}{2}N-n,2-a-n\\1+N-n,\tfrac{5}{2}-a-2n\end{matrix};1\right]\end{aligned}$$

by Lemma (12).

Proof of (8). In the left-hand side of (8), write $n-r$ for r, substitute for $\gamma_{n-r}(\tfrac{1}{2}x,a,b)$ from (1), then it becomes

$$\sum_{r=0}^{n}\sum_{s=0}^{n-r}(-2)^{n-r-s}\binom{n}{r}\frac{\Gamma(2-a)(-n+r;s)(a+n-r-1;s)}{s!\,\Gamma(2-a-n+r)}\left(\frac{b}{x}\right)^{r-s-n}.$$

Here we put $s = r + N - n$; then the last expression becomes

$$\frac{2^{2n}\,\Gamma(2-a)}{\Gamma(a+n-1)\,\Gamma(2-a-n)} \sum_{N=0}^{2n} (-2)^{-N} \frac{\Gamma(a+N-1)}{\Gamma(1+N-n)} (-2n;N)\left(\frac{b}{x}\right)^{-N} \times {}_2F_1\left[\begin{matrix} \frac{1}{2}N - n, \frac{1}{2} + \frac{1}{2}N - n; 1 \\ 1 + N - n \end{matrix}\right]$$

by Lemma (12). Here we sum the ${}_2F_1$ by Gauss's theorem (4) and so obtain (8). In (8) write $2x$ for x; then it becomes

$$(13)\quad \gamma_{2n}(2x, a-2n, b) = \sum_{r=0}^{n} \binom{n}{r} \frac{\Gamma(2-a)}{\Gamma(2-a-r)} \left(\frac{-4x}{b}\right)^{r} \gamma_r(x, a, b)$$

which may be taken as the duplication formula for Bessel polynomials.

Proof of (9). If $y = x$ in (2), then it becomes

$$(14)\quad \sum_{r=0}^{n} \binom{n}{r} \frac{\Gamma(2-a-n)}{\Gamma(2-a-n-r)} \left(\frac{-2x}{b}\right)^{r} \gamma_r(\tfrac{1}{2}x, a, b) = \{\gamma_n(x, a, b)\}^2.$$

To prove (9), substitute in the left of it for $\{\gamma_{n-r}(x, a, b)\}^2$ by the series in (14), then after some rearrangement the left-hand side of (9) becomes

$$\sum_{r=0}^{n} (-1)^r \frac{\Gamma(2-a-2n+2r)\,\Gamma(1-a-2n+r)}{r!(n-r)!\,\Gamma(1-a-2n+2r)\,\Gamma(2-a-n+r)} \left(\frac{b}{x}\right)^{-2n+2r}$$
$$\times \sum_{s=0}^{n-r} \frac{1}{s!(r-s)!\,\Gamma(2-a-2n+2r+s)} \left(\frac{b}{x}\right)^{2s} \gamma_{n-r-s}(\tfrac{1}{2}x, a, b)$$
$$= \frac{1}{n!}\left(\frac{b}{x}\right)^{-n} \sum_{p=0}^{n} (-2)^{n-p} \frac{\Gamma(2-a-2n)}{p!(n-p)!\,\Gamma(2-a-2n+p)\,\Gamma(2-a-n)}$$
$$\times \left(\frac{b}{x}\right)^{p} \gamma_{n-p}(\tfrac{1}{2}x, a, b)\, {}_4F_3\left[\begin{matrix} 1+a-2n, \frac{3}{2} - \frac{1}{2}a - n, -p, -n \\ \frac{1}{2} - \frac{1}{2}a - n, 2-a-n, 2-a-2n+p \end{matrix}; 1\right].$$

But the ${}_4F_3$ can be summed by (3) if we substitute in (3) $\alpha = 1 - a - 2n$, $d = -n$, $e = -p$. Thus the last expression reduces to

$$\frac{1}{n!}\left(\frac{b}{x}\right)^{-n} \sum_{p=0}^{n} \frac{(-2)^{n-p}}{p!(n-p)!\,\Gamma(2-a-n+p)} \left(\frac{b}{x}\right)^{p} \gamma_{n-p}(\tfrac{1}{2}x, a, b)$$
$$= \{(n!)^2\,\Gamma(2-a)\}^{-1} \sum_{r=0}^{n} \binom{n}{r} \frac{\Gamma(2-a)}{\Gamma(2-a-r)} \left(\frac{-2x}{b}\right)^{r} \gamma_r(\tfrac{1}{2}x, a, b)$$
$$= \{(n!)^2\,\Gamma(2-a)\}^{-1} \gamma_{2n}(x, a-2n, b)$$

by (8). Hence the proof of (9) is complete.

Proof of (10). To prove (10), substitute for $\gamma_r(x, a, b)$ in the left-hand side of (10) from (1), change the order of summation, then sum the innermost series by means of (3) and so obtain the right-hand side of (10) by a second application of (1).

Finally it may be noted that each of the explicitly summed series in the last proofs can be transformed to another explicitly summed series and the result in the two cases is the same.

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