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DECOMPOSITIONS OF COMPLETE SYMMETRIC DIGRAPHS INTO THE ORIENTED PENTAGONS

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Abstract

We show that the complete symmetric digraph DK_n, $n \geqslant 5$, can be decomposed into each of the four oriented pentagons if and only if $n \equiv 0$ or 1 (mod 5).

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1. Introduction

Hung and Mendelsohn (1973) have found a necessary and sufficient condition for the partitioning of the set of arcs of a complete symmetric digraph into each of the two oriented triangles. In a recent paper, Harary and others (1978) have considered the same problem for each of the four oriented quadrilaterals. In doing so they have made strong use of the fact that each orientation of a triangle and quadrilateral is self-converse. Also, Harary and others (1967) had earlier shown that the only graphs for which every orientation is self-converse are the two smallest complete graphs K_1 and K_2 and the three smallest cycles C_3 (the triangle), C_4 (the quadrilateral) and C_5 (the pentagon). The object of this paper is to settle the one remaining case, that is, to find necessary and sufficient conditions so that the set of arcs of a complete symmetric digraph can be partitioned into each of the four oriented pentagons shown in Figure 1.

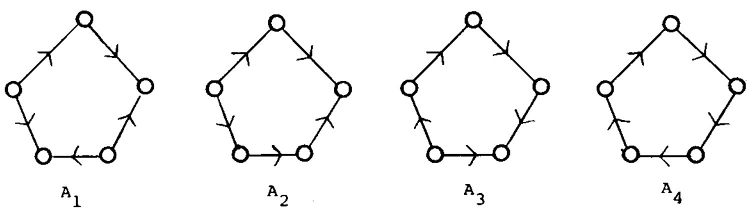

FIGURE 1.

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2. Definitions

For the definitions of graph, digraph and other related elementary concepts, we refer to Behzad and Chartrand (1971) or Harary (1969). We shall use the same notations as used in the above books. We shall denote the complete symmetric digraph with n vertices by DK_n. A digraph is said to be *self-converse* if it is isomorphic to its converse. The *union* $G_1 \cup G_2$ of two graphs G_1 and G_2 is a graph with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. For a connected graph G, nG denotes the graph with n components each of which is isomorphic to G. The *join* $G_1 + G_2$ of two graphs G_1 and G_2 is their union $G_1 \cup G_2$ together with all edges joining the vertices of $V(G_1)$ with the vertices of $V(G_2)$. It is easy to see that

$$K_m + K_n = K_{m+n} = K_m \cup K_n \cup K_{m,n}$$

with the vertex set of the complete bipartite graph $K_{m,n}$ chosen appropriately.

Given a graph G and graphs $H_1, H_2, \ldots, H_s$, if there exists a partition of $E(G)$ such that the resulting subgraphs of G are isomorphic to $H_1, H_2, \ldots, H_s$, we say that the graph G has been *decomposed* into the graphs $H_1, H_2, \ldots, H_s$. In particular, if every H_i, $i = 1, \ldots, s$, is isomorphic to some graph H, the decomposition of G is called an *isomorphic factorization* of G and we write $H | G$. If G cannot be isomorphically factored into graphs isomorphic to H, we write $H \nmid G$.

The following known results shall be used.

THEOREM 2.1. $C_5 | K_n$, $n \geqslant 5$, *if and only if* $n \equiv 1$ *or* $5 \pmod{10}$.

THEOREM 2.2. *For any odd* $n \geqslant 3$, K_n *can be decomposed into* 3*-cycles and* 5*-cycles.*

THEOREM 2.3. *For any* $n > 2$, $K_{2n} - I$, *where* I *is a* 1*-factor of* K_{2n}, *can be decomposed into* 3*-cycles and* 5*-cycles.*

Theorem 2.1 was proved by Rosa and Huang (1975) and independently proved by Bermond and Sotteau (1977). Theorem 2.2 is proved by observing that K_n can be decomposed into triangles when $n \equiv 1$ or $3 \pmod 6$ as is well known. When $n \equiv 5 \pmod 6$, it was shown by Wilson (1974) that K_n can be decomposed into triangles and exactly one K_5. The K_5 can be decomposed into two 5-cycles. Theorem 2.3 then follows quickly from the same results by deleting a vertex not contained in a K_5 and all edges incident with the vertex. The resulting graph has even order, contains a 1-factor from the triangles that contained the deleted vertex, and the remaining edges can be partitioned into triangles and 5-cycles.

3. Preliminary results

Throughout the rest of the paper A_1, A_2, A_3 and A_4 denote the four orientations of C_5 as shown in Figure 1.

THEOREM 3.1. *If* $A_i \mid DK_n$, $i = 1, 2, 3, 4$, *then* $n \equiv 0, 1, 5$ or $6 \pmod{10}$.

PROOF. For $i = 1, 2, 3, 4$, $A_i \mid DK_n$ implies that $5 \mid n(n-1)$, that is, $n \equiv 0$ or $1 \pmod 5$ or equivalently $n \equiv 0$, 1, 5 or 6 (mod 10).

Since each oriented C_5 is self-converse, we have the following results.

LEMMA 3.2. *If* $C_5 \mid K_n$, *then* $A_i \mid DK_n$ *for* $i = 1$, 2, 3 and 4.

Theorem 2.1 together with Lemma 3.2 give the following result.

THEOREM 3.3. *If* $n \equiv 1$ *or* 5 (mod 10), *then* $A_i \mid DK_n$ *for* $i = 1$, 2, 3 *and* 4.

The cases when $n \equiv 0$ or 6 (mod 10) are dealt with in Section 4. We now prove some lemmas to be used in that section.

LEMMA 3.4. $C_5 \mid K_{5\,5\,5}$.

PROOF. Let $\{u_1, u_2, u_3, u_4, u_5\}$, $\{v_1, v_2, v_3, v_4, v_5\}$ and $\{w_1, w_2, w_3, w_4, w_5\}$ be the three sets of independent vertices of $K_{5,5,5}$. The following gives a decomposition of $K_{5,5,5}$ into C_5's. Five of the C_5's are

$$u_1 v_1 u_5 v_5 w_3 u_1; \quad u_2 v_2 u_4 v_4 w_3 u_2; \quad u_3 v_3 u_1 w_2 v_5 u_3; \quad u_4 v_1 w_5 v_2 w_3 u_4; \quad u_4 v_5 w_4 v_1 w_2 u_4.$$

The rest then are obtained by rotating u_i into v_i into w_i into u_i for $i = 1, 2, 3, 4, 5$ in the above 5-cycles.

Define a graph Γ as follows: $V(\Gamma) = \bigcup_{i=1}^{5} V_i$ such that $|V_i| = 5$ for each $i = 1$, 2, 3, 4, 5 and the edge set of Γ consists of exactly all possible edges between vertices of V_i and V_{i+1}, $i = 1, \ldots, 5$, where the subscripts are taken modulo 5. In the notation of Harary (1969), Γ is the composition $C_5[\bar{K}_5]$ of the graphs C_5 and $\bar{K}_5$.

LEMMA 3.5. $C_5 \mid \Gamma$.

PROOF. Let

$$V_1 = \{a_1, a_2, a_3, a_4, a_5\}, \quad V_2 = \{b_1, b_2, b_3, b_4, b_5\},$$
$$V_3 = \{c_1, c_2, c_3, c_4, c_5\}, \quad V_4 = \{d_1, d_2, d_3, d_4, d_5\},$$
$$V_5 = \{e_1, e_2, e_3, e_4, e_5\}.$$

Following is a decomposition of Γ into C_5's where all subscripts are taken modulo 5 and $i = 1,2,3,4,5$.

$$a_i\, b_i\, c_i\, d_i\, e_i\, a_i,$$

$$a_i\, b_{i+1}\, c_{i+2}\, d_{i+3}\, e_{i+4}\, a_i,$$

$$a_i\, b_{i+2}\, c_{i+4}\, d_{i+6}\, e_{i+8}\, a_i,$$

$$a_i\, b_{i+3}\, c_{i+6}\, d_{i+9}\, e_{i+12}\, a_i,$$

$$a_i\, b_{i+4}\, c_{i+8}\, d_{i+12}\, e_{i+16}\, a_i.$$

LEMMA 3.6. *C_5 isomorphically factors the n-partite graph $K_{5,5,\dots,5}$ where n is odd.*

PROOF. With the n-partite graph $K_{5,5,\dots,5}$, we can associate a complete graph K_n with each vertex of K_n corresponding to an independent set of vertices of the n-partite graph and an edge of K_n corresponding to all the edges between two independent sets of vertices in the n-partite graph. Since n is odd, by Theorem 2.2 K_n can be decomposed into 3-cycles and 5-cycles. Under the correspondence between K_n and the n-partite graph, this implies that $K_{5,5,\dots,5}$ can be decomposed into factors that are either $K_{5,5,5}$ or Γ. The result then follows from Lemmas 3.4 and 3.5.

COROLLARY 3.7. *For each $i = 1,2,3,4$, $A_i \mid DK_{5,5,\dots,5}$ the directed n-partite graph with n odd.*

LEMMA 3.8. *$A_i \mid DK_6$ for $i = 1,2,3,4$.*

PROOF. Let *abcdea* be a 5-cycle. Henceforth, we agree to write the four orientations of it as

$$A_1\colon a \to b \to c \leftarrow d \to e \leftarrow a, \quad A_3\colon a \to b \to c \to d \to e \leftarrow a,$$

$$A_2\colon a \to b \to c \to d \leftarrow e \leftarrow a, \quad A_4\colon a \to b \to c \to d \to e \to a.$$

Let $V(DK_6) = \{u_1, u_2, u_3, u_4, u_5, u_6\}$. We list below the decomposition of DK_6 into each A_i. The direction of an edge is as given by the cycle at the top.

$A_1\colon a \to b \to c \leftarrow d \to e \leftarrow a$	$A_2\colon a \to b \to c \to d \leftarrow e \leftarrow a$
$u_2\ u_3\ u_4\ u_5\ u_6\ u_2$	$u_2\ u_6\ u_5\ u_4\ u_3\ u_2$
$u_1\ u_6\ u_5\ u_4\ u_3\ u_1$	$u_4\ u_3\ u_1\ u_6\ u_5\ u_4$
$u_6\ u_2\ u_5\ u_1\ u_4\ u_6$	$u_6\ u_2\ u_5\ u_1\ u_4\ u_6$
$u_6\ u_1\ u_2\ u_5\ u_3\ u_6$	$u_6\ u_3\ u_5\ u_2\ u_1\ u_6$
$u_2\ u_4\ u_6\ u_3\ u_1\ u_2$	$u_2\ u_1\ u_3\ u_6\ u_4\ u_2$
$u_3\ u_5\ u_1\ u_4\ u_2\ u_3$	$u_1\ u_5\ u_3\ u_2\ u_4\ u_1$

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A_3: $a \to b \to c \to d \to e \leftarrow a$	A_4: $a \to b \to c \to d \to e \to a$
$u_6\ u_5\ u_4\ u_3\ u_2\ u_6$	$u_2\ u_6\ u_5\ u_4\ u_3\ u_2$
$u_3\ u_4\ u_5\ u_6\ u_1\ u_3$	$u_3\ u_5\ u_1\ u_4\ u_6\ u_3$
$u_2\ u_6\ u_4\ u_1\ u_5\ u_2$	$u_4\ u_5\ u_6\ u_1\ u_2\ u_4$
$u_3\ u_5\ u_2\ u_1\ u_6\ u_3$	$u_5\ u_3\ u_1\ u_6\ u_2\ u_5$
$u_1\ u_2\ u_4\ u_6\ u_3\ u_1$	$u_6\ u_4\ u_2\ u_1\ u_3\ u_6$
$u_5\ u_1\ u_4\ u_2\ u_3\ u_5$	$u_1\ u_5\ u_2\ u_3\ u_4\ u_1$

LEMMA 3.9. *$A_i \mid DK_{10}$ for $i = 1, 2, 3, 4$.*

PROOF. We write

$$K_{10} = K_6 + K_4 = K_6 \cup K_4 \cup K_{4,6}.$$

Let $V(K_6) = \{u_0, u_1, u_2, u_3, u_4, u_5\}$ and $V(K_4) = \{v_1, v_2, v_3, v_4\}$. The two independent sets of vertices of $K_{4,6}$ are $V(K_6)$ and $V(K_4)$. By Lemma 3.8, we know each of $A_i \mid DK_6$ for $i = 1, 2, 3, 4$. This leaves the graph $K_4 \cup K_{4,6}$. It can be decomposed into the

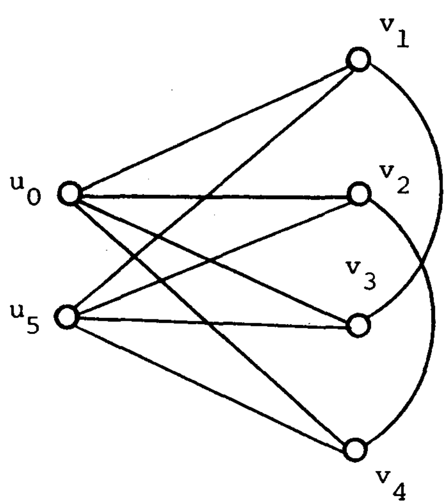

FIGURE 2

graph of Figure 2 and the following four C_5's: $u_1 v_1 v_4 u_3 v_2 u_1$, $u_2 v_2 v_1 u_4 v_3 u_2$, $u_3 v_3 v_4 u_2 v_1 u_3$ and $u_4 v_4 u_1 v_3 v_2 u_4$. Since each A_i is self-converse, $A_i \mid DC_5$ for $i = 1, 2, 3, 4$. Thus, the proof is complete if we decompose DH, where H is the graph in Figure 2, into each of the four A_i's.

For A_1 we have

$u_0 \to v_1 \leftarrow v_3 \leftarrow u_5 \to v_2 \leftarrow u_0, \quad u_0 \leftarrow v_2 \leftarrow v_4 \to u_5 \leftarrow v_3 \to u_0, \quad u_0 \to v_3 \leftarrow v_1 \leftarrow u_5 \to v_4 \leftarrow u_0$

and

$$u_0 \leftarrow v_1 \to u_5 \leftarrow v_2 \to v_4 \to u_0.$$

For A_2 we have

$u_0 \to v_1 \leftarrow v_3 \leftarrow u_5 \leftarrow v_2 \to u_0, \quad u_0 \to v_2 \leftarrow v_4 \leftarrow u_5 \leftarrow v_3 \to u_0, \quad u_0 \to v_3 \leftarrow v_1 \leftarrow u_5 \leftarrow v_4 \to u_0$

and

$$u_0 \to v_4 \leftarrow v_2 \leftarrow u_5 \leftarrow v_1 \to u_0.$$

For A_3 we have

$v_3 \to u_5 \to v_2 \to u_0 \to v_1 \leftarrow v_3, \quad v_4 \to u_5 \to v_3 \to u_0 \to v_2 \leftarrow v_4, \quad v_1 \to u_5 \to v_4 \to u_0 \to v_3 \leftarrow v_1$

and

$$v_2 \to u_5 \to v_1 \to u_0 \to v_4 \leftarrow v_2.$$

Finally, for A_4 we have the 5-cycles

$$u_0 v_1 v_3 u_5 v_2 u_0, \quad u_0 v_2 v_4 u_5 v_3 u_0, \quad u_0 v_3 v_1 u_5 v_4 u_0 \quad \text{and} \quad u_0 v_4 v_2 u_5 v_1 u_0.$$

4. Main results

THEOREM 4.1. *If* $n \equiv 6 \pmod{10}$, $A_i \mid DK_n$ *for* $i = 1, 2, 3$ *and* 4.

PROOF. Let $n = 10k+6$. We write

$$K_n = K_{10k+6} = K_{5(2k+1)+1} = [(2k+1)K_5 + K_1] \cup K_{5,5,\ldots,5}$$
$$= (2k+1)K_6 \cup K_{5,5,\ldots,5},$$

where the vertex set of the complete multipartite graph $K_{5,5,\ldots,5}$ is chosen appropriately. The result then follows from Corollary 3.7 and Lemma 3.8.

THEOREM 4.2. *If* $n \equiv 0 \pmod{10}$ *and* $n \neq 20$, *then* $A_i \mid DK_n$ *for* $i = 1, 2, 3$ *and* 4.

PROOF. The result has been proved for $n = 10$ in Lemma 3.9. So let $n = 10k$ with $k > 2$. We write

$$K_n = K_{10k} = kK_{10} \cup K_{10,10,\ldots,10}.$$

In view of Lemma 3.9, it suffices to show that $A_i \mid DK_{10,10,\ldots,10}$ for $i = 1, 2, 3, 4$. With $K_{10,10,\ldots,10}$ we can associate a graph $K_{2k} - I$, where I is a 1-factor of K_{2k}, as follows. Let $V_i = \{u_1^i, u_2^i, \ldots, u_{10}^i\}$, $1 \leqslant i \leqslant k$, be the maximal independent subsets of vertices in $K_{10,10,\ldots,10}$. Then let $S_i = \{i_1^i, u_2^i, \ldots, u_5^i\}$ and $S_{i+k} = \{u_6^i, u_7^i, \ldots, u_{10}^i\}$ for $i = 1, 2, \ldots, k$.

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We define $V(K_{2k}) = \{S_1, S_2, \ldots, S_{2k}\}$ and S_i adjacent to S_j if and only if $j \neq i+k$ or $i \neq j+k$. Notice that the edges $S_i S_{i+k}$ for $1 \leqslant i \leqslant k$ form a 1-factor for K_{2k} as just defined.

By Theorem 2.3, $K_{2k} - I$ can be decomposed into 3-cycles and 5-cycles. This amounts to the fact that the k-partite graph $K_{10,10,\ldots,10}$ can be decomposed into the factors $K_{5,5,5}$ and Γ. The result then follows from Lemmas 3.4 and 3.5.

THEOREM 4.3. *$A_i \mid DK_{20}$ for $i = 1, 2, 3, 4$.*

PROOF. We write K_{20} as $K_{20} = 2K_{10} \cup K_{10,10}$ with the vertex set of the complete bipartite graph chosen appropriately. We shall show that $A_i \mid D(K_{10} \cup K_{10,10})$ for $i = 1, 2, 3, 4$. This together with Lemma 3.9 will prove the result. Let the vertex set of the two K_{10}'s be $\{u_1, u_2, \ldots, u_{10}\}$ and $\{v_1, v_2, \ldots, v_{10}\}$. Then $K_{10} \cup K_{10,10}$ can be decomposed into the graph Λ shown in Figure 3 and twenty-five disjoint 5-cycles

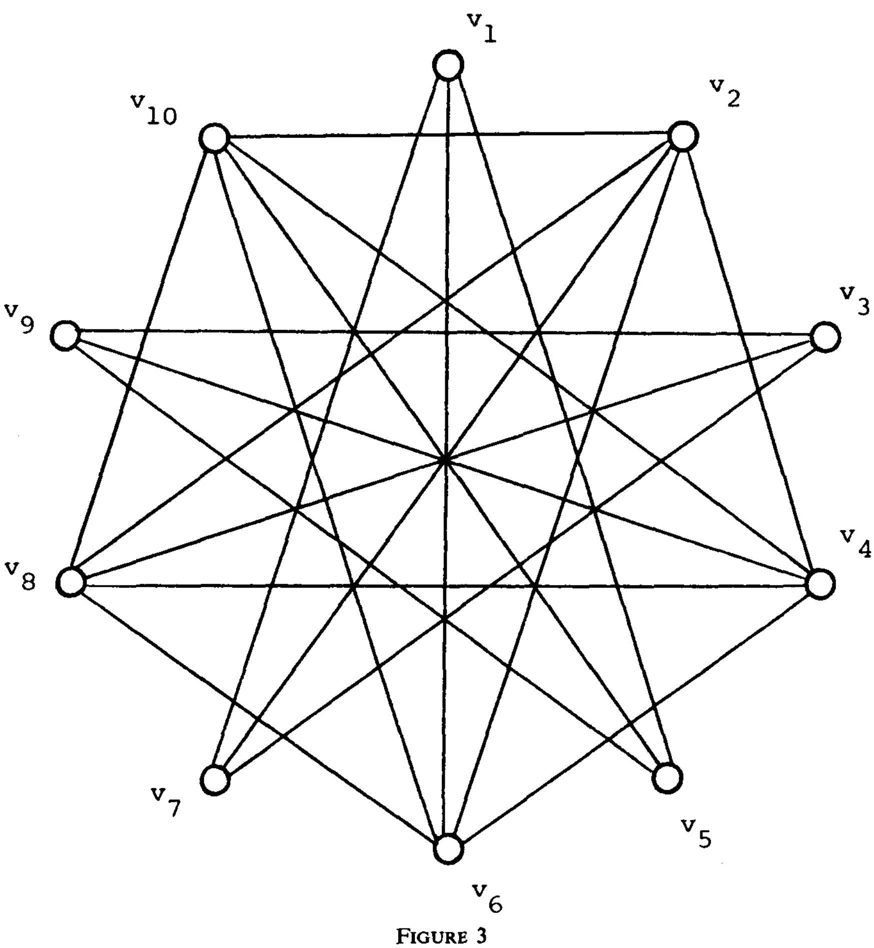

FIGURE 3

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as follows:

$$C: u_1, v_1, u_2, v_3, v_4, u_1; \quad C': u_1, v_3, u_7, v_2, v_5, u_1; \quad C'': u_1, v_8, u_{10}, v_7, v_9, u_1$$

and

$$\varphi^k C, \varphi^k C' \quad (1 \leqslant k \leqslant 9) \qquad \text{and} \qquad \varphi^{2k} C'' \quad (1 \leqslant k \leqslant 4),$$

where $\varphi = (1\ 2 \ldots 10)$ is a cyclic permutation of the ten subscripts and subscripts are taken modulo 10.

We show next that $A_i | D\Lambda$ for $i = 2, 3, 4$, by listing the copies of A_2, A_3 and A_4 respectively in the decomposition of $D\Lambda$. This together with the fact that each A_i $(i = 1, 2, 3, 4)$ is self-converse shall prove the result for $i = 2$, 3 and 4.

A_2: $a \to b \to c \to d \leftarrow e \leftarrow a \qquad a \to b \to c \to d \leftarrow e \leftarrow a$

v_8 v_6 v_4 v_2 v_{10} v_8 $\qquad$ v_8 v_2 v_4 v_9 v_3 v_8

v_6 v_2 v_8 v_4 v_{10} v_6 $\qquad$ v_4 v_6 v_1 v_5 v_{10} v_4

v_9 v_5 v_1 v_7 v_3 v_9 $\qquad$ v_7 v_2 v_6 v_8 v_3 v_7

v_2 v_7 v_1 v_6 v_{10} v_2 $\qquad$ v_5 v_9 v_4 v_8 v_{10} v_5

A_3: $a \to b \to c \to d \to e \leftarrow a \qquad a \to b \to c \to d \to e \leftarrow a$

v_2 v_4 v_6 v_8 v_{10} v_2 $\qquad$ v_4 v_2 v_8 v_3 v_9 v_4

v_4 v_8 v_2 v_6 v_{10} v_4 $\qquad$ v_{10} v_5 v_1 v_6 v_4 v_{10}

v_7 v_1 v_5 v_9 v_3 v_7 $\qquad$ v_3 v_8 v_6 v_2 v_7 v_3

v_{10} v_6 v_1 v_7 v_2 v_{10} $\qquad$ v_9 v_5 v_{10} v_8 v_4 v_9

A_4: $a \to b \to c \to d \to e \to a \qquad a \to b \to c \to d \to e \to a$

v_1 v_5 v_9 v_3 v_7 v_1 $\qquad$ v_4 v_2 v_8 v_3 v_9 v_4

v_2 v_6 v_{10} v_4 v_8 v_2 $\qquad$ v_4 v_{10} v_5 v_1 v_6 v_4

v_2 v_4 v_6 v_8 v_{10} v_2 $\qquad$ v_5 v_{10} v_8 v_4 v_9 v_5

v_3 v_8 v_6 v_2 v_7 v_3 $\qquad$ v_6 v_1 v_7 v_2 v_{10} v_6

Now to show that $A_1 | D(K_{10} \cup K_{10,10})$, let H be a graph defined by

$$H = \Lambda - C^* + C' + \varphi^4 C'',$$

where C^* is the 5-cycle $v_2, v_6, v_{10}, v_4, v_8, v_2$. Then $K_{10} \cup K_{10,10}$ can be decomposed into the graph H and twenty-four disjoint 5-cycles: C^*, C, C'', $\varphi^k C$, $\varphi^k C'$ $(1 \leqslant k \leqslant 9)$, $\varphi^2 C''$, $\varphi^6 C''$ and $\varphi^8 C''$.

Since A_1 is self-converse, it is enough to show that $A_1 | DH$. We list below the copies of A_1 in the decomposition of DH.

$$
\begin{array}{llllll@{\qquad}llllll}
\multicolumn{6}{l}{A_1\colon a \to b \to c \leftarrow d \to e \leftarrow a} & \multicolumn{6}{l}{a \to b \to c \leftarrow d \to e \leftarrow a} \\
v_5 & v_9 & v_4 & v_2 & v_{10} & v_5 & v_6 & v_4 & v_2 & u_4 & v_1 & v_6 \\
v_3 & v_8 & v_6 & v_4 & v_9 & v_3 & v_2 & v_7 & v_1 & v_3 & u_5 & v_2 \\
v_1 & v_6 & v_8 & v_{10} & v_5 & v_1 & u_1 & v_5 & v_2 & u_7 & v_3 & u_1 \\
v_8 & v_{10} & v_2 & v_7 & v_3 & v_8 & v_9 & v_3 & u_7 & v_2 & v_5 & v_9 \\
u_5 & v_2 & u_4 & v_1 & v_3 & u_5 & v_5 & v_1 & v_7 & v_3 & u_1 & v_5
\end{array}
$$

Finally, the results of the Theorems 3.1, 3.3, 4.1, 4.2 and 4.3 can be put together into a single theorem.

THEOREM 4.4. *$A_i | DK_n$, $n \geqslant 5$, for $i = 1, 2, 3$ or 4 if and only if $n \equiv 0$ or 1* (mod 5).

The above theorem can also be proved using results and techniques in the survey paper by Bermond and Sotteau (1975). The proof in our paper is elementary and easily extended to other values for the cycle length.

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