ON D. E. LITTLEWOOD'S ALGEBRA OF S-FUNCTIONS

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1. Introduction. Several papers have been written on the "new" multiplication of S-functions since Littlewood [**3**, p. 206] first suggested the problem. M. Zia-ud-Din [**13**] calculated the case $\{m\} \otimes \{n\}$ for $mn \leqslant 12$, making use of the tables of the characters of the symmetric group of degree mn. Later Thrall [**10**, pp. 378-382] developed explicit formulae for the cases $\{m\} \otimes \{2\}$, $\{m\} \otimes \{3\}$, $\{2\} \otimes \{m\}$ (where m is any integer). Recently Todd [**12**] has obtained a formula for the factors $\{\mu\} \otimes S_n = t_{n(\mu)}$ as sums of irreducible characters $\{\sigma\}$. This reduces the problem of calculating $\{\mu\} \otimes \{\lambda\}$ to the ordinary multiplication of S-functions [**3**, p. 94]. General solutions to the problem have also been obtained by Thrall [**10**; p. 375] and by Robinson [**7**; **8**]. For these general results, however, the actual calculations are quite laborious in most cases.

In this paper a method of computing the general case $\{m\} \otimes \{4\}$ is developed and a formula is obtained (independently of Todd's method) for expressing the factors $t_{n(m)}$ as sums of S-functions $\{\sigma\}$. This formula provides a very brief method of calculating $t_{n(m)}$ and is easily adapted to recursive computation. The method of calculating $\{m\} \otimes \{4\}$ is also extended to cover all the remaining partitions of four. This method has been applied to calculate the products $\{7\} \otimes \{4\}$, $\{7\} \otimes \{2,1^2\}$ in full.

2. Preliminary definitions and lemmas. Using Thrall's notation [**10**, p. 374], $t_{n(m)} = \{m\} \otimes S_n$,

$$\{m\} \otimes \{\mu\} = \sum_{\beta} \frac{\chi^{(\beta)}(\mu)}{\beta_1! \ldots \beta_r!} \left(\frac{t_{1(m)}}{1}\right)^{\beta_1} \ldots \left(\frac{t_{r(m)}}{r}\right)^{\beta_r}. \tag{1}$$

Hence, if the $t_{n(m)}$ are known as sums of S-functions the product $\{m\} \otimes \{\mu\}$ may be computed by the ordinary multiplication of S-functions.

In proving the direct and recursion formulae for $t_{n(m)}$ we will make use of the following three lemmas.

Definitions of Young diagram, n-hook, removal of an n-hook, star diagram and δ-number are given in [**9**].

LEMMA 1. *Let $(\sigma) = (\sigma_1, \ldots, \sigma_n)$ be a partition of mn into n or less parts, and suppose that the numbers $r_1 = \sigma_1 + n - 1$, $r_2 = \sigma_2 + n - 2$, $\ldots$, $r_n = \sigma_n$ are all incongruent* (mod n). *Then the necessary and sufficient condition on s and k that a hook of length ns with top right node lying in the kth row may be removed from $[\sigma]$ is that $r_k = \sigma_k + (n - k) \geqslant ns$.*

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Proof.[1] Since $r_1, \ldots, r_n$ are incongruent (mod n), the classes of congruent δ-numbers are exactly $r_1, \ldots, r_n$ and the corresponding diagrams are simply vertical lines of $r_1, \ldots, r_n$ nodes respectively. Hence for an ns-hook to be removable (with foot in the kth column), it is necessary and sufficient that $r_k \geqslant ns$ [**9**, p. 85, Theorem C].

LEMMA 2. *Let* (σ) *be a partition of* mn *into* n *or less parts; then* $[\sigma]$ *has no* n*-core*[2] *if and only if the numbers* $r_1, \ldots, r_n$ *are incongruent* (mod n).

Proof. This lemma follows at once from the criterion (7.12) given in [**11**, p. 722].

LEMMA 3. *Let* (σ) *be a partition of* mn *defined as in Lemma* 1. *Then a hook of length* kn *may be added to* $[\sigma]$ *commencing with the lower left node at the end of any of the* n *rows of* $[\sigma]$ *(including cases where* $\sigma_i = 0$) *and a diagram* $[\sigma']$ *associated with a partition of* $n(m+k)$ *into* n *or less parts will result.*

Proof. Let the top right node of the annexed hook lie in the $(p+1)$ th row of $[\sigma']$. Then if $[\sigma']$ is not a right diagram we have

$$(\sigma_p - \sigma_{p+1} + 1) + \ldots + (\sigma_{p+s-1} - \sigma_{p+s} + 1) \doteq nk$$

for some value of s. That is, $\sigma_p \equiv \sigma_{p+s} - s$ (mod n), which contradicts the assumption that the r_i are incongruent.

3. Formulae for $t_{n(m)}$. The following theorems give direct and recursion formulae for $t_{n(m)}$.

THEOREM 1. $t_{n(m)} = \sum \phi_\sigma\{\sigma\}$ *where* (σ) *ranges over all partitions of* nm; ϕ_σ *is zero if* (σ) *has more than* n *parts or if the Young diagram* $[\sigma]$ *associated with* (σ) *has a (non-zero)* n*-core. Otherwise* $\phi_\sigma = \theta_\sigma$ *where* θ_σ *is plus or minus one according as the sum of the leg-lengths of the removed* n*-hooks is even or odd.*

Proof.[3] Let $T_{n(m)} = \sum \phi_\sigma\{\sigma\}$ where $\sum \phi_\sigma\{\sigma\}$ satisfies all the conditions stated in the theorem. We take as an induction hypothesis that $T_{n(h)} = t_{n(h)}$ for all $h < m$. Now we have [**10**, p. 374]

$$t_{n(1)} = S_n = \{n\} - \{n-1,1\} + \{n-2,1^2\} + \ldots + (-1)^{n-1}\{1^n\} = T_{n(1)}.$$

Hence the theorem is true for $m = 1$. Now in general,

$$t_{n(m)} = \sum_\beta \frac{1}{\beta_1! \ldots \beta_m!}\left(\frac{S_n}{1}\right)^{\beta_1} \ldots \left(\frac{S_{nm}}{m}\right)^{\beta_m}.$$

We will show (i) $T_{n(m)}$, $t_{n(m)}$ have the same derivatives with respect to S_{nk} $(k = 1, \ldots, m)$ and (ii) $T_{n(m)}$ is a function of S_{nk} $(k = 1, \ldots, m)$ only.

[1] I am indebted to Professor R. A. Staal for this proof which is shorter than my original one.

[2] When all possible n-hooks have been removed from a diagram the resulting diagram is called its n-core. The n-core and θ_σ are independent of the order of removal of the hooks [**5**], [**6**].

[3] For a method of evaluating the θ_σ which does not lead to the recursion formula see [**1**].

Now [4, p. 107]

(2) $$k\frac{\partial t_{n(m)}}{\partial S_{nk}} = t_{n(m-k)} = \sum_{\mu} \phi_\mu\{\mu\},$$

where (μ) ranges over all partitions of $n(m-k)$ an the ϕ_μ are, by the induction hypothesis, as described in the theorem. Also [4, p. 133]

(3) $$(nk)\frac{\partial T_{n(m)}}{\partial S_{nk}} = \sum_{\sigma} \phi_\sigma \sum_{j=1}^{n} \{\sigma_1, \ldots, \sigma_j - nk, \ldots, \sigma_n\},$$

or in the language of hooks:

(4) $$(nk)\frac{\partial T_{n(m)}}{\partial S_{nk}} = \sum_{\sigma} \phi_\sigma \sum_{i} (-1)^{h_i}\{\sigma^i\},$$

where $[\sigma^i]$ is obtained from $[\sigma]$ by removing an nk-hook of leg-length h_i commencing in the ith row and the summation is over all values of i (rows) from which such a hook may be removed. Now multiplying (2) by n we have:

(5) $$(nk)\frac{\partial t_{n(m)}}{\partial S_{nk}} = n\sum_{\mu} \phi_\mu\{\mu\}.$$

Hence we must show the right sides of (4), (5) to be equal. For fixed (μ) we label the n values of $\phi_\mu\{\mu\}$ occurring on the right side of (5) by $\phi_\mu\{\mu\}_1, \ldots, \phi_\mu\{\mu\}_n$. Consider $\phi_\mu\{\mu\}_r$ (for $\phi_\mu \neq 0$), by Lemma 3 an nk-hook may be added to $[\mu]$ starting (bottom left node) at the rth row and a new diagram will result. Let this annexed hook terminate in the jth row, then denoting the augmented diagram by $[\sigma]$ we must show $\phi_\mu\{\mu\}_r = \phi_\sigma(-1)^{h_j}\{\sigma_j\}$ where $h_j = r - j$, that is, we must show $\phi_\mu = \theta_\sigma(-1)^{h_j}$. Now by Lemma 1 the nk-hook which is deleted from $[\sigma]$ to yield $[\mu]$ may be partitioned into k n-hooks which may be removed in order, starting at the top right node of the nk-hook. Again by Lemma 1, each deletion leaves a new diagram; hence the bottom left node of a given n-hook must lie in the same row as the top right node of its successor. Let the ith removed n-hook terminate in the q_ith row and commence in the q_{i-1}th row; then $q_0 = j$ and $q_k = r$. Now the sum of the leg-lengths of these removed hooks is

$$(q_1 - q_0) + \ldots + (q_k - q_{k-1}) = q_k - q_0 = r - j,$$

which is the leg-length (h_j) of the nk-hook. But by the induction assumption

$$\phi_\mu = \theta_\sigma(-1)^{(q_1-q_0)+\ldots+(q_k-q_{k-1})} = \theta_\sigma(-1)^{h_j},$$

as required. Similarly there corresponds to each

$$\theta_\sigma\ (-1)^{h_i}\{\sigma^i\}$$

a unique $\phi_\mu\{\mu\}_r$. To demonstrate (ii) we write [3, p. 86; 10, p. 374]

(6) $$T_{n(m)} = \sum_{\sigma} \phi_\sigma\{\sigma\} = \sum_{\sigma} \phi_\sigma \sum_{\rho} \frac{h_\rho}{(mn)!}\cdot \chi^\sigma_\rho S_\rho$$

where

$$S_\rho = (s_1)^{\rho_1} \ldots (s_{nm})^{\rho_{mn}}.$$

Now the coefficient of S_ρ on the right of (6) is

$$\frac{h_\rho}{(mn)!}\sum_\sigma \phi_\sigma \chi_\rho^\sigma.$$

Now ϕ_σ has been shown [**11**] to be expressable as

$$\phi_\sigma = \sum_\alpha c_\alpha \chi_\alpha^\sigma$$

where α ranges over partitions of mn of the form $(\beta)_n$ where (β) is a partition of m and $(\beta)_n$ is the partition of mn obtained from (β) on multiplying each element of (β) by n, that is, $(\beta)_n = (\beta_1 n, \ldots, \beta_m n)$. Hence from the orthogonality relations for the characters of the symmetric group the coefficient of

$$S_\rho = \frac{h_\rho}{(mn)!}\sum_\sigma \sum_\alpha c_\alpha \chi_\alpha^\sigma \chi_\rho^\sigma$$

is zero if (ρ) is not of the form $(\beta)_n$ also. Hence $T_{n(m)}$ is a function of the S_{nk} $(k = 1, \ldots, m)$ only.

THEOREM 2. *Let* $t_{n(m)} = \sum \phi_\sigma\{\sigma\}$, *then* $t_{n(m+1)}$ *is obtained recursively as follows. To each* $[\sigma]$ *associated with a partition* (σ) *for which* ϕ_σ *is not zero we add an n-hook in all possible ways whose top right node lies in the first row of the augmented diagram* $[\sigma']$. *Then* $t_{n(m+1)} = \sum \phi_{\sigma}'\{\sigma'\}$ *where* $\phi_\sigma' = \phi_\sigma\ (-1)^k$ *where* k *is the leg-length of the annexed hook.*

Proof. The proof follows at once from Lemmas 1, 2, 3 and Theorem 1.

In recent papers [**7**, **8**, **12**], Robinson and Todd have given independent methods for evaluating $\{\mu\} \otimes \{\lambda\}$ by step by step building processes. Robinson gives a systematic procedure (in place of Littlewood's more or less empirical methods) by means of which the irreducible components of $\{\mu\} \otimes \{\lambda\}$ can be determined. In this general method the recursion is from n to $n+1$. Todd gives a general method and also treats the restricted case $\{m\} \otimes S_n = t_{n(m)}$ studied here. He gives recursion formulae by means of which $t_{n(m)}$ may be determined if $t_{n-1(m)}$ and $t_{n(m-1)}$ are both known. In the above methods the quantity θ_σ is made use of throughout.

4. The product $\{m\} \otimes \{4\}$. We now develop a method for computing the general case $\{m\} \otimes \{4\}$. From the calculations for $\{m\} \otimes \{4\}$ (for a specific value of m) $\{m\} \otimes \{2, 1^2\}$ is obtained by inspection. A modification of this method is also given for computing $\{m\} \otimes \{3,1\}$ and $\{m\} \otimes \{1^4\}$. The remaining case, $\{m\} \otimes \{2^2\}$, follows immediately from the calculations for $\{m\} \otimes \{4\}$ and $\{m\} \otimes \{1^4\}$; hence the method applies to every partition of four.

Writing t_i for $t_{i(m)}$ we have, from (1) of §2;

$$\{m\} \otimes \{4\} = \frac{1}{24}(t_1^4 + 6t_1^2 t_2 + 3t_2^2 + 8t_3 t_1 + 6t_4).$$

Rearranging terms we have

$$(7) \qquad \{m\} \otimes \{4\} = \frac{1}{12}\left[\frac{3}{2}(t_1^{\ 2} + t_2)^2 - t_1^{\ 4} + 4t_3t_1 + 3t_4\right].$$

Now [**10**, p. 380]

$$\tfrac{1}{2}(t_1^{\ 2} + t_2) = \{m\} \otimes \{2\} = \sum_v \{2m - 2v, 2v\}, \qquad v \leqslant \tfrac{1}{2}m.$$

It remains to develop explicit formulae for t_1^4, t_3t_1 as sums of irreducible characters $\{\sigma\}$; t_4 being known by Theorem 1.

The following congruence relations will be used in the proofs which follow. Let

$$t_3t_1 = \sum_\lambda \theta_\lambda^{t_3t_1}\{\lambda\}, \qquad t_4 = \sum_\lambda \theta_\lambda^{t_4}\{\lambda\}, \qquad t_1^{\ 4} = \sum_\lambda \theta_\lambda^{t_1^4}\{\lambda\}.$$

Now

$$\theta_\lambda^{t_3t_1}, \quad \theta_\lambda^{t_4}, \quad \theta_\lambda^{t_1^4}$$

are integers or zero, hence we have, from (7),

$$(8) \qquad \theta_\lambda^{t_3t_1} \equiv \theta_\lambda^{t_1^4} \pmod 3,$$

$$(9) \qquad \theta_\lambda^{t_4} \equiv \theta_\lambda^{t_1^4} \pmod 2.$$

We now derive a formula for $t_1^4 = \{m\}^4$. Let (λ) be an arbitrary partition $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ of $4m$ into four or less parts. We proceed to calculate the coefficient of $\{\lambda\}$ in $\{m\}^4$. To illustrate a term in the product $\{m\}^4$ diagramatically we denote the Young diagrams $[m]_i$ by the respective notations:

x x x . . . x, o o o . . . o, * * * . . . *, - - - - . . . -.

Then a diagram $[\lambda]$, corresponding to $\{\lambda\}$ must appear in the product $[m]_1$ $[m]_2$ $[m]_3$ $[m]_4$ as follows:

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[λ] = x x x . . . x o o o . . . o * * * . . . * - - - - . . . -
      o o o . . . o * * * . . . * - - - - . . . -
      * * * . . . * - - - - . . . -
      - - - - . . . -
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Now labelling the set of nodes in the first row which arises from $[m]_i$ by u_{i1}, in the second row by u_{i2} etc., we have:

$$\begin{aligned} m &= u_{11} & \text{and} \quad \lambda_1 &= u_{11} + u_{21} + u_{31} + u_{41} \\ m &= u_{21} + u_{22} & \lambda_2 &= u_{22} + u_{32} + u_{42} \\ m &= u_{31} + u_{32} + u_{33} & \lambda_3 &= u_{33} + u_{43} \\ m &= u_{41} + u_{42} + u_{43} + u_{44} & \lambda_4 &= u_{44} \end{aligned}$$

By a repeated application of the rule for the ordinary multiplication of S-functions we see that the necessary and sufficient conditions that a set of integers u_{ij} form a Young diagram appearing in the product $\{m\}^4$ are the following:

(a) $\sum_i \quad u_{ij} = \lambda_j$
(b) $\sum_j \quad u_{ij} = m$
(c) $\quad u_{ij} \geqslant 0$
(d) $u_{22} + u_{32} \leqslant u_{11} + u_{21}$
(e) $\lambda_2 \leqslant u_{11} + u_{21} + u_{31}$
(f) $u_{33} \leqslant u_{22}$
(g) $\lambda_3 \leqslant u_{22} + u_{32}$
(h) $\lambda_4 \leqslant u_{33}$

Conditions (a), (b), (c) follow from the geometry of the Young diagram; (d), . . . , (h) follow from the rule for multiplying S-functions of type $\{m\}$. Now it follows from conditions (a), (b) that the quantities u_{33}, u_{32}, u_{22} determine all the u_{ij} uniquely. Relabelling these quantities i, j, k respectively, we write all the u_{ij} in terms of the quantities $i, j, k, \lambda_1, \lambda_2, \lambda_3, \lambda_4, m$:

$$u_{11} = m \qquad u_{31} = m - (i + j)$$
$$u_{21} = m - k \qquad u_{44} = \lambda_4$$
$$u_{22} = k \qquad u_{43} = \lambda_3 - i$$
$$u_{32} = j \qquad u_{42} = \lambda_2 - (k + j)$$
$$u_{33} = i \qquad u_{41} = \lambda_1 + (i + j + k) - 3m$$

Now rewriting (d), . . . , (h) we have:

(d′) $k \leqslant \frac{1}{2}(2m - j)$
(e′) $k \leqslant 3m - \lambda_2 - (i + j)$
(f′) $i \leqslant k$
(g′) $\lambda_3 - j \leqslant k$
(h′) $\lambda_4 \leqslant i$

Combining these inequalities with $u_{ij} \geqslant 0$, we obtain the following limits for i, j, k:

$$\max (\lambda_4, \lambda_3 + \lambda_4 - m) \leqslant i \leqslant \min (m, \lambda_3)$$
$$0 \leqslant j \leqslant \min(m - i, \lambda_2 - i)$$
$$N \leqslant k \leqslant M$$

where
$$M = \min (\tfrac{1}{2}(2m - j), \lambda_2 - j, 3m - (\lambda_2 + i + j))$$
$$N = \max (i, \lambda_3 - j, 3m - (\lambda_1 + i + j))$$

Now setting $K_{ij} = \max (0, 1 + M - N)$ we have

THEOREM 3.

$$t_1^{\,4} = \{m\}^4 = \sum_\lambda \theta_\lambda^{t_1^4}\{\lambda\} \text{ where } \theta_\lambda^{t_1^4} = \sum_{i,j} K_{ij}$$

and i, j range over the values indicated above.

This formula illustrates the fact (which is easily proved directly in the general case) that if

$$\{m\}^n = \sum_\lambda \theta_\lambda^{t_1^n}\{\lambda\},$$

$$\{m + k\}^n = \sum_{\bar\lambda} \theta_{\bar\lambda}^{t_1^n}\{\bar\lambda\}$$

then if $(\bar\sigma)$ is a partition of $n(m + k)$ with $\bar\sigma_n \geqslant k$, and if $(\sigma) = (\bar\sigma_1 - k, \ldots, \bar\sigma_n - k)$, then

$$\theta_\sigma^{t_1^n} = \theta_{\bar\sigma}^{t_1^n}.$$

Hence we have a recursion formula for $\{m+1\}^n$ in terms of $\{m\}^n$ for all partitions (σ) of $(m+1)(n)$ with $\sigma_n \geqslant 1$.

The following theorem enables us to compute the quantity $t_3 t_1$ by inspection.

THEOREM 4.

$$t_3 t_1 = \sum_{\lambda} \theta_{\lambda}^{t_3 t_1} \{\lambda\}$$

where $\theta_{\lambda}{}^{t_3 t_1} = 1,\ 0,\ -1$ *according as* $\theta_{\lambda} t_1{}^4$ *is congruent to* $1,\ 0,\ -1$ *respectively* (mod 3).

Proof. The congruence (mod 3) has been established (8). It remains to be shown that $\theta_{\lambda}{}^{t_3 t_1}$ is always 1, 0, or -1. To show this we let [10, p. 381] $t_3 = \sum g(\lambda')\{\lambda'\}$ where $g(\lambda')$ is $1, 0, -1$ according as $(1+\lambda_1-\lambda_2)$ is congruent to $1, 0, -1$ (mod 3), and (λ') ranges over all partitions of $3m$ into three or fewer parts. Now $t_1 = \{m\}$, hence

$$t_3 t_1 = \sum_{\lambda'} g(\lambda')\{\lambda'\}\{m\} = \sum_{\lambda} \theta_{\lambda}^{t_3 t_1}\{\lambda\}. \tag{10}$$

Consider a partition (λ) of $4m$, then

$$\theta_{\lambda}^{t_3 t_1} = \sum_{\lambda'} g(\lambda'),$$

where the summation is over all partitions (λ') from which (λ) can be obtained on multiplying $\{\lambda'\}$ by $\{m\}$. Now consider which diagrams $[\lambda']$ are obtained from $[\lambda]$ on deleting m nodes as indicated in (10) above. Since (λ') is a partition of $3m$ into three or fewer parts, this amounts to deleting $(m-\lambda_4)$ nodes from the first three rows of $[\lambda]$ in accordance with the rule for multiplying $\{\lambda'\}\ \{m\}$. Four cases arise:

(i) $\lambda_1 - \lambda_2 \geqslant m - \lambda_4, \quad \lambda_2 - \lambda_3 \geqslant m - \lambda_4$
(ii) $\lambda_1 - \lambda_2 \geqslant m - \lambda_4, \quad \lambda_2 - \lambda_3 < m - \lambda_4$
(iii) $\lambda_1 - \lambda_2 < m - \lambda_4, \quad \lambda_2 - \lambda_3 \geqslant m - \lambda_4$
(iv) $\lambda_1 - \lambda_2 < m - \lambda_4, \quad \lambda_2 - \lambda_3 < m - \lambda_4$

We will consider (i) in detail. The number of nodes which may be deleted from λ_3 is $0, 1, 2, \ldots, \min(\lambda_3 - \lambda_4, m - \lambda_4) = s$. We first delete zero nodes from λ_3, $m - \lambda_4 - r$ from λ_1 and r from λ_2 $(r = 0, 1, \ldots, m - \lambda_4)$. This gives rise to the set of values for $g(\lambda')$ whose sum is indicated as T_1 below. We next delete one node from λ_3, $m - \lambda_4 - (r+1)$ from λ_1 and r from λ_2 $(r = 0, 1, \ldots, m - \lambda_4 - 1)$, giving rise to a set of values for $g(\lambda')$ with sum indicated as T_2 below. This process is continued to the $s = \min(\lambda_3 - \lambda_4, m - \lambda_4)$ step. Denoting the set of values $0, -1, 1$, by x_1, x_2, x_3, not necessarily respectively but in the same cyclic order, the sums $T_1, \ldots, T_s$ must then appear as follows:

$$\begin{aligned}
T_1 &= x_1 + x_2 + x_3 + x_1 + x_2 + x_3 + x_1 + \ldots + x_i && (m - \lambda_4 + 1) \text{ terms}\\
T_2 &= \quad x_3 + x_1 + x_2 + x_3 + x_1 + x_2 + \ldots + x_j && (m - \lambda_4) \text{ terms}\\
T_3 &= \qquad x_2 + x_3 + x_1 + x_2 + x_3 + \ldots + x_k && (m - \lambda_4 - 1) \text{ terms}\\
&\quad \ldots
\end{aligned}$$

Now since $x_1 + x_2 + x_3 = 0$, we have at once:

$$\theta_\lambda^{t_3 t_1} = \sum_{\lambda'} g(\lambda') = T_1 + \ldots + T_s = 0$$

if $s \equiv 0 \pmod 3$, since each set of three rows has total sum zero. If $s \equiv 1 \pmod 3$ the sum is simply T_s which is obviously 1, -1 or zero in all cases. If $s \equiv 2 \pmod 3$ we partition the final two rows T_{s-1}, T_s as indicated below:

$$\begin{array}{lcl} T_{s-1} & = & \overline{x_1 + x_2} \,|\, + x_3 \,|\, + x_1 + x_2 \,|\, + x_3 + \ldots + x_i, \\ T_s & = & \phantom{x_1 +{}} x_3 \,|\, + x_1 + x_2 \,|\, + x_3 \,|\, + x_1 + \ldots + x_j. \end{array}$$

Hence $T_1 + \ldots + T_s = T_{s-1} + T_s$ is 0, x_1, or $x_1 + x_3$ which is obviously one of 0, 1, or -1 for all values of x_1, x_3. Thus the proof for case (i) is complete. The cases (ii), (iii), (iv) give rise to a similar type of array of values as displayed above for case (i); again by direct calculation the total sum is seen to be 0,1, or -1. This completes the proof of the theorem.

The remaining term of (7),

$$\frac{1}{4}(t_1^2 + t_2)^2 = [\sum \{2m - 2v, 2v\}]^2,$$

is computed directly by the ordinary multiplication of S-functions. This calculation is somewhat lengthy although it is a considerable simplification of the direct calculation of t_2^2 and $t_1^2 t_2$ independently.

5. The remaining partitions of four. We first consider the case $\{m\} \otimes \{2, 1^2\}$:

$$\begin{aligned} \{m\} \otimes \{2, 1^2\} &= \frac{1}{24}(3t_1^4 - 6t_1^2 t_2 - 3t_2^2 + 6t_4) \\ &= \frac{1}{12}\Big[-\frac{3}{2}(t_1^2 + t_2)^2 + 3t_1^4 + 3t_4\Big]. \end{aligned}$$

Hence $\{m\} \otimes \{2, 1^2\}$ may be computed by inspection from the calculations for $\{m\} \otimes \{4\}$.

To compute $\{m\} \otimes \{1^4\}$, $\{m\} \otimes \{3, 1\}$ we modify the above method as follows:

$$\begin{aligned} \{m\} \otimes \{1^4\} &= \frac{1}{24}(t_1^4 - 6t_1^2 t_2 + 3t_2^2 + 8t_3 t_1 - 6t_4) \\ &= \frac{1}{12}\Big[\frac{3}{2}(t_1^2 - t_2)^2 - t_1^4 + 4t_3 t_1 - 3t_4\Big]. \end{aligned}$$

This calculation follows at once from the results for $\{m\} \otimes \{4\}$ except for the term $\frac{1}{4}(t_1^2 - t_2)^2$. Now

$$t_1^2 = \{m\}^2 = \{2m\} + \{2m - 1, 1\} + \{2m - 2, 2\} + \ldots + \{m, m\}$$

and by Theorem 1 we have

$$t_2 = \{2m\} - \{2m - 1, 1\} + \ldots + (-1)^m \{m, m\}.$$

Hence the term

$$\frac{1}{2}(t_1^2 - t_2) = \sum_v \{2m - v, v\}, \qquad (v = 1, 3, \ldots, m'),$$

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where m' is the greatest odd integer $\leqslant m$. The term $\frac{1}{4}(t_1^2 - {}_2)^2$ is now computed by the ordinary multiplication of S-functions.

Now

$$\{m\} \otimes \{3, 1\} = \frac{1}{12}\left[3t_1^4 - \frac{3}{2}(t_1^2 - t_2)^2 - 3t_4\right],$$

hence this case follows by inspection from the calculations for $\{m\} \otimes \{1^4\}$.

For the remaining case $\{m\} \otimes \{2^2\}$ we have

$$\{m\} \otimes \{2^2\} = \frac{1}{12}\left[3(t_1^4 + t_2^2) - 2t_1^4 - 4t_3t_1\right].$$

Here the coefficient of $t_1^2t_2$ is zero but the quantity t_2^2 must be calculated. We do this indirectly by making use of the results already obtained for $\{m\} \otimes \{4\}$, $\{m\} \otimes \{1^4\}$, and the following identity:

$$\tfrac{1}{2}\left[(t_1^2 + t_2)^2 + (t_1^2 - t_2)^2\right] = (t_1^4 + t_2^2).$$

6. Conclusion. By means of the method developed here and the earlier work of Thrall, the cases $\{m\} \otimes \{2\}$, $\{m\} \otimes \{3\}$, $\{m\} \otimes \{4\}$ may be computed directly. The next case, $\{m\} \otimes \{5\}$, is considerably more complicated and does not readily lend itself to direct calculation.

The author has used this method to compute the products $\{7\} \otimes \{4\}$, $\{7\} \otimes \{2,1^2\}$ in full, *Some Results in Littlewood's Algebra of S-functions*, thesis (microfilmed), University of Michigan, 1950. The cases $\{5\} \otimes \{4\}$, $\{6\} \otimes \{4\}$ have been computed recently by another method by Foulkes [**2**].

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