Can. J. Math., Vol. XXXIV, No. 3, 1982, pp. 621–661

THE GENERALIZED ORTHOCOMPLETION AND STRONGLY PROJECTABLE HULL OF A LATTICE ORDERED GROUP

RICHARD N. BALL

The central result is the existence and uniqueness for an arbitrary l-group G of two hulls, $\bar{G}$ and $\bar{G}^{\omega}$, which in the representable case coincide with the orthocompletion and strongly projectable hull of G. This is done by introducing two notions of extension, written $\preccurlyeq$ and $\preccurlyeq_{\omega}$, and proving that each G has a maximal $\preccurlyeq$ extension $\bar{G}$ and a maximal $\preccurlyeq_{\omega}$ extension $\bar{G}^{\omega}$. Two constructions of $\bar{G}$ and $\bar{G}^{\omega}$ are given: an l-permutation construction leads to descriptive structural information, and a construction by "consistent maps" leads to a strong universal mapping property.

The notion of a strongly projectable hull has a lengthy history. The concept of an orthocompletion, together with the first proof of its existence and uniqueness, is due to Bernau [**4**]. Conrad summarized and extended all these results in an important paper [**10**]. The chief novelty of the present work is that the ideas apply to non-representable as well as to representable l-groups. When specialized to the representable case, the construction of Section 2 is related to the nice constructions of Bleier in [**6**] and [**7**].

The notation, which is multiplicative even for the representable case, is standard. G is understood to be an l-group whose complete Boolean algebra of polars will be designated $\mathscr{P}_G$ or simply $\mathscr{P}$. The symbols $\vee$, $\wedge$, $\perp$, $0_{\mathscr{P}}$, and $1_{\mathscr{P}}$ refer respectively to supremum, infimum, complementation, least element and greatest element in $\mathscr{P}$. The symbols $\vee$ and $\wedge$ also refer to supremum and infimum of elements of G; the reader must be prepared to distinguish the two meanings from context.

1. Extensions. The crucial concept is the following. For l-groups G and H define $G \preccurlyeq H$ to mean that G is an l-subgroup of H in which the polars of G and H are in one-to-one correspondence by intersection, and such that

$$\vee \{(hg^{-1})^{\perp} | \ g \in G\} = 1_{\mathscr{P}} \quad \text{for all } h \in H.$$

Similarly, for κ a fixed infinite cardinal number, define $G \preccurlyeq_{\kappa} H$ to mean that G is an l-subgroup of H in which the polars of G and H are in one-to-one correspondence by intersection, and such that for every $h \in H$ there

Received August 16, 1980 and in revised form September 30, 1981.

is some set $X \subseteq G$ of cardinality less than κ satisfying

$$\vee \{(hg^{-1})^{\perp} | g \in X\} = 1_{\mathscr{P}}.$$

Clearly $G \preccurlyeq_{\kappa} H$ implies $G \preccurlyeq H$; just as clearly, $G \preccurlyeq G$.

To say that the polars of G and H are in one-to-one correspondence is to say that every nontrivial polar of H has nontrivial intersection with G (Proposition 5.3, [**1**]), and the correspondence is actually a Boolean isomorphism. Therefore, in the first four sections we shall blur the distinction between the polars of G and those of H, using the same symbol $\mathscr{P}$ for both sets.

LEMMA 1.1. *If $G \preccurlyeq H$ then G is order dense in H.*

Proof. Assume $G \preccurlyeq H$ and $1 < h \in H$. Find $g \in G$ such that

$$Q = (hg^{-1})^{\perp} \wedge h^{\perp\perp} \neq 0_{\mathscr{P}}.$$

It must be possible to find x with $1 < x \in Q \cap G$; let

$$y = (x \wedge g) \vee 1 \in G.$$

To show $y \leqq h$ consider an arbitrary prime P such that $P < Py$. Then $P < Px$ and $x \in (hg^{-1})^{\perp}$ imply $(hg^{-1}) \in P$ or $Ph = Pg$. That is, $Py \leqq Pg = Ph$, and since P was arbitrary, $y \leqq h$. To show $y > 1$ consider any minimal prime T such that $x \notin T$. Then $x \in (hg^{-1})^{\perp}$ implies $(hg^{-1}) \in T$ or $Th = Tg$. Furthermore, $x^{\perp\perp} \not\subseteq T$ and $x^{\perp\perp} \subseteq h^{\perp\perp}$ imply $h^{\perp\perp} \not\subseteq T$, which implies $h \notin T$ by the minimality of T. Therefore $Ty = Tx \wedge Tg > T$ and $y > 1$.

LEMMA 1.2. *Suppose $G \leqq H \leqq K$. Then $G \preccurlyeq K$ if and only if $G \preccurlyeq H$ and $H \preccurlyeq K$, and similarly for $\preccurlyeq_{\kappa}$.*

Proof. Since the implication from left to right is clear, suppose $G \preccurlyeq H \preccurlyeq K$. Clearly G is order dense in K and polars in G and K correspond. Consider $k \in K$. Then for any fixed $h \in H$,

$$\begin{aligned} \vee_G (kg^{-1})^{\perp} = \vee_G (kh^{-1}hg^{-1})^{\perp} &\supseteq \vee_G ((kh^{-1})^{\perp} \wedge (hg^{-1})^{\perp}) \\ &= (kh^{-1})^{\perp} \wedge (\vee_G (hg^{-1})^{\perp}) = (kh^{-1})^{\perp}. \end{aligned}$$

But because $\vee \{(kh^{-1})^{\perp} | h \in H\} = 1_{\mathscr{P}}$, it follows that

$$\vee \{(kg^{-1})^{\perp} | g \in G\} = 1_{\mathscr{P}}.$$

The argument for $\preccurlyeq_{\kappa}$ is similar and depends on the fact that $\kappa^2 = \kappa$.

LEMMA 1.3. *Suppose $\mathscr{C}$ is a collection of l-groups totally ordered by $\preccurlyeq$ ($\preccurlyeq_{\kappa}$). Then $C \preccurlyeq \cup \mathscr{C}$ ($C \preccurlyeq_{\kappa} \cup \mathscr{C}$) for any $C \in \mathscr{C}$.*

Proof. Suppose $\mathscr{C}$ is totally ordered by $\preccurlyeq$. Then every $C \in \mathscr{C}$ is order dense in $\cup \mathscr{C}$, from which it follows that the polar of C and $\cup \mathscr{C}$ correspond by intersection. Given a particular $C \in \mathscr{C}$ and $x \in \cup \mathscr{C}$ we may assume $x \in D$ for some $D \in \mathscr{C}$ such that $C \preccurlyeq D$. Because $C \preccurlyeq D$, $\bigvee_G (xg^{-1})^\perp = 1_\mathscr{P}$ must hold in D. Because polars in D and $\bigcup \mathscr{C}$ correspond, this must be true in $\bigcup \mathscr{C}$ also. The argument for $\preccurlyeq_\kappa$ is analagous.

The next lemma is 5.12 of [**1**], proved here for completeness. $|H|$ denotes the cardinality of H and $G^\mathscr{P}$ denotes the collection of maps from $\mathscr{P}$ to G.

LEMMA 1.4. *$G \preccurlyeq H$ implies $|H| \leqq |G^\mathscr{P}|$.*

Proof. Well order G and let $*$ be some object not in G. With each $h \in H$ associate the map f_h: $\mathscr{P} \rightarrow G \cup \{*\}$ defined by declaring the image of $P \in \mathscr{P}$ under f_h, written Pf_h, to be the first $g \in G$ such that $P \subseteq (hg^{-1})^\perp$. If no such g exists, then Pf_h is defined to be $*$. If h and k are different members of H, then $(hk^{-1})^{\perp\perp} \neq 0_\mathscr{P}$ and hence one may find $g \in G$ such that

$$(hk^{-1})^{\perp\perp} \wedge (hg^{-1})^\perp = R \neq 0_\mathscr{P}.$$

$R \subseteq (hg^{-1})^\perp$ implies $Rf_h \neq *$; assume $Rf_h = g$. Then $Rf_k \neq g$, for if $Rf_k = g$ then $R \subseteq (kg^{-1})^\perp$, which yields $hg^{-1}, kg^{-1} \in R^\perp$ or $(hk^{-1}) \in R^\perp$, contrary to $R \subseteq (hk^{-1})^{\perp\perp}$. The point is that $f_h \neq f_k$.

The preceding lemmas, together with a straightforward Zorn's Lemma argument, guarantee the existence of at least one maximal $\preccurlyeq$ extension and at least one maximal $\preccurlyeq_\kappa$ extension for each l-group and each infinite cardinal κ.

THEOREM 1.4. *For every l-group G there is at least one l-group H such that $G \preccurlyeq H$ and H has no proper $\preccurlyeq$ extension. For every l-group G and every infinite cardinal κ there is at least one l-group H such that $G \preccurlyeq_\kappa H$ and H has no proper $\preccurlyeq_\kappa$ extension.*

2. Uniqueness of the maximal extensions. The purpose of this section is the explicit construction of the maximal $\preccurlyeq$ and $\preccurlyeq_\kappa$ extensions of a given l-group G. A second construction by l-permutation group techniques will be outlined in Section 4. The present construction has several advantages over the l-permutation construction. First, it avoids the details of a particular Holland representation, which after all must be irrelevant since the definition of the maximal extension is independent of representation. Secondly, it is particularly simple when specialized to the representable case, essentially coinciding with Bleir's constructions in [**6**] and [**7**]. Finally, this is the appropriate construction for the universal mapping properties of Section 5.

A set $\mathscr{S} \subseteq \mathscr{P}$ is *large* if $P \subseteq Q \in S$ implies $P \in S$ and if $\vee \mathscr{S} = 1_\mathscr{P}$. If $\mathscr{S} \subseteq \mathscr{P}$ and k is a mapping from $\mathscr{S}$ into G then Pk denotes the image of $P \in \mathscr{S}$ under k, and dom (k) denotes $\mathscr{S}$. A map k: $\mathscr{S} \rightarrow G$ is *consistent* if its domain $\mathscr{S}$ is a large set of polars such that

$$(Pk)(Qk)^{-1} \in (P \wedge Q)^{\perp} \quad \text{for all } P, Q \in \mathscr{S}.$$

Given a convex l-subgroup C and element $g \in G$, C^g is the conjugate $g^{-1}Cg$. If $\mathscr{S} \subseteq \mathscr{P}$ and k: $\mathscr{S} \rightarrow G$ we shall write P^k for P^{Pk} and $\mathscr{S}^k$ for $\{P^k | P \in \mathscr{S}\}$.

LEMMA 2.1. *Let k: $\mathscr{S} \rightarrow G$ be a consistent map and suppose $P \subseteq Q \in \mathscr{S}$. Then*

$$P^k = P^{Pk} = P^{Qk}.$$

Proof. $(Pk)(Qk)^{-1} \in P^{\perp}$ and so $(Pk)(Qk)^{-1}$ commutes with every member of P. Therefore $P^{Pk} = P^{Qk}$.

Of particular interest are those consistent maps having the properties isolated in Lemmas 2.2 and 2.4.

LEMMA 2.2. *If k: $\mathscr{S} \rightarrow G$ is a consistent map then $\mathscr{S}^k$ is large if and only if $\vee \mathscr{S}^k = 1_{\mathscr{P}}$.*

Proof. Suppose $\vee \mathscr{S}^k = 1_{\mathscr{P}}$ and consider a polar $P \subseteq Q^{Qk}$ for some $Q \in \mathscr{S}$. There is a unique polar R such that $R^{Qk} = P$, and $R \in \mathscr{S}$ because $R \subseteq Q \in \mathscr{S}$. Therefore $R^k = R^{Rk} = R^{Qk} = P$, proving $P \in \mathscr{S}^k$ and proving $\mathscr{S}^k$ large.

Given a consistent map k: $\mathscr{S} \rightarrow G$ and prime T, define the *T-support of k* to be

$$T\text{-supp }(k) = \{Tx | T^x \not\supseteq P, \quad \text{some } P \in \mathscr{S}\}.$$

The point of the next lemma is that k can be interpreted as having a consistent action on each member of T-supp (k).

LEMMA 2.3. *Suppose k: $\mathscr{S} \rightarrow G$ is a consistent map, T a prime with $Tx \in T$-supp (k), and $P, Q \in \mathscr{S}$ such that $T^x \not\supseteq P$, $T^x \not\supseteq Q$. Then $Tx(Pk) = Tx(Qk)$.*

Proof. Since T^x is prime and fails to contain P or Q, $T^x \not\supseteq P \wedge Q$. Therefore

$$(Pk)(Qk)^{-1} \in (P \wedge Q)^{\perp} \subseteq T^x,$$

so $T^xPk = T^xQk$.

The previous lemma makes reasonable the notation Txk for $Tx(Pk)$ in case T is a prime such that $T^x \not\supseteq P \in$ dom (k) and k is a consistent map.

LEMMA 2.4. *For a consistent map k: $\mathscr{S} \rightarrow G$ the following are equivalent:*

(a) *If $P, Q \in \mathscr{S}$ and $1 \leqq s \leqq (Pk)(Qk)^{-1} \vee 1$ then $P^s \wedge Q = P \wedge Q$.*

(b) *There is no disjoint pair of nonzero polars $P, Q \in \mathscr{S}$ and element s satisfying $1 \leqq s \leqq (Pk)(Qk)^{-1} \vee 1$ and $P^s = Q$.*

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(c) *For every prime T, $Ta < Tb$ in T-*supp (k) *implies $Tak < Tbk$.*

Proof. (a) clearly implies (b). Suppose that (a) fails for $P, Q \in \mathscr{S}$ and element $1 \leqq s \leqq (Pk)(Qk)^{-1} \vee 1$. Observe that $(Pk)(Qk)^{-1} \in (P \wedge Q)^{\perp}$ implies $s \in (P \wedge Q)^{\perp}$. Therefore

$$P^s = (P \wedge Q)^s \vee R^s = (P \wedge Q) \vee R^s,$$

where $R = P \wedge Q^{\perp} \in \mathscr{S}$. The only way $P^s \wedge Q$ could fail to equal $P \wedge Q$ is if $R^s \wedge Q \neq 0_{\mathscr{P}}$. Let $V = R^s \wedge Q \in \mathscr{S}$ and let U be the unique polar satisfying $U^s = V$. Since $U \subseteq R$ it follows that $U \in \mathscr{S}$. Finally, let

$$u = s \wedge [(Uk)(Vk)^{-1} \vee 1].$$

The argument will be completed by showing $U^s = U^u$, for then U, V, and u violate condition (b). This is done by showing $su^{-1} \in U^{\perp}$, which in turn is accomplished by showing $Ts = Tu$ for any prime T such that $T \not\supseteq U$. For such a prime T it must be true that $T < Ts$, for otherwise $T = T^s \not\supseteq U^s = V$ together with $T \not\supseteq U$ implies $T \not\supseteq U \wedge V = 0_{\mathscr{P}}$, a contradiction. Since $1 \leqq s \leqq (Pk)(Qk)^{-1} \vee 1$, we have $Ts(Qk) \leqq T(Pk)$. But $T(Pk) = T(Uk)$ because $(Pk)(Uk)^{-1} \in U^{\perp} \subseteq T$ since $T \not\supseteq U$. Likewise $Ts(Qk) = Ts(Vk)$ because $(Qk)(Vk)^{-1} \in V^{\perp} \subseteq T^s$ since $T^s \not\supseteq U^s = V$. Combining the last three conditions gives

$$Ts(Vk) \leqq T(Uk) \quad \text{or} \quad T < Ts \leqq T(Uk)(Vk)^{-1}.$$

Hence

$$Ts = T(s \wedge [(Uk)(Vk)^{-1} \vee 1]) = Tu.$$

Now suppose k satisfies (a) and that T is a prime with cosets $Ta < Tb$ in T-supp (k). Without loss of generality assume $a < b$. If there is a single polar $P \in \mathscr{S}$ such that $T^a, T^b \not\supseteq P$ then

$$Tak = Ta(Pk) < Tb(Pk) = Tbk.$$

Now suppose there is no single $P \in \mathscr{S}$ such that $T^a, T^b \not\supseteq P$. Then there are polars P and Q in $\mathscr{S}$ such that $T^a \not\supseteq P$ and $T^b \not\supseteq Q$. Let

$$s = z^{-1}b \wedge ((Pk)(Qk)^{-1} \vee 1).$$

Since $Ta(Pk) \geqq Tb(Qk)$, it follows that

$$T^a(Pk)(Qk)^{-1} \geqq T^a(a^{-1}b),$$

hence

$$T^a s = T^a(a^{-1}b) \quad \text{and} \quad T^{as} = T^b.$$

But $T^a \not\supseteq P$ implies $T^b = T^{as} \not\supseteq P^s$, which together with $T^b \not\supseteq Q$ implies $T^b \not\supseteq P^s \wedge Q$. By (a) applied to s, $P^s \wedge Q = P \wedge Q$. But this contradicts the case hypothesis that no single polar P exists such that $T^a, T^b \not\supseteq P$.

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Suppose condition (b) fails for disjoint P, $U \in \mathscr{S}$ and element s satisfying

$$1 \leqq s \leqq (Pk)(Qk)^{-1} \vee 1 \quad \text{and} \quad P^s \wedge Q \neq 0_{\mathscr{P}}.$$

Let T be any prime such that $T^s \not\supseteq P^s \wedge Q$. Then $T \not\supseteq P$ and $T^s \not\supseteq Q$ imply T, $T^s \in T$-supp (k). Furthermore $T < Ts$, for if not then $T = T^s \not\supseteq Q$ together with $T \not\supseteq P$ implies $T \not\supseteq P \wedge Q = 0_{\mathscr{P}}$, a contradiction. Therefore $Tk = T(Pk)$ and $Tsk = Ts(Qk)$. Since

$$T < Ts \leqq T[(Pk)(Qk)^{-1} \vee 1],$$

we have

$$Tk = T(Pk) \geqq Ts(Qk) = Tsk,$$

which violates condition (c).

We shall use the symbol K for the set of consistent maps having the properties isolated in Lemmas 2.2 and 2.4. To reiterate, these properties for a consistent map $k: \mathscr{S} \rightarrow G$ are the following:

(1) $\vee \mathscr{S}^k = 1$.

(2) If $P, Q \in \mathscr{S}$ and $1 \leqq s \leqq (Pk)(Qk)^{-1} \vee 1$ then $P^s \wedge Q = P \wedge Q$.

Let us say that consistent maps k and m are *equivalent*, written $k \sim m$, provided there is a large set $\mathscr{T} \subseteq \operatorname{dom}(k) \cap \operatorname{dom}(m)$ such that $(Pk)(Pm)^{-1} \in P^{\perp}$ for all $P \in \mathscr{T}$.

LEMMA 2.5. *$\sim$ is an equivalence relation on K.*

Proof. Suppose $\mathscr{S}$ and $\mathscr{R}$ are large sets. Because any Boolean algebra $\mathscr{P}$ is infinitely distributive,

$$\begin{aligned} 1_{\mathscr{P}} = 1_{\mathscr{P}} \wedge 1_{\mathscr{P}} &= (\vee \mathscr{S}) \wedge (\vee \mathscr{R}) \\ &= \vee \{S \wedge R | S \in \mathscr{S}, R \in \mathscr{R}\} = \vee \{T | T \in \mathscr{S} \cap \mathscr{R}\}, \end{aligned}$$

proving $\mathscr{S} \cap \mathscr{R}$ large. Therefore if $k \sim m \sim n$ by virtue of large sets $\mathscr{S} \subseteq \operatorname{dom}(k) \cap \operatorname{dom}(m)$ and $\mathscr{R} \subseteq \operatorname{dom}(m) \cap \operatorname{dom}(n)$ such that $(Sk)(Sm)^{-1} \in S^{\perp}$ for all $S \in \mathscr{S}$ and $(Rm)(Rn)^{-1} \in R^{\perp}$ for all $R \in \mathscr{R}$, then

$$(Tk)(Tn)^{-1} = (Tk)(Tm)^{-1}(Tm)(Tn)^{-1} \in T^{\perp}$$

for all T in the large set $\mathscr{S} \cap \mathscr{R}$. This proves that $\sim$, which is clearly reflexive and symmetric, is also transitive.

LEMMA 2.6. *Consistent maps k and m are equivalent if and only if $(Pk)(Pm)^{-1} \in P^{\perp}$ for all $P \in \operatorname{dom}(k) \cap \operatorname{dom}(m)$.*

Proof. Suppose $\mathscr{T}$ is a large subset of $\operatorname{dom}(k) \cap \operatorname{dom}(m)$ such that $(Qk)(Qm)^{-1} \in Q^{\perp}$ for all $Q \in \mathscr{T}$, and suppose $P \in \operatorname{dom}(k) \cap \operatorname{dom}(m)$. For each $Q \in \mathscr{T}$ such that $Q \subseteq P$ it is true that $(Pk)(Qk)^{-1}$, $(Pm)(Qm)^{-1}$

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$\in Q^{\perp}$, and hence that

$$(Pk)(Pm)^{-1} = (Pk)(Qk)^{-1}(Qk)(Qm)^{-1}(Qm)(Pm)^{-1} \in Q^{\perp}.$$

But $\vee \mathscr{T} = 1_{\mathscr{P}}$ implies $\vee \{Q \in \mathscr{T} | Q \subseteq P\} = P$, so that

$$(Pk)(Pm)^{-1} \in \wedge \{Q^{\perp} | Q \in \mathscr{T}, Q \subseteq P\} = P^{\perp}.$$

LEMMA 2.7. *If k and m are equivalent consistent maps then k has property* (1) *or* (2) *if and only if m has the same property.*

Proof. Let dom $(k) = \mathscr{S}$ and dom $(m) = \mathscr{R}$. Suppose $\vee \mathscr{S}^k = 1_{\mathscr{P}}$ and consider an arbitrary nonzero polar Q. Then there must be some $P \in \mathscr{S}$ such that $0_{\mathscr{P}} \neq P^k \subseteq Q$. Let $R \in \mathscr{S} \cap \mathscr{R}$ satisfy $0_{\mathscr{P}} \neq R \subseteq P$. Now $k \sim m$ implies $(Rk)(Rm)^{-1} \in R^{\perp}$, so

$$R^m = R^k = R^{Pk} \subseteq P^{Pk} \subseteq Q.$$

This proves $\vee \mathscr{R}^m = 1_{\mathscr{P}}$.

Now suppose k enjoys property (2) but m does not. Consider disjoint $P, Q \in \mathscr{R}$ and element s such that

$$1 \leqq s \leqq (Pm)(Qm)^{-1} \vee 1 \quad \text{and} \quad P^s = Q.$$

Let

$$u = s \wedge [(Pk)(Qk)^{-1} \vee 1].$$

To show that $P^s = P^u$ we need only show $su^{-1} \in P^{\perp}$, which can be done by showing $Tu = Ts$ for any prime T such that $T \not\supseteq P$. For such a prime T it is true that $T < Ts$, for otherwise $T = T^s \not\supseteq P^s = Q$ together with $T \not\supseteq P$ implies $T \not\supseteq P \wedge Q = 0_{\mathscr{P}}$, a contradiction. Therefore

$$T < Ts \leqq T[(Pm)(Qm)^{-1} \vee 1] \quad \text{implies} \quad Ts(Qm) \leqq T(Pm).$$

But $T(Pm) = T(Pk)$ because $(Qm)(Qk)^{-1} \in Q^{\perp} \subseteq T^s$ since $T \not\supseteq P$. Likewise $Ts(Qm) = Ts(Qk)$ because $(Qm)(Qk)^{-1} \in Q^{\perp} \subseteq T^s$ since $T^s \not\supseteq P^s = Q$. Combining the last three conditions gives

$$Ts(Qk) \leqq T(Pk) \quad \text{or} \quad Ts = T[s \wedge [(Pk)(Qk)^{-1} \vee 1]] = Tu.$$

We have thus shown $P^s = P^u$. But since

$$1 \leqq u \leqq (Pk)(Qk)^{-1} \vee 1,$$

property (2) applied to k is contradicted by $P^u = P^s = Q$. Therefore m must also enjoy property (2).

To rephrase the preceding lemma, any consistent map equivalent to a member of K is itself a member of K.

LEMMA 2.8. *Suppose $k: \mathscr{S} \to G$ is a consistent map. Then there is a unique set $\mathscr{R}$ such that $\mathscr{S} \subseteq \mathscr{R} \subseteq \mathscr{P}$ and such that $\mathscr{R}$ is maximal with respect to*

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the existence of a consistent map $m: \mathscr{R} \to G$ *extending* k. k *is equivalent to every such* m, *and* $k \in K$ *if and only if each* $m \in K$.

Proof. Let $\mathscr{R}$ be the union of $\mathscr{S}$ with the set of all polars Q not in $\mathscr{S}$ for which there is some $g \in G$ such that $g(Pk)^{-1} \in P^{\perp}$ for all $P \in \mathscr{S}$ such that $P \subseteq Q$. Define $Qm = g$. To show m consistent consider $Q, R \in \mathscr{R}$ and suppose $Qm = g$, $Rm = h$. For each $P \in \mathscr{S}$ such that $P \subseteq Q \wedge R$,

$$(Qm)(Rm)^{-1} = gh^{-1} = g(Pk)^{-1}(Pk)h^{-1} \in P^{\perp}.$$

Since $\vee \mathscr{S} = 1_{\mathscr{P}}$,

$$\vee \{P \in \mathscr{S} \mid P \subseteq Q \wedge R\} = Q \wedge R.$$

Therefore

$$\begin{aligned}(Qm)(Rm)^{-1} &\in \wedge \{P^{\perp} \mid P \in \mathscr{S}, P \subseteq Q \wedge R\} \\ &= (\vee \{P \in \mathscr{S} \mid P \subseteq Q \wedge R\})^{\perp} = (Q \wedge R)^{\perp}.\end{aligned}$$

$\mathscr{R}$ is maximal, since it clearly contains the domain of every consistent extension of k. k and m are equivalent since they agree on $\mathscr{S}$.

We shall refer to the set $\mathscr{R}$ of Lemma 2.8 as the *maximal domain of* k. This concept has particular importance in Sections 2, 3, and 4.

For each $k \in K$ let $[k] = \{m \in K \mid k \sim m\}$, the equivalence class of k, and let $\bar{G}$ be $\{[k] \mid k \in K\}$. The task of the rest of this section is to endow $\bar{G}$ with group and lattice operations in such a way as to make it the maximal $\preccurlyeq$ extension of G. Before doing so, however, it is profitable to list several more technical lemmas.

LEMMA 2.9. *Suppose* $g \in G$ *and* $P \in \mathscr{P}$. *Denote* $g^{+} = g \vee 1$ *and* $g^{-} = g \wedge 1$ *by* a *and* b, *respectively. Then*

$$P^{g} = [P \wedge g^{\perp}] \vee [P \wedge a^{\perp\perp}]^{a} \vee [P \wedge b^{\perp\perp}]^{b}.$$

Proof. Since $\vee \{g^{\perp}, a^{\perp\perp}, b^{\perp\perp}\} = 1_{\mathscr{P}}$,

$$P = [P \wedge g^{\perp}] \vee [P \wedge a^{\perp\perp}] \vee [P \wedge b^{\perp\perp}]$$

and

$$P^{g} = [P \wedge g^{\perp}]^{g} \wedge [P \wedge a^{\perp\perp}]^{g} \vee [P \wedge b^{\perp\perp}]^{g}.$$

But $[P \wedge g^{\perp}]^{g} = [P \wedge g^{\perp}]$ because g commutes with all members of $g^{\perp}$,

$$[P \wedge a^{\perp\perp}]^{g} = [P \wedge a^{\perp\perp}]^{a}$$

because $bg^{-1} = a^{-1}$ commutes with all members of $b^{\perp\perp}$.

COROLLARY 2.10. *Suppose* P *and* Q *are disjoint polars and* g *is an element of an* l*-group* G, *and suppose* g^{+} *and* g^{-} *are denoted* a *and* b. *If* $P^{a} \wedge Q = P^{b} \wedge Q = 0_{\mathscr{P}}$ *then* $P^{g} \wedge Q = 0_{\mathscr{P}}$.

LEMMA 2.11. *Suppose $k \in K$ and P and Q are disjoint members of* dom (k). *Then P^k and Q^k are disjoint.*

Proof. By Property (2) applied twice,

$$P^a \wedge Q = P^b \wedge Q = 0_{\mathscr{P}},$$

where $a = (Pk)(Qk)^{-1} \vee 1$ and $b = (Pk)(Qk)^{-1} \wedge 1$. By Corollary 2.10, $P^k \wedge Q^k = 0_{\mathscr{P}}$.

LEMMA 2.12. *Suppose $k \in K$ and $P, Q \in$* dom (k). *Then*

$$(P \wedge Q)^k = P^k \wedge Q^k.$$

Proof. $P = (P \wedge Q) \vee (P \wedge Q^\perp)$ implies $P^{Pk} = (P \wedge Q)^{Pk} \vee (P \wedge Q^\perp)^{Pk}$, which by Lemma 2.1 can be written

$$P^k = (P \wedge Q)^k \vee (P \wedge Q^\perp)^k.$$

Similarly,

$$Q^k = (P \wedge Q)^k \vee (Q \wedge P^\perp)^k.$$

Since P and $Q \wedge P^\perp$ are disjoint members of dom (k), Lemma 2.11 implies P^k and $(Q \wedge P^\perp)^k$ are disjoint. Similarly, Q^k and $(P \wedge Q^\perp)^k$ are disjoint. Therefore

$$P^k \wedge Q^k = (P \wedge Q)^k.$$

LEMMA 2.13. *Suppose $k, m \in K$ and let*

$$\mathscr{S} = \{R \in \text{dom } (k) | \, R^k \in \text{dom } (m)\}.$$

Then $\mathscr{S}$ is large.

Proof. If $P \subseteq R \in \mathscr{S}$ then $P \in$ dom (k) and $P^k = P^{Rk} \subseteq R^k \in$ dom (m), showing $P \in \mathscr{S}$. Now fix $Q \in$ dom (k) and let $(Qk)^{-1} = g$. Since dom $(m)^g$ is large,

$$\vee \{R \subseteq Q | \, R \in \text{dom } (m)^g\} = Q.$$

But $\{R \subseteq Q| \, R \in \text{dom } (m)^g\} \subseteq \mathscr{S}$, which proves $\vee \mathscr{S} \supseteq Q$ for all $Q \in$ dom (k). That is,

$$\vee \mathscr{S} \supseteq \vee \text{ dom } (k) = 1_{\mathscr{P}}.$$

PROPOSITION 2.14. *Suppose $k, m \in K$ and $\mathscr{S}$ is as in Lemma* 2.13. *Define $f: \mathscr{S} \rightarrow G$ by $Rf = (Rk)(R^k m)$ for all $R \in \mathscr{S}$. Then $f \in K$.*

Proof. To verify the consistency of f consider $P, Q \in \mathscr{S}$ and let $x = (Qk)(Pk)^{-1}$. Note that $x \in (P \wedge Q)^\perp$ since $P, Q \in$ dom (k). Similarly

$$y = (P^k m)(Q^k m)^{-1} \in (P^k \wedge Q^k)^\perp.$$

https://doi.org/10.4153/CJM-1982-042-5 Published online by Cambridge University Press

Therefore

$$z = (Pk)y(Pk)^{-1} \in (P \wedge Q^x)^\perp.$$

But $x \in (P \wedge Q)^\perp$ implies $P \wedge Q = P^x \wedge Q^x$, whence

$$P \wedge Q \subseteq P \wedge Q^x \quad \text{and} \quad (P \wedge Q^x)^\perp \subseteq (P \wedge Q)^\perp.$$

Therefore $(Pf)(Qf)^{-1} = zx^{-1} \in (P \wedge Q)^\perp$.

To verify property (1) we must prove $\vee \mathscr{S}^f = 1$. Consider $0_{\mathscr{P}} \neq V \in \mathscr{P}$. Because dom $(m)^m$ is large there is some $Q \in$ dom (m) with $0_{\mathscr{P}} \neq Q^m \subseteq V$. Because dom $(k)^k$ is large there is some $P \in$ dom (k) with $0_{\mathscr{P}} \neq P^k \subseteq Q$. By definition $P \in \mathscr{S}$, and $P^f = (P^k)^x$ where $x = P^k m$. But $P^k \subseteq Q$ implies $(P^k)^x = (P^k)^{Qm}$, hence $P^f \subseteq V$. This shows $\vee \mathscr{S}^f = 1_{\mathscr{P}}$.

To verify property (2) suppose $Ta < Tb$ in T-supp (f) for some prime T. More explicitly, suppose $T^a \not\supseteq P$ and $T^b \not\supseteq Q$ for $P, Q \in \mathscr{S}$. Then $Ta < Tb$ in T-supp (k) since $P, Q \in$ dom (k), and because k satisfies (2) it follows that $Ta(Pk) < Tb(Qk)$. Now

$$T^{a(Pk)} \not\supseteq P^k \in \text{dom } (m) \quad \text{and} \quad T^{b(Qk)} \not\supseteq Q^k \in \text{dom } (m),$$

hence

$$Ta(Pk)(P^k m) < Tb(Qk)(Q^k m)$$

because m satisfies (2). That is, $Taf < Tbf$.

For $k, m \in K$ we shall denote the function f of Proposition 2.14 by km.

LEMMA 2.15. *Given $k, m \in K$, define $[k]\,[m] = [km]$. This multiplication is well defined and associative.*

Proof. Suppose $j \sim k$ and $m \sim n$ in K. Let $\mathscr{R}$ denote the intersection of the domains of the products km and jn. Then for $P \in \mathscr{R}$, $P^j = P^k$ because $(Pj)(Pk)^{-1} \in P^\perp$. Therefore

$$a = (P^k m)(P^j n)^{-1} \in (P^k)^\perp = (P^\perp)^{Pk},$$

say $a = y^{Pk}$ where $y \in P^\perp$. Then

$$(Pkm)(Pjn)^{-1} \in (P^k)a(Pj)^{-1} = y(Pk)(Pj)^{-1} \in P^\perp,$$

which proves $km \sim jn$.

It remains to show $([k]\,[m])\,[n] = [k]\,([m]\,[n])$; in fact, the stronger formula $(km)n = k(mn)$ holds for all $k, m, n \in K$. Both sides of the last equation work out to be the function with value abc at Q, where $a = Qk$, $b = Q^a m$, and $c = Q^{ab} n$, and where Q is a polar such that $Q \in$ dom (k), $Q^a \in$ dom (m), and $Q^{ab} \in$ dom (n).

Define $i\colon \mathscr{P} \to G$ to be the map which takes each $P \in \mathscr{P}$ to $1 \in G$.

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Clearly $i \in K$ and $ik = ki = k$ for all $k \in K$. Thus $\bar{G}$ has an identity $[i]$, which we shall denote 1.

LEMMA 2.16. *Given $k \in K$ define h:* dom $(k)^k \rightarrow G$ *by* $(P^k)h = (Pk)^{-1}$ *for all* $P \in$ dom (k). *Then* $h \in K$, *and* $hk \sim kh \sim i$.

Proof. To verify the consistency of h consider $P^k, Q^k \in$ dom $(k)^k$. Then

$$\begin{aligned}(P^k h)(Q^k h)^{-1} &= (Pk)^{-1}(Qk) = [(Qk)(Pk)^{-1}]^{Qk} \in ((P \wedge Q)^{\perp})^{Qk} \\ &= ((P \wedge Q)^{Qk})^{\perp} = ((P \wedge Q)^k)^{\perp} = (P^k \wedge Q^k)^{\perp}.\end{aligned}$$

Property (1) holds for h since

$$\text{dom } (h)^h = (\text{dom } (k)^k)^h = \text{dom } (k).$$

To verify property (2) consider $Ta < Tb$ in T-dom (h); say $T^a \not\supseteq P^k$ and $T^b \not\supseteq Q^k$ for $P, Q \in$ dom (k). If $x = a(Pk)^{-1}$ and $y = b(Qk)^{-1}$ then $T^x \not\supseteq P$ and $T^y \not\supseteq Q$, so that $Tx, Ty \in T$-dom (k). Now $Tx = Ty$ implies by Lemma 2.3 that $Ta = Tx(Pk) = Ty(Qk) = Tb$, and $Ty < Tx$ implies by property (2) applied to k that $Tb = Tyk < Txk = Ta$. The only remaining possibility is that $Tah = Ta(Pk)^{-1} = Tx < Ty = Tb(Qk)^{-1} = Tbh$. Finally, it is immediate from the definitions that dom (kh) and dom (hk) are dom (k) and dom $(k)^k$, respectively, and that Pkh and Qhk are both 1 for polars P and Q in the appropriate domains.

Let us denote the map h of Lemma 2.16 by k^{-1}. This is, of course, a slight abuse of the notion of inverse, since k^{-1} is not an inverse of k in K, but $[k^{-1}]$ is an inverse of $[k]$ in $\bar{G}$. The reader should observe that the inverse operation is well defined in the sense that $k \sim m$ implies $k^{-1} \sim m^{-1}$.

Having made $\bar{G}$ a group, it remains to impose a compatible lattice structure. This can be done in the simplest way.

PROPOSITION 2.17. *Given $k, m \in K$ define f:* dom $(k) \cap$ dom $(m) \rightarrow G$ *by* $Pf = Pk \vee Pm$. *Then* $f \in K$.

Proof. To verify the consistency of f consider $P, Q \in$ dom (f). Then $(Pf)(Qf)^{-1}$ can be written as either

$$[a \wedge (Pk)(Qm)^{-1}] \vee [(Pm)(Qk)^{-1} \wedge b]$$

or as

$$[a \vee (Pm)(Qk)^{-1}] \wedge [(Pk)(Qm)^{-1} \vee b],$$

where $a = (Pk)(Qk)^{-1}$ and $b = (Pm)(Qm)^{-1}$. Thus

$$a \wedge b \leqq (Pf)(Qf)^{-1} \leqq a \vee b,$$

and since $a, b \in (P \wedge Q)^{\perp}$, it follows that

$$(Pf)(Qf)^{-1} \in (P \wedge Q)^{\perp}.$$

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To verify that f has property (2) consider $Ta < Tb$ in T-supp (f). Since Ta and Tb must also be in T-supp (k) and T-supp (m) we have $Tak < Tbk$ and $Tam < Tbm$. It follows at once that

$$Taf = Tak \vee Tam < Tbk \vee Tbm = Tbf.$$

To verify $\vee$ dom $(f)^f = 1_{\mathscr{P}}$ consider an arbitrary polar $V \neq 0_{\mathscr{P}}$. Because dom (k) is large, there is a polar $P \in$ dom (k) such that $0_{\mathscr{P}} \neq P^k \subseteq V$. Because dom (m) is large, there is a polar $Q \in$ dom (m) such that $0_{\mathscr{P}} \neq Q^m \subseteq P^k$. Without loss of generality we may assume P, $Q \in$ dom (f) and $Q^m = P^k$. If $P \not\subseteq [(Pm)(Pk)^{-1} \vee 1]^{\perp\perp}$ then there is some $R \in$ dom (f) such that

$$0_{\mathscr{P}} \neq R \subseteq [(Pm)(Pk)^{-1} \vee 1]^{\perp} \cap P, \quad \text{and}$$
$$R^f = R^{Rm \vee Rk} = R^{Pm \vee Pk} = R^{Pk} \subseteq P^{Pk} \subseteq V,$$

in which case we are done. Therefore assume that

$$P \subseteq [(Pm)(Pk)^{-1} \vee 1]^{\perp\perp}$$

and by a similar argument that

$$Q \subseteq [(Qk)(Qm)^{-1} \vee 1]^{\perp\perp}.$$

Now P and Q may not coincide, for then

$$P \subseteq [(Pm)(Pk)^{-1} \vee 1]^{\perp\perp} = [(Qm)(Qk)^{-1} \vee 1]^{\perp\perp}$$
$$\subseteq [(Qk)(Qm)^{-1} \vee 1]^{\perp} \supseteq Q,$$

a contradiction. Let us suppose $Q \wedge P^{\perp} \neq 0_{\mathscr{P}}$; the argument for $P \wedge Q^{\perp} \neq 0_{\mathscr{P}}$ is analagous. Let T be any prime such that $T \not\supseteq Q \wedge P^{\perp}$ and let $x = (Qm)(Pk)^{-1}$. Then $T^x \not\supseteq P$, meaning that T and Tx are different members of T-dom (f). If $T < Tx$ then

$$T(Qk) = Tk < Txk = Tx(Pk) = T(Qm).$$

This is a contradiction because $T \not\supseteq Q \subseteq [(Qk)(Qm)^{-1} \vee 1]^{\perp\perp}$ implies $T(Qk) > T(Qm)$. On the other hand $Tx < T$ implies

$$T^x < T^x x^{-1} \quad \text{and}$$
$$T^x(Pm) = T^x m < T^x x^{-1} m = T^x x^{-1}(Qm) = T^x(Pk).$$

But this is also a contradiction because

$$T^x \not\supseteq P \subseteq [(Pm)(Pk)^{-1} \vee 1]^{\perp\perp} \quad \text{implies} \quad T^x(Pm) > T^x(Pk).$$

One has no recourse but to concede that $\vee$ dom $(f)^f = 1_{\mathscr{P}}$.

Denote the map f of Proposition 2.17 by $k \vee m$; the function $k \wedge m$ is defined dually. It is routine to verify that $\sim$ respects $\vee$ and $\wedge$ in the sense that $k \sim k'$ and $m \sim m'$ imply $k \vee m \sim k' \vee m'$ and $k \wedge m \sim$

$k' \wedge m'$. Therefore we can define $[k] \vee [m] = [k \vee m]$ and $[k] \wedge [m] = [k \wedge m]$.

THEOREM 2.18. *$\bar{G}$ is a lattice ordered group.*

Proof. Six identities and their duals must be verified:

$$x \vee x = x, x \vee y = y \vee x, (x \vee y) \vee z = x \vee (y \vee z),$$
$$x \wedge (x \vee y) = x, x(y \vee z) = xy \vee xz, \quad \text{and} \quad (x \vee y)z = xz \vee yz.$$

Each of the first five follows directly from the fact that the same law holds in G. To prove the sixth, consider $k, m, n \in K$ and let $h = kn \vee km \in K$. Let

$$\mathcal{S} = \{P \in \operatorname{dom}(k) \cap \operatorname{dom}(m) | P \subseteq [(Pk)(Pm)^{-1} \vee 1]^{\perp\perp} \quad \text{or}$$
$$P \subseteq [(Pk)(Pm)^{-1} \vee 1]^{\perp}\}.$$

Since each $P \in \operatorname{dom}(k) \cap \operatorname{dom}(m)$ can be written

$$(P \wedge [(Pk)(Pm)^{-1} \vee 1]^{\perp\perp}) \vee (P \wedge [(Pk)(Pm)^{-1} \vee 1]^{\perp}),$$

it follows that $\mathcal{S}$ is large. Let $f: \mathcal{S} \rightarrow G$ be defined by

$$Pf = Pk \quad \text{if} \quad P \subseteq [(Pk)(Pm)^{-1} \vee 1]^{\perp\perp}, \text{ and}$$
$$Pf = Pm \quad \text{if} \quad P \subseteq [(Pk)(Pm)^{-1} \vee 1]^{\perp}.$$

Then $f \sim k \vee m$; we shall show $fn \sim h$. For that purpose let

$$\mathcal{R} = \{P \in \mathcal{S} | P^k, P^m \in \operatorname{dom}(n)\},$$

also a large set. The claim is that $(Pfn)(Ph)^{-1} \in P^{\perp}$ for all $P \in \mathcal{R}$. To establish this claim it is enough to prove $T(Pfn) = T(Ph)$ for all primes T such that $T \not\supseteq P$. Two cases arise; the first is when

$$P \subseteq [(Pk)(Pm)^{-1} \vee 1]^{\perp\perp}.$$

If T is any prime such that $T \not\supseteq P$ then $T(Pk) \geqq T(Pm)$. Since $T^{Pk} \not\supseteq P^k \in \operatorname{dom}(n)$ and $T^{Pm} \not\supseteq P^m \in \operatorname{dom}(n)$, $T(Pk)$ and $T(Pm)$ are in T-supp (n). Therefore

$$T(Pk)(P^k n) = T(Pk)n \geqq T(Pm)n = T(Pm)(P^m n),$$

meaning

$$T(Ph) = T[(Pk)(P^k n) \vee (Pm)(P^m n)] = T(Pk)(P^k n)$$
$$= T(Pf)(P^f n) = T(Pfn).$$

The second case arises when $P \subseteq [(Pk)(Pm)^{-1} \vee 1]^{\perp}$. If T is any prime such that $T \not\supseteq P$ then $T(Pm) \geqq T(Pk)$. By an argument directly analagous to the first case we obtain $T(Ph) = T(Pfn)$.

THEOREM 2.19. *With each $g \in G$ associate the equivalence class of the map k_g: $\mathscr{P} \to G$ defined by $Pk_g = g$ for all $P \in \mathscr{P}$. Then, under this association, $G \preccurlyeq \bar{G}$.*

Proof. A little reflection reveals the map $g \to [k_g]$ to be an l-homomorphism. To see that this map is one-to-one consider $g \neq h$ in G and let

$$P = (gh^{-1})^{\perp\perp} \in \text{dom }(k_g) \cap \text{dom }(k_h).$$

Since $(Pk_g)(Pk_h)^{-1} = gh^{-1} \notin P^\perp$, $[k_g] \neq [k_h]$. Therefore G is an l-subgroup of $\bar{G}$, and we shall henceforth use the symbols $[k_g]$ and g interchangeably for elements of G.

To prove G order dense in $\bar{G}$ consider $1 < x = [k] \in \bar{G}$. By replacing k with $k \vee k_1$ if necessary, we may assume k: dom $(k) \to G^+$. Since $x \neq 1$, there is some polar $P \in$ dom (k) such that $Pk^{\perp\perp} \cap P \neq 0_{\mathscr{P}}$, since otherwise k is equivalent to k_1. Without loss of generality $P \subseteq Pk^{\perp\perp}$. Find $g \in G$ such that $1 < g \leqq Pk$ and $g \in P$. Then $k_g \wedge k = k_g$, so $1 < g \leqq x$.

Finally consider an arbitrary $x = [k] \in \bar{G}$, let $P \in$ dom (k) satisfy $P \neq 0_{\mathscr{P}}$, and let $Pk = g \in G$. We claim $P \subseteq (xg^{-1})^\perp$. To see this, consider $1 < p \in P$, let

$$(kp \wedge kk_g{}^{-1}) \vee k_1 = h,$$

and let

$$\mathscr{S} = \{Q \in \text{dom }(k) |\, Q \subseteq P \quad \text{or} \quad Q \subseteq P^\perp\},$$

a large set. For $Q \in \mathscr{S}$ such that $Q \subseteq P^\perp$ it is true that

$$(Qk_p)(Qk_1)^{-1} = p \in P \subseteq Q^\perp$$

and therefore $(Qh)(Qk_1)^{-1} \in Q^\perp$. For $Q \in \mathscr{S}$ such that $Q \subseteq P$ it is true that

$$Qkk_g{}^{-1} = (Qk)(Q^k k_g)^{-1} = (Qk)(Pk)^{-1}(Pk)(g^{-1}) = (Qk)(Pk)^{-1} \in Q^\perp,$$

implying $(Qh)(Qk_1)^{-1} \in Q^\perp$. We have proved $h \sim k_1$ or $p \wedge xg^{-1} = 1$, meaning $P \subseteq (xy^{-1})^\perp$. But since

$$\vee \text{ dom }(k) = 1_{\mathscr{P}} \quad \text{in } G,$$

$$\vee \{P^{\perp\perp} |\, P \in \text{dom }(k)\} = 1_{\mathscr{P}} \quad \text{in } \bar{G},$$

hence

$$\wedge \{P^\perp |\, P \in \text{dom }(k)\} = 0_{\mathscr{P}} \quad \text{in } \bar{G}$$

and therefore

$$\wedge \{(xg^{-1})^{\perp\perp} |\, g \in G\} = 0_{\mathscr{P}} \quad \text{in } \bar{G}.$$

THEOREM 2.20. *Suppose G is a fixed l-group. Then for any l-group H, $G \preccurlyeq H$ if and only if there is some l-monomorphism θ: $H \to \bar{G}$ over G.*

Proof. Suppose $G \preccurlyeq H$, and fix $h \in H$. Define

$$\mathscr{S}_h = \{P \text{ a polar of } G |\ hg^{-1} \in P^{\perp} \quad \text{for some } g \in G\}.$$

$\mathscr{S}_h$ is large because $\wedge \{(hg^{-1})^{\perp\perp} |\ g \in G\} = 0_{\mathscr{P}}$ in H implies

$$\vee \{(hg^{-1})^{\perp} \cap G |\ g \in G\} = 1_{\mathscr{P}} \quad \text{in } G.$$

Define $k_h: \mathscr{S}_h \to G$ to be any function such that $h(Pk_h)^{-1} \in P^{\perp}$ for all $P \in \mathscr{S}_h$, and define $h\theta = [k_h]$. θ is well defined, for if $m: \mathscr{S}_h \to G$ is a second function satisfying $h(Pm)^{-1} \in P^{\perp}$ for all $P \in \mathscr{S}_h$ then

$$(Pm)(Pk_h)^{-1} = (Pm)h^{-1}h(Pk_h)^{-1} \in P^{\perp} \quad \text{for all } P \in \mathscr{S}_h$$

and so $k_h \sim m$.

We claim $k_h \in K$. To verify the consistency of K_h simply observe that for $P, Q \in \mathscr{S}_h$,

$$(Pk_h)(Qk_h)^{-1} = ((Pk_h)h^{-1})(h(Qk_h)^{-1}) \in P^{\perp} \vee Q^{\perp} = (P \wedge Q)^{\perp}.$$

Property (1) holds because $\mathscr{S}_h$ large implies $\{P^{\perp\perp} |\ P \in \mathscr{S}_h\}^h$ large, and it can easily be shown that $\{(P^{\perp\perp})^h \cap G |\ P \in \mathscr{S}_h\}$ is $\mathscr{S}_h$ conjugated by k_h. To verify property (2) consider $Ta < Tb$ in T-supp (k_h). More explicitly, suppose $T^a \not\supseteq P \in \mathscr{S}_h$ and $T^b \not\supseteq Q \in \mathscr{S}_h$. Let R be any prime of H such that $R \cap G = T$. The crucial point is that $Ra < Rb$ and therefore $Rah < Rbh$. But $R^a \not\supseteq P^{\perp\perp}$ implies $h(Pk_h)^{-1} \in P^{\perp} \subseteq R^a$ and so $R^a h = R^a(Pk_h)$ or $Rah = Ra(Pk_h)$. Likewise $Rbh = Rb(Qk_h)$. Combining these conditions gives

$$Ra(Pk_h) < Rb(Qk_h),$$

which implies

$$Tah = Ta(Pk_h) < Tb(Qk_h) = Tbh.$$

To show that θ preserves the lattice operations consider $f, h \in H$, $P \in \mathscr{S}_f \cap \mathscr{S}_h$, and let $Pk_f = r$, $Pk_h = t$. By expanding $(f \vee h)(r \vee t)^{-1}$ in two ways one sees that it is bounded below by $fr^{-1} \wedge ht^{-1}$ and above by $fr^{-1} \vee ht^{-1}$. Since $fr^{-1}, ht^{-1} \in P^{\perp}$, it follows that

$$(f \vee h)(r \vee t)^{-1} \in P^{\perp}.$$

But since $r \vee t = Pk_j$ where $j = f \vee h$, this shows $\mathscr{S}_j \subseteq \mathscr{S}_f \cap \mathscr{S}_h$ and $k_j \sim k_f \vee k_h$, so

$$f\theta \vee h\theta = (f \vee h)\theta.$$

A dual argument settles the issue of $\wedge$.

To show that θ preserves group operations consider $f, h \in H$, $P \in \mathscr{S}_f$ such that $P^r \in \mathscr{S}_h$, where $r = Pk_f$ and $t = P^r k_h$. Since $fr^{-1} \in P^{\perp}$ and $ht^{-1} \in (P^r)^{\perp}$,

$$fh(rt)^{-1} = (fr^{-1})r(ht^{-1})r^{-1} \in P^{\perp}.$$

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Therefore $P \in \mathscr{S}_j$, where $j = fh$, and since Pk_j can be taken to be rt and since $rt = Pk_f k_h$, it follows that $k_j \sim k_f k_h$ and θ preserves products. To show that for $h \in H$, $(h\theta)^{-1} = (h^{-1})\theta$, let P be any polar of $\mathscr{S}_h$, let $\mathscr{R}$ be $\mathscr{S}_h$ conjugated by k_h, and let $Pk_h = r$. Then $hr^{-1} \in P^\perp$ implies $h^{-1}r \in (P^\perp)^r = (P^r)^\perp$. This shows that $\mathscr{R} \subseteq \mathscr{S}_j$, where $j = h^{-1}$, and that $k_j \sim k_h^{-1}$. Therefore

$$(h\theta)^{-1} = [k_h]^{-1} = [k_h^{-1}] = [k_j] = (h^{-1})\theta.$$

THEOREM 2.21. *Every maximal $\preccurlyeq$ extension of G is isomorphic to $\bar{G}$ over G.*

Proof. If H is a maximal $\preccurlyeq$ extension of G then there is an l-monomorphism $\theta: H \to \bar{G}$ over G. θ is onto, since otherwise H would have a proper $\preccurlyeq$ extension.

For each cardinal number κ define $\bar{G}^\kappa$ to be those $[k] \in \bar{G}$ such that the maximal domain of k contains a subset $\mathscr{S}$ of cardinality less than κ such that $\vee \mathscr{S} = 1_{\mathscr{P}}$. For example, $\bar{G}^\omega$ consists of those $[k] \in \bar{G}$ such that the ideal of $\mathscr{P}$ generated by the maximal domain of k is all of $\mathscr{P}$. $\bar{G}^\kappa$ is well defined because equivalent consistent maps have the same maximal domain.

LEMMA 2.22. $\bar{G}^\kappa \leqq \bar{G}$.

Proof. Suppose $[k], [m] \in \bar{G}^\kappa$; say $\mathscr{R} \subseteq \operatorname{dom}(k)$ and $\mathscr{S} \subseteq \operatorname{dom}(m)$ are sets of cardinality less than κ such that $\vee \mathscr{S} = \vee \mathscr{R} = 1_{\mathscr{P}}$. Consider the sets

$$\{P \wedge (Pk)Q(Pk)^{-1} | \, P \in \mathscr{R}, Q \in \mathscr{S}\},$$
$$\{P^k | \, P \in \mathscr{R}\}, \quad \text{and} \quad \{P \wedge Q | \, P \in \mathscr{R}, Q \in \mathscr{S}\}.$$

These sets have supremum $1_{\mathscr{P}}$ and have cardinality less than κ. Since they are contained in $\operatorname{dom}(km)$, $\operatorname{dom}(k^{-1})$ and $\operatorname{dom}(k \vee m)$, respectively, it follows that $[k]\,[m]$, $[k]^{-1}$, $[k] \vee [m] \in \bar{G}^\kappa$.

LEMMA 2.23. $G \preccurlyeq_\kappa \bar{G}^\kappa$.

Proof. Fix $g \in G$ and let $k_g: \mathscr{P} \to G$ have the meaning of Theorem 2.19. Then $\{1_{\mathscr{P}}\}$ is a maximal disjoint subset of the maximal domain of k_g of cardinality 1. This shows $G \leqq \bar{G}^\kappa$. Now fix $h = [k] \in \bar{G}^\kappa$; suppose $\mathscr{R} \subseteq \operatorname{dom}(k)$ is a set of cardinality less than κ such that $\vee \mathscr{R} = 1_{\mathscr{P}}$. Let $X = \{Pk | \, P \in \mathscr{R}\}$. Observe that for any $P \in \operatorname{dom}(k)$, $h(Pk)^{-1} \in P^\perp$ or $P \subseteq (h(Pk)^{-1})^\perp$. Therefore

$$\vee \{(hx^{-1})^\perp | \, x \in X\} \supseteq \vee \mathscr{R} = 1_{\mathscr{P}}.$$

This shows $G \preccurlyeq_\kappa \bar{G}^\kappa$.

THEOREM 2.24. *Suppose G is a fixed l-group. Then for any l-group H, $G \preccurlyeq_\kappa H$ if and only if there is an l-monomorphism θ: $H \to \bar{G}^\kappa$ over G.*

Proof. Observe that if $G \preccurlyeq_\kappa H$ then $G \preccurlyeq H$, from which by Theorem 2.20 we get an l-monomorphism θ: $H \to \bar{G}$. An inspection of the proof of this theorem reveals that θ actually maps H into $\bar{G}^\kappa$.

THEOREM 2.25. *Every maximal $\preccurlyeq_\kappa$ extension is l-isomorphic to $\bar{G}^\kappa$ over G.*

3. Maximal extensions in relation to the orthocompletion, polar completion, and strongly projectable hull. If G is representable then every consistent map is in K, and the group and lattice operations are componentwise in the following sense. For k, $m \in K$ the domain of km is simply dom $(k) \cap$ dom (m), and $Pkm = (Pk)(Pm)$ for all polars $P \in$ dom (km). Likewise dom (k^{-1}) = dom (k). and $Pk^{-1} = (Pk)^{-1}$ for all $P \in$ dom (k^{-1}). The lattice operations, of course, are generally componentwise in this sense. A consequence of componentwise operations is that G and $\bar{G}$ generate the same variety of l-groups when G is representable. Some notation is necessary to prove this. For each $g \in G$ let $\mathbf{g}$ be a constant symbol. $W(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ is a *word with constants from G* if it is an expression built up from the variable symbols $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ and constant symbols $\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_m$, $\mathbf{g}_i \in G$, using the group and lattice operations. An *equation with constants from G* is a formula of the form

$$\forall\, \mathbf{x}_1, \ldots, \mathbf{x}_n\, W(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = 1,$$

where W is a word with constants from G.

PROPOSITION 3.1. *If G is a representable l-group then an equation ψ with constants from G holds in G if and only if it holds in $\bar{G}$.*

Proof. Suppose $W(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ is a word with constants from G such that the corresponding equation ψ holds in G. Let $x_1 \ldots x_n$ be members of $\bar{G}$ and suppose $x_i = [k_i]$ and that the intersection of the domains of the k_i's is $\mathscr{S}$. Then because all operations are componentwise, $W(x_1, x_2, \ldots x_n) = [h]$, where dom $(h) = \mathscr{S}$, and $Ph = W(Px_1, Px_2, \ldots Px_n)$ for all $P \in \mathscr{S}$. But because ψ holds in G, $Ph = 1$ for all $P \in \mathscr{S}$ and $[h] = 1$. That is, ψ holds in $\bar{G}$.

COROLLARY 3.2. *If G is representable then G and $\bar{G}$ generate the same variety.*

An interesting open question is whether G and $\bar{G}$ generate the same variety. The simple proof for the representable case given above will not work in general because the group operations are not componentwise.

PROPOSITION 3.3. *Every polar of G is a cardinal summand if and only if G is representable and has no proper $\preccurlyeq_\omega$ extensions.*

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Proof. Suppose that G is representable, $G = \bar{G}^\omega$, $1 < g \in G$, and P is a polar of G. Let

$$\mathscr{S} = \{Q \in \mathscr{P} | \, Q \subseteq P \quad \text{or} \quad Q \subseteq P^\perp\},$$

a large set. Define $k: \mathscr{S} \rightarrow G$ by $Qk = x$ if $Q \subseteq P$ and $Qk = 1$ if $Q \subseteq P^\perp$. Clearly $k \in K$, and $h = [k] \in \bar{G}^\omega = G$. Just as clearly, $h \in P$ and $gh^{-1} \in P^\perp$, showing G to be the cardinal sum of P with $P^\perp$.

Now suppose that every polar of G is a cardinal summand. Then every polar is plainly normal, hence G is representable. Consider $x = [h] \in \bar{G}^\omega$ and let $\mathscr{R}$ be a finite subset of dom (h) such that $\vee \mathscr{R} = 1$. Without loss of generality we may assume $\mathscr{R}$ to consist of pairwise disjoint polars. For each $P \in \mathscr{R}$ there exists by projectability an element $g_p \in P$ such that $xg_p^{-1} \in P^\perp$. Because $\mathscr{R}$ is finite, the product of all the g_p's is an element y of G. But it is easily verified that $k \sim k_y$, where k_y has the meaning of Theorem 2.19. Therefore $x = y \in G$.

An l-group whose every polar is a cardinal summand is termed *strongly projectable* in the literature. Given a representable l-group G, a second l-group H is called a *strongly projectable hull* of G if G is order dense in H, if H is strongly projectable, and if $G \leqq K < H$ implies K is not strongly projectable. A proof that every representable l-group has a strongly projectable hull which is unique up to l-isomorphism over G may be found in [**10**].

PROPOSITION 3.4. *Any strongly projectable hull of a representable l-group G is l-isomorphic to $\bar{G}^\omega$ over G.*

Proof. By Proposition 3.1 every $\preccurlyeq$ extension of a representable l-group G is representable. Hence $\bar{G}^\omega$ is a strongly projectable hull of G.

Is there a projectability condition which makes sense for every l-group G, which reduces to strong projectability in the representable case, and which is equivalent to $G = \bar{G}^\omega$? A candidate for such a condition is the subject of Proposition 3.5. The *normalizer* of a polar P is $\{x | \, P^x = P\}$, known to be an l-subgroup of G. The *convex normalizer* of P, written CN (P), is the largest convex l-subgroup of G contained in the normalizer of P.

PROPOSITION 3.5. *If $G = \bar{G}^\omega$ then each polar P is a cardinal summand of its convex normalizer.*

Proof. Suppose $P \in \mathscr{P}$ and $1 < g \in$ CN (p). Let

$$\mathscr{S} = \{Q \in \mathscr{P} | \, Q \subseteq P \quad \text{or} \quad Q \subseteq P^\perp\},$$

a large set, and define k: $\mathscr{S} \rightarrow G$ by declaring $Qk = g$ for $Q \subseteq P$ and $Qk = 1$ for $Q \in P^\perp$. k is clearly consistent, and in fact is in K. To verify property (1) it is only necessary to observe that $\mathscr{S}^k = \mathscr{S}$. To verify

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property (2) consider $Q, R \in \mathscr{S}$ and $1 \leqq s \leqq (Qk)(Rk)^{-1} \vee 1$. The only nontrivial case occurs when $Q \subseteq P$ and $R \subseteq P^{\perp}$, in which case $1 \leqq s \leqq g$. Because $g \in \mathrm{CN}\,(P)$, $Q^s = Q$ and $Q^s \wedge R = Q \wedge R = 0_{\mathscr{P}}$. Thus $k \in K$, and $x = [k] \in \bar{G}^{\omega} = G$. But it can easily be shown that $x \in P$ and $gx^{-1} \in P^{\perp}$, thus confirming P to be a cardinal summand of $\mathrm{CN}\,(P)$.

It should be pointed out that the convex normalizer of any polar in a representable l-group G is G itself, so that the condition of Proposition 3.5 reduces to strong projectability in that case. The interesting and important open question is whether the converse of Proposition 3.5 is true.

If $G = \bar{G}^{\omega}$ it is not necessarily true that each polar is a cardinal summand of its normalizer. For example, take G to be $A(\mathbf{R})$, the order preserving permutations of the real numbers $\mathbf{R}$, and let P be the polar consisting of those permutations whose support is contained in $\cup \{(2n, 2n + 1) | \, n \text{ an integer}\}$, where $(2n, 2n + 1)$ is the open interval between $2n$ and $2n + 1$. Let $g \in G$ be defined by $(r)g = r + 2$ for all $r \in \mathbf{R}$. Then $P^g = P$, yet g cannot be gotten as a product of permutations, one from P and one from $P^{\perp}$. This is in spite of the fact that $G = \bar{G}^{\omega}$.

$\bar{G}$ has all the projectability properties set out above, and in addition has strong convergence completeness properties. The next several propositions refer to Section 5 of [**1**].

PROPOSITION 3.6. *$\bar{G}$ is complete with respect to the polar Cauchy structure.*

Proof. As was mentioned in the proof of Proposition 5.13 in [**1**], $G \preccurlyeq G^p$ and therefore $G \preccurlyeq G^{ip}$, where G^p and G^{ip} are the polar Cauchy completion and iterated polar Cauchy completion of G, respectively. It follows that $(\bar{G})^p = \bar{G}$.

It is possible to characterize G^p inside $\bar{G}$. Given G, let L consist of those elements $[k] \in \bar{G}$ such that the maximal domain of k contains a large ideal. This notion is well defined, for equivalent members of K have the same maximal domain.

PROPOSITION 3.7. *G^p is l-isomorphic to L over G.*

Proof. Given $x \in L$ find $k \in K$ and ideal $\mathscr{E}$ of polars such that $x = [k]$, $\mathscr{E} \subseteq \mathrm{dom}\,(k)$, and $\vee\, \mathscr{E} = 1_{\mathscr{P}}$. For each polar $P \in \mathscr{E}$, let

$$F(P) = \{g \in G | \, (Pk)g^{-1} \in P^{\perp}\}.$$

Let $\mathscr{F}$ be the filter on G generated by the sets $F(P)$ for $P \in \mathscr{E}$. It is straightforward to verify that $\mathscr{F}$ is a polar Cauchy filter. Define $x\theta$ to be the Cauchy equivalence class of $\mathscr{F}$. The verification that θ is a well defined l-isomorphism from L onto G^p over G is routine.

Is there an l-group G such that $(G^p)^p \neq G^p$? This question was first posed in [**1**] and remains unresolved. Though the question at first glance

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seems to have nothing to do with $\bar{G}$, Proposition 3.7 makes it clear that it really concerns l-subgroups of $\bar{G}$ and ought to be easier to settle in this light.

PROPOSITION 3.8. *For any l-group G, $\bar{G}$ and G^p are l-isomorphic over G if and only if the maximal domain of each $k \in K$ contains a large ideal.*

Proof. If the maximal domain of every $k \in K$ contains a large ideal then $L = \bar{G}$ and the previous proposition provides an l-isomorphism from $\bar{G}$ onto G^p. If G^p and $\bar{G}$ are l-isomorphic over G then L and $\bar{G}$ are l-isomorphic over G. Therefore L has no proper $\preccurlyeq$ extensions and hence $L = \bar{G}$.

An element $b \in G$ is *basic* if $b > 1$ and $b^\perp$ is prime. G has a *basis* if each $1 < g \in G$ exceeds a basic element. It is known that G has a basis if and only if $\mathscr{P}$ is atomic.

PROPOSITION 3.9. *If G has a basis then $\bar{G} = G^p$.*

Proof. The ideal $\mathscr{E}$ generated by the atoms of $\mathscr{P}$ is the unique minimal large ideal of $\mathscr{P}$. Since every large set must contain $\mathscr{E}$, $G^p = \bar{G}$ by the previous proposition.

Each Cauchy completion studied to date has had a relation completeness with respect to the adjoining of suprema of subsets of a particular sort [**2**]. The polar completion G^{ip} is complete with respect to containing suprema for sets of the following kind. A subset $Z \subseteq G^+$ is type $\mathscr{Y}$ if

$$\wedge \{(Zz^{-1} \vee 1)^{\perp\perp} | z \in Z\} = \wedge \{(z^{-1}Z \vee 1)^{\perp\perp} | z \in Z\} = 0_{\mathscr{P}}$$

(See Propositions 5.15–5.23 and Section 3 of [**1**]). Observe that every pairwise disjoint subset of G^+ is type $\mathscr{Y}$. G is *sup $\mathscr{Y}$-complete* if every type $\mathscr{Y}$ subset of G has a supremum in G. G is *laterally complete* if every pairwise disjoint subset of G^+ has a supremum in G. The example following Corollary 5.23 of [**1**] shows that lateral completeness is a strictly weaker property than sup $\mathscr{Y}$-completeness.

PROPOSITION 3.10. *$\bar{G}$ is sup $\mathscr{Y}$-complete and therefore laterally complete.*

An l-group which is both laterally complete and strongly projectable is termed *orthocomplete* in the literature ([**4**], [**10**]).

PROPOSITION 3.11. *G is orthocomplete if and only if G is representable and has no proper $\preccurlyeq$ extension.*

Proof. Suppose G is representable and $G = \bar{G}$. By Proposition 3.3 G is strongly projectable, and by Proposition 3.10 G is laterally complete. Now suppose G to be orthocomplete and therefore representable, and consider $1 \leqq x = [k] \in \bar{G}$. Assume without loss of generality that $Pk \geqq 1$ for all $P \in \text{dom}(k)$. Obtain by Zorn's Lemma a set $\mathscr{R} \subseteq \text{dom}(k)$ maximal (in the containment order) with respect to consisting of pairwise disjoint polars. Note $\vee \mathscr{R} = 1_{\mathscr{P}}$. For each $P \in \mathscr{R}$ find by projectability an element $g_p \in P$ such that $xg_p^{-1} \in P^\perp$. The set $\{g_p | P \in \mathscr{R}\}$ is pairwise disjoint and has a supremum y in G. But it is clear that $k \sim k_y$, where k_y has the meaning of Theorem 2.19. Therefore $x = [k_y] = y \in G$.

H is an *orthocompletion* of the representable l-group G if H is orthocomplete, if G is order dense in H, and if $G \leqq K < H$ implies K is not orthocomplete. Bernau proves in [**4**] that every representable l-group has an orthocompletion which is unique up to l-isomorphism over G.

PROPOSITION 3.12. *Any orthocompletion of a representable l-group G is l-isomorphic to $\bar{G}$ over G.*

Proof. By Proposition 3.1 every $\preccurlyeq$ extension of a representable l-group G is representable. Hence $\bar{G}$ is the orthocompletion of G.

Conrad proves in [**10**] that the orthocompletion of a representable l-group G is the lateral completion of its strongly projectable hull. The most likely analog of this result for an arbitrary l-group G is unresolved; namely, whether $(\bar{G}^\omega)^p = \bar{G}$. In light of Proposition 3.8 the question is this: if $G = \bar{G}^\omega$, must every $k \in K$ have a maximal domain containing a large ideal?

If G is a representable l-group with a basis then $\bar{G}$ can be described quite precisely. Let A be a maximal set of pairwise disjoint basic elements and for each $a \in A$ let π_a be the natural l-homomorphism from G onto the totally ordered group $G/a^\perp$. Let

$$\pi : G \rightarrow \prod_{a \in A} G/a^\perp$$

be the resulting product l-monomorphism.

PROPOSITION 3.13. *Suppose G is a representable l-group with a basis and that A is a maximal pairwise disjoint set of basic elements. Then the natural l-monomorphism π: $G \rightarrow \prod_{a \in A} G/a^\perp$ has a unique extension $\hat{\pi}$ from $\bar{G}$ onto $\prod_{a \in A} G/a^\perp$.*

Proof. Consider $f = [k] \in \bar{G}$ and $a \in A$. Because $a^{\perp\perp}$ is an atom of $\mathscr{P}$ it must be in dom (k); therefore define $f\hat{\pi}$ to be the element of $\prod G/a^\perp$ whose value at index a is $(a^{\perp\perp}k)\pi_a$. It is easy to see that $\hat{\pi}$ is well defined and, by virtue of the componentwise nature of all operations, is an l-homomorphism which extends π. To show $\hat{\pi}$ one-one observe that if $f_1 \neq f_2$ in $\bar{G}$ then $f_i = [k_i]$ where

$$(a^{\perp\perp}k_1)(a^{\perp\perp}k_2)^{-1} \notin a^\perp \quad \text{for some } a \in A.$$

But this directly implies that $f_1\hat{\pi}$ and $f_2\hat{\pi}$ disagree at index a. To show $\hat{\pi}$ onto consider an arbitrary $1 < z \in \prod_{a \in A} G/a^\perp$. For each $a \in A$ fix $1 < g(a) \in G$ such that $(a)z = g(a)\pi_a$. Now $a^{\perp\perp}$ is a cardinal summand of the strongly projectable l-group $\bar{G}$, so for each $a \in A$ there is a unique $1 \leqq h(a) \in \bar{G}$ such that

$$h(a) \in a^{\perp\perp} \quad \text{and} \quad h(a)(g(a))^{-1} \in a^{\perp} \text{ in } \bar{G}.$$

Furthermore, the different $h(a)$'s are disjoint because the different a's are. The lateral completeness of $\bar{G}$ assures the existence of $\bigvee_{a \in A} h(a) = t$ in $\bar{G}$, and clearly $t\hat{\pi} = z$.

It remains to show the uniqueness of $\hat{\pi}$. For that purpose consider an l-monomorphism $\theta: \bar{G} \to \prod G/a^{\perp}$ extending π, an arbitrary $f = [k] \in \bar{G}$, and $a \in A$. Let $a^{\perp\perp}k = g \in G$. We claim $|fg^{-1}| \wedge a = 1$ in $\bar{G}$. For if

$$m = (kk_g^{-1} \vee k_g k^{-1}) \wedge k_a,$$

where k_g and k_a have the meaning of Theorem 2.19, then $a^{\perp\perp}m = 1$ by construction and for all $b \in A$ such that $b \neq a$ it is true that $1 \leqq b^{\perp\perp}m \leqq a$, hence $b^{\perp\perp}m \in b^{\perp}$. This proves that $m \sim i$ and therefore that $|fg^{-1}| \wedge a = 1$ in $\bar{G}$. By applying θ to this equality we obtain

$$|(f\theta)(g\pi)^{-1}| \wedge a\pi = 1 \text{ in } \prod G/a^{\perp},$$

which implies that the elements $f\theta$ and $g\pi$ of $\prod G/a^{\perp}$ have the same value at index a. But $f\hat{\pi}$ and $g\pi$ agree at index a, hence $f\hat{\pi}$ and $f\theta$ agree at a. Since a was an arbitrary index, $f\hat{\pi} = f\theta$.

The following theorem is closely related to Theorems 3.2 and 3.4 of [**9**]. $\sum_{\alpha \in A} T_\alpha$ denotes

$$\left\{ f \in \prod_{\alpha \in A} T_\alpha \,\middle|\, (\alpha)f = 1 \text{ for all but finitely many } \alpha \right\}.$$

G is said to be a *large* l-subgroup of H if $G \leqq H$ such that every nontrivial convex l-subgroup of H has nontrivial intersection with G. In particular, if G is order dense in H then G is a large l-subgroup of H,

THEOREM 3.14. *For an l-group G the following are equivalent.*

(a) *G is representable and has a basis.*

(b) *$\bar{G}$ is a product of totally ordered groups.*

(c) *$\bar{G}$ is representable and completely distributive.*

(d) *$\bar{G}^{\omega}$ is representable and completely distributive.*

(e) *G is a large l-subgroup of an l-group which is completely distributive and strongly projectable.*

(f) *G is a large l-subgroup of a product of totally ordered groups.*

(g) *There is a collection $\{T_\alpha | \alpha \in A\}$ of totally ordered groups and an l-monomorphism σ such that*

$$\sum_{\alpha \in A} T_\alpha \leqq G\sigma \leqq \prod_{\alpha \in A} T_\alpha.$$

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Proof. We have proven the implications from (a) to (b) to (c) to (d) to (e). To show that (e) implies (a) suppose that G is a large l-subgroup of the completely distributive strongly projectable l-group H. Then G is representable because H is, and the polars of G and H correspond by intersection. Therefore we need only show that H has a basis. For that purpose consider arbitrary $1 < h \in H$ and find an order closed prime Q of H which does not contain h. There must be some $f \in H$ such that $h > f \geqq h \wedge q$ for all $q \in Q$. Let $b = hf^{-1} > 1$ and observe that $b \leqq h$ since $f \geqq 1$. We claim that b is basic and prove it by showing $b^{\perp\perp}$ to be an atom of $\mathscr{P}$. Consider disjoint polars $P_1, P_2 \subseteq b^{\perp\perp}$, and by the strong projectability of H find elements h_1, h_2, and h_3 in H such that $h_1 \in P_1$, $h_2 \in P_2$, $h_3 \in P_1{}^{\perp} \cap P_2{}^{\perp}$, and $h = h_1h_2h_3$. Because h_1 and h_2 are disjoint, at least one must lie in the prime Q, say $h_1 \in Q$. By assumption $h > f \geqq h \wedge h_1 = h_1$, hence

$$1 < b = hf^{-1} \leqq hh_1{}^{-1} = h_2h_3 \in P_1{}^{\perp}.$$

But then $P_1 \subseteq b^{\perp\perp} \subseteq P_1{}^{\perp}$ implies $P_1 = 0_{\mathscr{P}}$, proving $b^{\perp\perp}$ to be an atom.

Thus far we have proven the equivalence of conditions (a) through (e). To show that (a) implies (g) let σ be the π of Proposition 3.13. Since (g) implies (f) and (f) implies (e), the proof is complete.

COROLLARY 3.15. *For a strongly projectable l-group G the following are equivalent.*

(a) *G is completely distributive.*

(b) *G has a basis.*

(c) *There is a collection $\{T_\alpha | \alpha \in A\}$ of totally ordered groups and an l-monomorphism σ such that*

$$\sum_{\alpha \in A} T_\alpha \leqq G\sigma \leqq \prod_{\alpha \in A} T_\alpha.$$

If G is orthocomplete then these conditions are equivalent to G being a product of totally ordered groups.

4. An l-permutation construction of $\bar{G}$. Suppose G is an l-subgroup of $A(S)$, the l-group of order preserving permutations of the chain S. Holland's Representation Theorem [**12**] assures that this supposition involves no loss of generality. The purpose of this section is to represent $\bar{G}$ as the l-group of all order preserving "near permutations" of S. Light is thereby shed on the l-permutation structure of $\bar{G}$; moreover, the reader familiar with l-permutation techniques may find this construction more intuitive than that of Section 2. We shall assume familiarity with the basic concepts; notation and terminology undefined herein may be found in [**11**].

It is important to realize that G need not act transitively on S. Nevertheless the usual proof of the Holland Representation Theorem allows us to assume that S has convex G orbits. That is, if $s_1 < s_2 < s_3$ in G and if there is some $g \in G$ taking s_1 to s_3 then there is some $x \in G$ taking s_1 to s_2.

With each $g \in G$ is associated its support

$$S(g) = \{s \in S \mid (s)g \neq s\},$$

with each convex l-subgroup Q is associated its support

$$S(Q) = \cup \{S(g) \mid 1 < g \in Q\},$$

and with each $\mathscr{S} \subseteq \mathscr{P}$ is associated its support

$$S(\mathscr{S}) = \cup \{S(P) \mid P \in \mathscr{S}\}.$$

Given any map $f: S \to S$ let

$$\mathscr{S}(f) = \{P \in \mathscr{P} \mid \text{there is some } g \in G \text{ such that } (s)f = s(g) \text{ for all } s \in S(P)\}.$$

For each $f: S \to S$ choose $k(f)$ to be a particular map from $\mathscr{S}(f)$ to G such that

$$(s)f = (s)Pk(f) \quad \text{for all } s \in S(P) \text{ and all } P \in \mathscr{S}(f).$$

LEMMA 4.1. *For any function $f: S \to S$, the following are equivalent.*
(a) *$\mathscr{S}(f)$ is large.*
(b) *$k(f)$ is consistent.*
(c) *$S(\mathscr{S}(f))$ meets $S(g)$ for all $1 < g \in G$.*

Proof. Suppose $\mathscr{S}(f)$ is large and consider $P, Q \in \mathscr{S}(f)$, $s \in S(P \wedge Q) = S(P) \cap S(Q)$. Then

$$(s)Pk(f) = (s)f = (s)Qk(f).$$

Therefore $(Pk(f))(Qk(f))^{-1} \in (P \wedge Q)^{\perp}$. That is, $k(f)$ is consistent. The remaining implications are obvious.

Suppose f and h are maps from S into S with $\mathscr{S}(f)$ and $\mathscr{S}(h)$ large. f and h are *equivalent*, written $f \approx h$, provided

$$s(f) = (s)h \quad \text{for all } s \in S(\mathscr{S}(f) \cap \mathscr{S}(h)).$$

LEMMA 4.2. *Suppose f and h are maps from S into S with $\mathscr{S}(f)$ and $\mathscr{S}(h)$ large. Then $f \approx h$ if and only if $k(f) \sim k(h)$.*

Proof. $f \approx h$ if and only if

$$(s)Pk(f) = (s)f = (s)h = (s)Pk(h)$$

for all $s \in S(P)$ and all $P \in \mathscr{S}(f) \cap \mathscr{S}(h)$. That is, if and only if

$$(Pk(f))(Pk(h))^{-1} \in P^{\perp}$$

for all $P \in \mathscr{S}(f) \cap \mathscr{S}(\mathscr{R})$, or $k(f) \sim k(h)$.

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Lemma 4.2 shows that $\approx$ is an equivalence relation on

$$\{f: S \to S | \mathscr{S}(f) \text{ large}\};$$

we shall write the equivalence class of f as $\langle f \rangle$. Of most interest are those functions $f: S \to S$ which enjoy the following two properties.

(3) $\mathscr{S}(f)$ is large, and $S(\mathscr{S}(f))f = S(\mathscr{T})$ for some large $\mathscr{T} \subseteq \mathscr{P}$. Here

$$S(\mathscr{S}(f))f = \{(s)f | s \in S(\mathscr{S}(f))\}.$$

(4) $\mathscr{S}(f)$ is large, and $s_1 < s_2$ in $S(\mathscr{S}(f))$ implies $(s_1)f < (s_2)f$.

LEMMA 4.3. *A map $f: S \to S$ has property* (3) *if and only if $k(f)$ has property* (1), *and f has property* (4) *if and only if $k(f)$ has property* (2).

Proof. The first assertion is a result of the fact that $S(P)g = S(P^g)$ for any polar P and $g \in G$. Therefore

$$S(\mathscr{S}(f))f = S(\mathscr{S}(f)^{k(f)}).$$

Now suppose $f: S \to S$ has property (4), that P and Q are disjoint non-zero polars of $\mathscr{S}(f)$ with $P^r = Q$ for some r satisfying

$$1 \leqq r \leqq (Pk(f))(Qk(f))^{-1} \vee 1.$$

Choose $s_1 \in S(P)$ and let $s_2 = (s_1)r \in S(Q)$. Now $s_1 \neq s_2$ because $P \wedge Q = 0_{\mathscr{P}}$, so assume $s_1 < s_2$, the other case being proved similarly. Then

$$(s_2)f = (s_2)Qk(f) = (s_1)r(Qk(f)) \leqq (s_1)(Pk(f))(Qk(f))^{-1}(Qk(f))$$
$$= (s_1)Pk(f) = (s_1)f,$$

contradicting property (4). Now suppose $s_1 < s_2$ where $s_1 \in S(P)$ and $s_2 \in S(Q)$ for $P, Q \in \mathscr{S}(f)$. Suppose in addition that $s_1 < s_2 < (s_1)f$, the other cases being argued similarly. Since $(s_1)f = (s_1)Pk(f)$, the convexity of G-orbits guarantees that $(s_1)x = s_2$ for some $1 < x \in G$. Let

$$T = \{g \in G | (s_1)g = s_1\},$$

a prime convex l-subgroup of G. Now $T \not\supseteq P$ and

$$T^x = \{g \in G | (s_2)g = s_2\} \not\supseteq Q,$$

so $T < Tx$ in T-supp $(k(f))$. By property (2) applied to $k(f)$,

$$TPk(f) = Tk(f) < Txk(f) = TxQk(f).$$

But since

$$(s_1)f = (s_1)Pk(f) \quad \text{and} \quad (s_2)f = (s_2)Qk(f) = (s_1)xQk(f),$$

it follows that $(s_1)f < (s_2)f$.

It follows from Lemma 2.7, 4.2, and 4.3 that one member of an equivalent pair of functions has property (3) or property (4) only when the other member has the same property. Let M designate the set of all functions having both these properties, and let $M/\approx$ be $\{\langle f\rangle | f \in M\}$. G will be regarded as a subset of $M/\approx$ by the convention that $g \in G$ is interpreted as $\langle g\rangle$. Then Lemma 4.3 asserts that $f \in M$ if and only if $k(f) \in K$.

LEMMA 4.4. *If $f, h \in M$ then $fh \in M$ and $k(fh) \sim k(f)k(h)$. Moreover, $f \approx f'$ and $h \approx h'$ imply $fh \approx f'h'$.*

Proof. Let

$$\mathscr{R} = \{P \in \mathscr{S}(f) | P^{k(f)} \in \mathscr{S}(h)\},$$

a large set. For $P \in \mathscr{R}$ and $s \in S(P)$,

$$(s)fh = ((s)Pk(f))(P^{k(f)}k(h)).$$

This shows at once that $P \in \mathscr{S}(fh)$ and that $k(fh) \sim k(f)k(h)$. Now f, $h \in M$ implies $k(f), k(h) \in K$, which implies

$$k(f)k(h) \in K \quad \text{and} \quad k(fh) \in K,$$

whence $fh \in M$. The last assertion follows by a straightforward argument.

LEMMA 4.5. *Suppose $f \in M$ and let $h: S \rightarrow S$ be any function such that fh is the identity on $S(\mathscr{S}(f))$. Then $h \in M$ and $k(h) \sim k(f)^{-1}$. If f and f' are equivalent members of M and if h and h' correspond in the aforementioned sense to them, then h and h' are equivalent.*

Proof. For any $P \in \mathscr{S}(f)$ and any $s \in S(P^{k(f)}) = S(P)f$ there is a unique $r \in S(P)$ such that $(r)f = s$. Since $s = (r)f = (r)Pk(f)$,

$$(s)h = (r)fh = r = (s)(Pk(f))^{-1}.$$

This demonstrates that $\mathscr{S}(f)^{k(f)} \subseteq \mathscr{S}(h)$ and that $k(h) \sim k(f)^{-1} \in K$. Therefore $h \in M$. The last assertion follows from straightforward arguments.

LEMMA 4.6. *If $f, h \in M$ then $f \vee h \in M$ and $k(f \vee h) \sim k(f) \vee k(h)$. If $f \approx f'$ and $h \approx h'$ then $f \vee h \approx f' \vee h'$. And dually for $\wedge$.*

Proof. For $P \in \mathscr{S}(f) \cap \mathscr{S}(h)$ and $s \in S(P)$,

$$(s)(f \vee h) = (s)f \vee (s)h = (s)Pk(f) \vee (s)Pk(h)$$
$$= (s)(Pk(f) \vee Pk(h)).$$

This shows at once that $P \in \mathscr{S}(f \vee h)$ and that $k(f \vee h) \sim k(f) \vee k(h)$. Therefore $f \vee h \in M$. The equivalence assertion follows from straightforward arguments.

LEMMA 4.7. *The map θ given by $\langle f\rangle\theta = [k(f)]$ is a well defined l-isomorphism from $M/\approx$ onto $\bar{G}$ over G.*

Proof. For $g \in G, \mathscr{S}(g) = \mathscr{P}$ and $k(g) \sim k_g$, where k_g has the meaning of Theorem 2.19. This shows that θ is the identity on G. To show θ onto consider $[k] \in \bar{G}$. Let s_0 be a fixed element of S, and define $f: S \to S$ by declaring $(s)f = (s)Pk$ for each $P \in \text{dom}\ (k)$ and each $s \in S(P)$, and $(s)f = s_0$ otherwise. The fact that k is consistent assures that the definition of $(s)f$ is independent of the choice of P. Since $S(f) \supseteq \text{dom}\ (k)$ and $k(f) \sim k$, we get $f \in M$ and $\langle f\rangle\theta = [k]$.

Because of Lemma 4.7 we shall suppress the notation $M/\approx$, using $\bar{G}$ instead. Now suppose $G \leqq A(S)$ such that the G-orbits of S are convex. Let $L = \{x \in \bar{G} |\ x = \langle f\rangle$ for some $f \in M$ such that $S(\mathscr{S}(f))$ and $S(\mathscr{S}(f))f$ are both dense in the order topology on $S\}$. It is left as an exercise for the reader to show that L is an l-subgroup of $\bar{G}$. In the next theorem, $\bar{S}$ stands for the chain obtained by completing S by Dedekind cuts.

THEOREM 4.8. *Suppose $G \leqq A(S)$ such that the G-orbits of S are convex. Then there is a unique l-monomorphism $\theta: L \to A(\bar{S})$ over G.*

Proof. Consider $\langle f\rangle \in L$, where $S(\mathscr{S}(f))$ and $S(\mathscr{S}(f))f$ are dense in the order topology on S. Define $\hat{f}: \bar{S} \to \bar{S}$ as follows. For $r \in \bar{S}$ let $(r)\hat{f}$ be the supremum in $\bar{S}$ of

$$\{(s)f |\ s \in S,\ s \leqq r\}.$$

The density of $S(\mathscr{S}(f))$ and of $S(\mathscr{S}(f))f$ implies $\hat{f} \in A(\bar{S})$. The map θ defined by $\langle f\rangle\theta = \hat{f}$ is easily seen to be a well defined l-monomorphism. Now suppose $\psi: \bar{G} \to A(\bar{S})$ is an arbitrary l-monomorphism extending θ. Consider $x = \langle f\rangle \in \bar{G}$, $P \in \mathscr{S}(f)$, $s \in S(P)$, and let $g = Pk(f)$. Then $xg^{-1} \in P^{\perp}$ implies $(x\psi)g^{-1} \in P\psi^{\perp}$. Therefore

$$(s)(x\psi) = (s)g = (s)f = (s)\hat{f} = (s)(x\theta).$$

Since $x\psi$ and $x\theta$ agree on the dense set $\mathscr{S}(f)$, they are identical order preserving permutations of $\bar{S}$.

COROLLARY 4.9. *Suppose $G \leqq A(S)$ such that the G-orbits of S are convex and such that $S(\mathscr{S})$ is dense in the order topology on S whenever S is large. Then there is a unique l-monomorphism $\theta: \bar{G} \to A(\bar{S})$.*

COROLLARY 4.10. *Suppose $S = \bar{S}$ and $A(S)$ is doubly transitive. Then $\overline{A(S)} = A(S)$.*

The following corollary implicitly refers to McCleary's Classification Theorem ([**14**], [**15**], [**11**]).

COROLLARY 4.11. *Suppose $G \leqq A(S)$ is transitive and o-primitive. If G is l-isomorphic to an l-subgroup of the real numbers then $\bar{G} = G$. If G has* *period* $p \in A(\bar{S})$ *then there is a unique l-monomorphism*

$$\theta\colon \bar{G} \rightarrow \{x \in A(\bar{S}) \mid xp = px\}$$

over G. If G is doubly transitive with an element of bounded support, then there is a unique l-monomorphism $\theta\colon \bar{G} \rightarrow A(\bar{S})$ *over G.*

Proof. Either construction makes it clear that $G = \bar{G}$ for any totally ordered group G. If G has period p then it is easy to show that $S(\mathscr{S})$ is dense in the order topology on S whenever $\mathscr{S}$ is large. Therefore there is a unique l-monomorphism $\theta\colon \bar{G} \rightarrow A(\bar{S})$. We must show that for each $\langle f \rangle \in \bar{G}$, $\langle f \rangle\theta = \hat{f}$ commutes with p. The crucial observation is that for any polar P, $S(P)$ is periodic in the sense that $s \in S(P)$ implies $(s)p^n \in S(P)$ for all n. This is because $(s)q > s$ for $q \in P$ implies $(s)qp^n = (s)p^nq > (s)p^n$. Therefore

$$(s)fp = (s)(Pk(f))p = (s)p(Pk(f)) = (s)pf.$$

Therefore $\hat{f}p = p\hat{f}$. Finally, if G is doubly transitive with an element of bounded support, it is clear that $S(\mathscr{S})$ is dense in the order topology on S whenever $\mathscr{S}$ is large.

The only transitive o-primitive l-permutation groups not covered in Corollary 4.11 are the pathologically o-2-transitive ones. The point of the next proposition is that an l-monomorphism $\theta\colon \bar{G} \rightarrow A(\bar{S})$ cannot be obtained by the simple continuity argument of Theorem 4.8. Does any such θ exist in the pathologically o-2-transitive case? Is it unique?

PROPOSITION 4.12. *A transitive o-primitive l-permutation group* $G \leqq A(S)$ *has the property that* $S(\mathscr{S})$ *is dense in the order topology on S for all large* $\mathscr{S}$ *if and only if G is not pathologically o-2-transitive.*

Proof. Suppose G is doubly transitive with no element of bounded support. For any interval $I = \{s \in S \mid a \leqq s \leqq b\}$ we shall show the existence of a large $\mathscr{S} \subseteq \mathscr{P}$ such that $I \cap S(\mathscr{S}) = \emptyset$. Let $\mathscr{S}_0 = \emptyset$. Now suppose $\mathscr{S}_\alpha$ has been defined for all $\alpha < \beta$ such that

$$S(\cup \mathscr{S}_\alpha) \cap I = \emptyset.$$

If $\vee (\cup \mathscr{S}_\alpha) = 1_{\mathscr{P}}$, let

$$\mathscr{S}_\beta = \cup \{\mathscr{S}_\alpha \mid \alpha < \beta\}.$$

If not, there is some element $1 < x \in G$ such that $\cup \mathscr{S}_\alpha \subseteq x^\perp$. By n-fold transitivity, we may obtain y such that $1 < y \leqq x$ and $y^{\perp\perp} \cap I = \emptyset$. Define

$$\mathscr{S}_\beta = \{y^{\perp\perp}\} \cup (\cup \mathscr{S}_\alpha).$$

Because G has no element of bounded support, there must be some ordinal

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γ for which $\vee \mathscr{S}_\gamma = 1_{\mathscr{P}}$. The required large set $\mathscr{S}$ is

$$\{P|\ P \subseteq Q, \text{ some } Q \in \mathscr{S}_\gamma\}.$$

We turn now to the smaller l-group $\bar{G}^\omega$. In many instances $\bar{G}^\omega$ can be uniquely embedded in $A(S)$, not just in $A(\bar{S})$. The crucial observation is the following.

LEMMA 4.13. *Suppose M is a minimal prime and $\mathscr{R}$ is a finite set of principal polars such that $\vee \mathscr{R} = 1_{\mathscr{P}}$. Then there is at least one $P \in \mathscr{R}$ such that $M \not\supseteq P$.*

Proof. Suppose $\mathscr{R} = \{x_i^{\perp\perp}|\ 1 \leqq i \leqq n\}$, where $x_i > 1$ for all i. Let

$$x = x_1 \vee x_2 \vee \ldots \vee x_n.$$

It is well known that a prime P is minimal if and only if for each $1 < p \in P$ there is some $q \notin P$ such that $p \wedge q = 1$. Since $x^{\perp\perp} = \vee \mathscr{R} = 1_{\mathscr{P}}$, $x \notin M$ and $x_i \notin M$ for at least one i.

THEOREM 4.14. *Suppose $G \leqq A(S)$ such that the G-orbits of S are convex and such that for each $s \in S$ the stabilizer subgroup $G_s = \{g \in G|\ (s)g = s\}$ is minimal among the prime convex l-subgroups of G. Suppose in addition that every polar of G is principal. Then there is a unique l-monomorphism θ: $\bar{G}^\omega \rightarrow A(S)$ over G.*

Proof. We claim that $\bar{G}^\omega \leqq L$ in this case. If $x \in \bar{G}^\omega$ then $x = \langle f\rangle$ for $f \in M$ such that $\mathscr{S}(f)$ contains a finite set $\mathscr{R}$ such that $\vee \mathscr{R} = 1_{\mathscr{P}}$. For any $s \in S$ there is by the previous lemma some $P \in \mathscr{R}$ with $G_s \not\supseteq P$. That is, $s \in S(P) \subseteq S(\mathscr{S}(f))$, or $S(\mathscr{S}(f)) = S$. Since $\mathscr{S}(f)^{k(f)}$ contains the finite set $\mathscr{R}^{k(f)}$, a similar argument shows $S(\mathscr{S}(f))f = S$. Therefore $x \in L$, and the map θ of Theorem 4.8 carries x to $f \in A(S)$.

Theorem 4.14 can be sharpened a bit. The elements of $(\bar{G}^\omega)\theta$ actually respect the convex G congruences on S.

LEMMA 4.15. *Suppose $G \leqq A(S)$ and $f \in A(S)$ such that there is a finite $X \subseteq G$ with the property that for every $s \in S$, $(s)f = (s)x$ for some $x \in X$. Then f respects every convex G congruence on S.*

Proof. Let $\mathscr{C}$ be a convex G congruence on S and consider $s_1 < s_2$ in S such that $s_1 \mathscr{C} s_2$. Let

$$A = s_1\mathscr{C} = \{s \in S|\ s\mathscr{C}s_1\},\ B = (s_1)f\mathscr{C}, \quad \text{and}$$

$$Y = \{Y \in X|\ Ay = B,\ s(f) = s(g) \quad \text{for some } s \in A\}.$$

Find $r \in S$ such that $(s_2)y < (r)f \in B$ for all $y \in Y$. Such an r must always exist because $\{(s_2)y|\ y \in Y\}$ is a finite subset of B, a nonempty convex subchain of S without first or last element. Observe that $r > s_2$,

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for if x is an element of X such that $(r)f = (r)x$ then in fact $x \in Y$, hence

$$(r)f = (r)y > (s_2)y.$$

Then $s_1 < s_2 < r$ implies

$$(s_1)f < (s_2)f < (r)f,$$

which by the convexity of B proves $(s_2)f \in B$.

Given $G \leqq A(S)$ let

$$G^0 = \{f \in A(S) | f \text{ respects each convex } G \text{ congruence on } S\}$$

(see Section 4 of [**3**], pp 247 and 278 of [**11**]).

PROPOSITION 4.16. *The map θ of Theorem 4.14 carries $\bar{G}^\omega$ into G^0.*

Proof. Given $x \in \bar{G}^\omega$ there is some $f \in M$ such that $x = \langle f \rangle$ and $\mathscr{S}(f)$ contains a finite set $\mathscr{R}$ with $\vee \mathscr{R} = 1_{\mathscr{P}}$. The proof of Theorem 4.14 shows

$$S(\mathscr{S}(f)) = S(\mathscr{R}) = S.$$

Hence $X = \{Pk(f) | P \in \mathscr{R}\}$ has the property that for every $s \in S$ there is some $x \in X$ with $(s)f = (s)x$. Since $f = x\theta$, the previous lemma provides the result.

By dropping the requirement in Theorem 4.14 that stabilizer subgroups G_s be minimal prime, we lose the uniqueness of the map θ.

THEOREM 4.17. *Suppose $G \leqq A(S)$ such that the G-orbits of S are convex. If every polar of G is principal then there is at least one l-monomorphism $\theta: \bar{G}^\omega \to G^0$ over G.*

Proof. Let $J \subseteq S$ be such that there is exactly one point of J in each G orbit of S. For each $j \in J$ let M_j be minimal among those prime convex l-subgroups contained in $G_j = \{g \in G | (j)g = j\}$. Let R be

$$\{(M_jx, j) | x \in G, j \in J\},$$

ordered by declaring $(M_jx, j) \leqq (M_iy, i)$ when $j < i$ or when $j = i$ and $M_jx \leqq M_iy$. Define $n: R \to S$ by

$$(M_jx, j)n = (j)x \quad \text{for all } (M_jx, j) \in R.$$

n is well defined, order preserving, and onto. Define $\psi: G \to A(R)$ by declaring

$$(M_jx, j)(g\psi) = (M_jxg, j) \quad \text{for all } (M_jx, j) \in R \quad \text{and} \quad g \in G,$$

an l-monomorphism such that

$$((r)n)g = ((r)g\psi))n \quad \text{for all } r \in R \quad \text{and} \quad g \in G.$$

Moreover, the G orbit of $(M_j x, j)$ in R is

$$\{(M_j y, j) \mid y \in G\}$$

which is convex in R. By Propositions 4.14 and 4.16 there is a unique l-monomorphism $\hat{\psi}$: $\bar{G}^\omega \to (G\psi)^0$ extending ψ. Finally, define θ: $\bar{G}^\omega \to A(S)$ by

$$(s)(x\theta) = ((r)(x\hat{\psi}))n$$

where $x \in \bar{G}^\omega$, $s \in S$, and r is some element of R such that $(r)n = s$. This action is well defined, since each member of $\bar{G}^\omega\hat{\psi} \leqq (G\psi)^0$ respects the convex $G\psi$ congruence $\mathscr{C}$ on R defined by $r_1\mathscr{C}r_2$ if $(r_1)n = (r_2)n$. That θ actually carries $\bar{G}^\omega$ into G^0 is clear upon recollecting that the convex G congruences on S are in one-to-one correspondence with the convex $G\psi$ congruences on R which are coarser than $\mathscr{C}$.

COROLLARY 4.18. *For every $G \leqq A(S)$ such that the G-orbits of S are convex, $\overline{(G^0)}^\omega = G^0$.*

Proof. G^0 is laterally complete, hence every polar of G^0 is principal.

Theorems 4.14 and 4.17 have direct application to representable l-groups.

THEOREM 4.19. *Suppose G is representable with every polar principal and that H is the strongly projectable hull of G. Then every l-monomorphism mapping G onto a subdirect cardinal product $\prod T_\alpha$ of totally ordered groups has at least one extending l-monomorphism $\hat{\theta}$: $H \to \prod T_\alpha$. For each α let θ_α: $G \to T_\alpha$ be the projection map and let M_α be its kernel. If each M_α is minimal among the prime convex l-subgroups of G then $\hat{\theta}$ is unique.*

We close this section with characterizations of $\bar{G}$ and $\bar{G}^\omega$ in the representable case. These characterizations do not depend on any of the structural analyses of the present section but only on the construction of Section 2 and the componentwise nature of the operations. Parts of these results are slight modifications of those in [**6**] and [**7**].

Suppose G is a subdirect product of totally ordered groups; say $G \leqq \prod_{\alpha \in A} T_\alpha$. For $h \in \prod T_\alpha$ let $h_\alpha \in T_\alpha$ designate the value of h on coordinate α. For each polar P of G define its support $S(P)$ to be

$$\{\alpha \in A \mid (\alpha)g \neq 1 \quad \text{for some } g \in P\}.$$

For $\mathscr{S} \subseteq \mathscr{P}$ define

$$S(\mathscr{S}) = \cup \{S(P) \mid P \in \mathscr{S}\}.$$

Let U be the collection of all $h \in \prod T_\alpha$ for which there is some large $\mathscr{S} \subseteq \mathscr{P}$ such that for each $P \in \mathscr{S}$ there is some $g \in G$ with $(\alpha)h = (\alpha)g$ for all $\alpha \in S(P)$. Let V be the collection of those $h \in U$ for which the defining set $\mathscr{S}$ contains a finite subset $\mathscr{R}$ such that $\vee \mathscr{R} = 1_{\mathscr{P}}$. Finally, let N be the collection of all $h \in U$ for which there is some large $\mathscr{S} \subseteq \mathscr{P}$ such that $(\alpha)h = 1$ for all $\alpha \in S(\mathscr{S})$. Then it is routine to verify that U and V are l-subgroups of $\prod T_\alpha$ and that N is an l-ideal of U.

THEOREM 4.20. *$\bar{G}$ is l-isomorphic to U/N over G and $\bar{G}^\omega$ is l-isomorphic to $V/(N \cap V)$ over G. If each $M_\alpha = \{g \in G \mid (\alpha)g = 1\}$ is minimal prime then $N \cap V = 1$ and $\bar{G}^\omega$ is l-isomorphic to V.*

5. Universal mapping properties. This section investigates natural classes of l-homomorphisms $\theta: G \to H$ which can be uniquely extended to l-homomorphisms $\hat{\theta}: \bar{G} \to \bar{H}$ or at least $\hat{\theta}: \bar{G}^\omega \to \bar{H}^\omega$. The Boolean algebras of polars of G and H need no longer correspond, and we use symbols $\mathscr{P}_G$ and $\mathscr{P}_H$, respectively, to designate them whenever the context fails to make this distinction clear. The *induced map* of an l-homomorphism $\theta: G \to H$ is the map from $\mathscr{P}_G$ into $\mathscr{P}_H$ which takes each $P \in \mathscr{P}_G$ to $P\theta^{\perp\perp} \in \mathscr{P}_H$.

LEMMA 5.1. *The induced map of any l-homomorphism $\theta: G \to H$ takes $0_{\mathscr{P}}$ of G to $0_{\mathscr{P}}$ of H and preserves $\wedge$'s. Therefore $P^\perp\theta \subseteq P\theta^\perp$ for all $P \in \mathscr{P}_G$.*

Proof. Consider $P, Q \in \mathscr{P}_G$. Then $P \wedge Q \subseteq P, Q$ implies

$$(P \wedge Q)\theta^{\perp\perp} \subseteq P\theta^{\perp\perp} \wedge Q\theta^{\perp\perp}.$$

If the opposite inclusion failed there would be

$$1 < h \in P\theta^{\perp\perp} \wedge Q\theta^{\perp\perp} \wedge (P \wedge Q)\theta^\perp.$$

Then $h \in P\theta^{\perp\perp}$ implies the existence of $1 < p \in P$ such that $1 < h \wedge p\theta$. And $h \wedge p\theta \in Q\theta^{\perp\perp}$ implies the existence of $1 < q \in Q$ such that

$$1 < h \wedge p\theta \wedge q\theta = x.$$

But $1 < x \leqq h$ and $x \leqq (p \wedge q)\theta \in (P \wedge Q)\theta$ contradicts

$$h \in (P \wedge Q)\theta^\perp.$$

To verify the last assertion consider $P \in \mathscr{P}_G$ and observe

$$0_{\mathscr{P}} = (0_{\mathscr{P}})\theta^{\perp\perp} = (P \wedge P^\perp)\theta^{\perp\perp} = P\theta^{\perp\perp} \wedge P^\perp\theta^{\perp\perp}.$$

Therefore $P^\perp\theta \subseteq P\theta^\perp$.

LEMMA 5.2. *For an l-homomorphism $\theta: G \to H$ the following are equivalent.*
(a) *For every finite $\mathscr{S} \subseteq \mathscr{P}_G$ such that $\vee \mathscr{S} = 1_{\mathscr{P}}$ in G, it is true that*

$$\vee \{P\theta^{\perp\perp} \mid P \in \mathscr{S}\} = 1_{\mathscr{P}} \text{ in } H.$$

(b) *For each $P \in \mathscr{P}_G$, $P^\perp\theta^\perp = P\theta^{\perp\perp}$.*
(c) *The induced map is a Boolean homomorphism.*

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Proof. To verify that (a) implies (b) take $\mathcal{S} = \{P, P^\perp\}$. Then $\vee \mathcal{S} = 1_\mathcal{P}$ in G implies $P\theta^{\perp\perp} \vee P^\perp\theta^{\perp\perp}$ in H. By the previous lemma the latter pair is disjoint, therefore complementary. (b) implies (c) is an immediate consequence of the fact that any map between Boolean algebras which preserves $\wedge$'s, complements, greatest element (take $\mathcal{S} = \{1_\mathcal{P}\} \subseteq \mathcal{P}_G$) and least element is a Boolean homomorphism. (c) implies (a) is obvious.

An l-homomorphism which enjoys the property isolated in Lemma 5.2 will be termed Boolean. Several comments about Boolean l-homomorphisms are in order. First, a necessary but not sufficient condition for an l-homomorphism $\theta: G \rightarrow H$ with kernel M to be Boolean is that $\{P \in \mathcal{P}_G | P \subseteq M\}$ is an ideal. Second, the composition and product of Boolean l-homomorphisms are themselves Boolean. Finally, if H is totally ordered and every polar of G is principal then an l-homomorphism $\theta: G \rightarrow H$ is Boolean if and only if its kernel is minimal prime.

PROPOSITION 5.3. *Every representable l-group G in which every polar is principal has a Boolean l-monomorphism $\theta: G \rightarrow \prod T_\alpha$ onto a subdirect product of totally ordered groups with a unique extension to a Boolean l-monomorphism $\hat{\theta}: \bar{G}^\omega \rightarrow \prod T_\alpha$.*

Proof. See Theorem 4.19.

PROPOSITION 5.4. *A representable l-group G is strongly projectable if and only if every l-epimorphism on G is Boolean.*

Proof. Suppose G is representable, $G = \bar{G}^\omega$ and N is an l-ideal of G. Let $\theta: G \rightarrow G/N$ be the natural l-homomorphism. For any $P \in \mathcal{P}$ and $1 < x \in G$, $x \wedge P \subseteq N$ and $x \wedge P^\perp \subseteq N$ imply $x \in N$ because the sets $x \wedge P$ and $x \wedge P^\perp$ contain the projections of x on the cardinal summands P and $P^\perp$. Thus a positive object in $G\theta$ cannot be disjoint from both $P\theta$ and $P^\perp\theta$. Therefore $P\theta^{\perp\perp} = P^\perp\theta^\perp$. Now suppose G is representable but not strongly projectable. Then there is some $P \in \mathcal{P}$ and $1 < x \in G$ such that x does not lie in the convex l-subgroup N generated by P and $P^\perp$. Let $\theta: G \rightarrow G/N$ be the natural l-homomorphism. Since $1 < x\theta \in P\theta^\perp \wedge P^\perp\theta^\perp$, we cannot have $P\theta^{\perp\perp} = P^\perp\theta^\perp$.

In [**1**] an l-homomorphism $\theta: G \rightarrow H$ is called p-continuous if

$$\wedge \{P\theta^{\perp\perp} | P \in \mathcal{F}\} = 0_\mathcal{P} \quad \text{in } H$$

for all filters $\mathcal{F} \subseteq \mathcal{P}_G$ such that $\wedge \mathcal{F} = 0_\mathcal{P}$ in G.

LEMMA 5.5. *For an l-homomorphism $\theta: G \rightarrow H$ the following are equivalent.*

(a) *For every $\mathcal{S} \subseteq \mathcal{P}_G$ such that $\vee \mathcal{S} = 1_\mathcal{P}$ in G, it is true that*

$$\vee \{P\theta^{\perp\perp} | P \in \mathcal{S}\} = 1_\mathcal{P} \quad \textit{in } H.$$

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(b) *The map induced by θ is a complete Boolean homomorphism.*
(c) *θ is Boolean and p-continuous.*

Proof. To verify that (a) implies (b) consider $\mathscr{S} \subseteq \mathscr{P}_G$ with $\vee \mathscr{S} = Q$. Let $\mathscr{R}$ be $\mathscr{S} \cup \{Q^\perp\}$. Then $\vee \mathscr{R} = 1_{\mathscr{P}}$ in G implies

$$(\vee \{P\theta^{\perp\perp} | P \in \mathscr{S}\}) \vee Q^\perp\theta^{\perp\perp} = 1_{\mathscr{P}} \quad \text{in } H.$$

Since each $P\theta^{\perp\perp}$ is disjoint from $Q^\perp\theta^{\perp\perp} = Q\theta^\perp$, $\vee (P\theta^{\perp\perp})$ and $Q\theta^\perp$ are complementary. That is,

$$\vee (P\theta^{\perp\perp}) = Q\theta^{\perp\perp} = (\vee P)\theta^{\perp\perp}.$$

The preservation of infima in $\mathscr{P}_G$ is proved simply by passage to the complement. That (b) implies (c) is obvious. To prove that (c) implies (a) consider $\mathscr{S} \subseteq \mathscr{P}_G$ with $\vee \mathscr{S} = 1_{\mathscr{P}}$. Let

$$\mathscr{F} = \{P^\perp | P \subseteq \vee \mathscr{S}' \text{ for } \mathscr{S}' \text{ a finite subset of } \mathscr{S}\}.$$

Then $\mathscr{F}$ is a filter on $\mathscr{P}_G$ for which $\wedge \mathscr{F} = 0_{\mathscr{P}}$ in G. Therefore

$$\wedge \{P\theta^{\perp\perp} | P \in \mathscr{F}\} = 0_{\mathscr{P}} \quad \text{in } H.$$

Therefore

$$\wedge \{P\theta^\perp | P \in \mathscr{S}\} = 0_{\mathscr{P}} \quad \text{or} \quad \vee \{P\theta^{\perp\perp} | P \in \mathscr{S}\} = 1_{\mathscr{P}} \quad \text{in } H.$$

An l-homorphism which enjoys the property isolated in Lemma 5.5 will be termed *completely Boolean*. The composition and product of completely Boolean l-homomorphisms are again completely Boolean. Completely Boolean l-homomorphisms from G into a totally ordered group are quite rare. In fact, θ is such an l-homomorphism if and only if the kernel of θ is $b^\perp$ for some basic $b \in G$.

An l-homomorphism $\theta: G \to H$ *preserves type* $\mathscr{Y}$ *suprema* (*preserves disjoint suprema*) if for every type $\mathscr{Y}$ (pairwise disjoint) subset $D \subseteq G$ such that $\vee D = g$ in G it is true that

$$\vee \{d\theta | d \in D\} = g\theta \quad \text{in } H.$$

By applying Proposition 3.1 of [**1**] to type $\mathscr{Y}$ (pairwise disjoint) sets, one is led to the conclusion that an l-epimorphism preserves type $\mathscr{Y}$ (disjoint) suprema if and only if the supremum of any type $\mathscr{Y}$ (pairwise disjoint) subset of the kernel is itself in the kernel.

LEMMA 5.6. *Suppose X is a subset and g an element of G such that $g \geqq X$. Then X is type $\mathscr{Y}$ and $\vee X = y$ if and only if*

$$\vee \{(yx^{-1})^\perp | x \in X\} = 1_{\mathscr{P}}.$$

Proof. Suppose $\bigvee_X (yx^{-1})^\perp = 1_{\mathscr{P}}$. Consider an arbitrary $z < y$. Then there must be some $x \in X$ such that $(yz^{-1})^\perp = P \neq 0_{\mathscr{P}}$. Let $1 < p \in P$ satisfy $p \leqq yz^{-1}$, and find a value T of p. Then $p \notin T$ implies $yx^{-1} \in$

$p^\perp \subseteq T$, hence $Tz < Ty = Tx$. This proves $z \ngeqq x$, so that $\vee X = y$. To verify that X is type $\mathscr{Y}$, fix $x \in X$ and observe that for any $z \in X$,

$$yx^{-1} \geqq zx^{-1} \vee 1,$$

hence

$$(yx^{-1})^\perp \subseteq \wedge \{(zx^{-1} \vee 1)^\perp | z \in X\} = (Xx^{-1} \vee 1)^\perp.$$

Therefore

$$\wedge \{(Xx^{-1} \vee 1)^{\perp\perp} | x \in X\} = 0_{\mathscr{P}}.$$

A similar argument, using the fact that

$$1_{\mathscr{P}} = (\bigvee_X (yx^{-1})^\perp)^y = \bigvee_X (x^{-1}y),$$

yields

$$\wedge \{(x^{-1}X \vee 1)^{\perp\perp} | x \in X\} = 0_{\mathscr{P}}.$$

Now suppose X is type $\mathscr{Y}$ and $\vee X = y$. Choose a particular $x \in X$ such that

$$(Xx^{-1} \vee 1)^\perp = P \neq 0_{\mathscr{P}}.$$

We claim $yx^{-1} \in P^\perp$. For if not then there is some $p \in P$ such that $1 < p \leqq yx^{-1}$, or $x \leqq p^{-1}y < y$. We shall prove $X \leqq p^{-1}y < y$. Consider $z \in X$ and arbitrary prime T. If $p \in T$ then

$$Tz \leqq Ty = Tp^{-1}y.$$

If $p \notin T$ then

$$P^\perp = (Xx^{-1} \vee 1)^{\perp\perp} \subseteq T,$$

hence $zx^{-1} \vee 1 \in T$ or $Tz \leqq Tx$, implying $Tz \leqq Tp^{-1}y$. In both cases $Tz \leqq Tp^{-1}y$, hence $z \leqq p^{-1}y$, proving $X \leqq p^{-1}y < y$. But this contradicts $\vee X = y$ and proves $yx^{-1} \in P^\perp$ or $P \subseteq (yx^{-1})^\perp$. Therefore

$$\vee \{(Xx^{-1} \vee 1)^\perp | x \in X\} = 1_{\mathscr{P}}$$

implies

$$\bigvee_X (yx^{-1})^\perp = 1_{\mathscr{P}}.$$

PROPOSITION 5.7. *Every completely Boolean l-epimorphism $\theta: G \to H$ preserves type $\mathscr{Y}$ suprema.*

Proof. Let N be the kernel of θ and let X be a type $\mathscr{Y}$ subset of N^+ with supremum y. Let

$$\mathscr{S} = \{(yx^{-1})^\perp | x \in X\}.$$

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Observe that for $x \in X$ and $1 < p \in (yx^{-1})^{\perp}$,

$$p \wedge y \leqq px \wedge y = (p \wedge yx^{-1})x = x,$$

implying

$$y \wedge (yx^{-1})^{\perp} \subseteq N \quad \text{for all } x \in X.$$

Since θ is completely Boolean,

$$\vee \{(yx^{-1})^{\perp}\theta^{\perp\perp} | x \in X\} = 1_{\mathscr{P}} \quad \text{in } H.$$

It follows that $y\theta = 1$, or $y \in N$.

COROLLARY 5.8. *Every completely Boolean l-epimorphism preserves disjoint suprema.*

To demonstrate the failure of the converse of Proposition 5.7, let G be $\overleftarrow{U \times \mathbf{R}}$, the lexicographic extension of some non-totally ordered l-group U by the real numbers $\mathbf{R}$, and let $\theta: G \to \mathbf{R}$ be the l-epimorphism with kernel $U \times 0$. U is actually order closed and therefore θ preserves all suprema that exist in G. Yet θ is not even Boolean.

LEMMA 5.9. *Suppose $\theta: G \to H$ is a completely Boolean l-homomorphism with kernel N. Then there is a polar $Q \in \mathscr{P}_G$ such that for all $P \in \mathscr{P}_G$, $P \subseteq N$ if and only if $P \subseteq Q$.*

Proof. Suppose not. Let

$$Q = \vee \{P \in \mathscr{P}_G | P \subseteq N\}$$

and consider $1 < x \in Q - N$. Let

$$\mathscr{S} = \{P \in \mathscr{P}_G | P \subseteq N \quad \text{or} \quad P = Q^{\perp}\}.$$

Observe that $\vee \mathscr{S} = 1_{\mathscr{P}}$ in G, yet

$$1 < x\theta \in (\vee \{P\theta^{\perp\perp} | P \in \mathscr{S}\})^{\perp}$$

because $x \wedge P \subseteq N$ for all $P \in \mathscr{S}$. This contradicts the complete Booleanness of θ.

PROPOSITION 5.10. *Suppose $G = \bar{G}^{\omega}$. Then an l-epimorphism θ on G with kernel N is completely Boolean if and only if it has the following two properties.*

(a) *θ preserves type $\mathscr{Y}$ suprema.*

(b) *There is a polar $Q \in \mathscr{P}_G$ such that for all $P \in \mathscr{P}_G$, $P \subseteq N$ if and only if $P \subseteq Q$.*

Proof. Assume θ has properties (a) and (b), and consider some large set $\mathscr{S} \subseteq \mathscr{P}_G$ and some $1 < x \in G$ such that $x \wedge P \subseteq N$ for all $P \in \mathscr{S}$. We must show $x \in N$. The polar Q, maximal with respect to being contained in N, is normal and therefore a cardinal summand of G, as can be verified by reference to the proof of Proposition 3.3. Therefore $x = yz$ for unique $y \in Q^{\perp}$, $z \in Q$. We claim y to be the supremum of a type $\mathscr{Y}$ subset of N, which by Lemma 5.6 follows from showing

$$\vee \{(yn^{-1})^{\perp} | \, n \in N\} = 1_{\mathscr{P}}.$$

So consider an arbitrary $0_{\mathscr{P}} \neq V \subseteq Q^{\perp}$ and find $P \in \mathscr{S}$ such that $0_{\mathscr{P}} \neq P \subseteq V$. There must be some $1 < p \in P$ such that $p \nleqq y$, since otherwise $P \wedge x \subseteq N$ implies $P \subseteq N$, in contradiction to $P \subseteq Q^{\perp}$. Let

$$n = p \wedge y \in N$$

and let

$$a = p(y \wedge p)^{-1} = py^{-1} \vee 1 \in P.$$

Observe that $yn^{-1} = 1 \vee yp^{-1}$, which is disjoint from a. This shows

$$(yn^{-1})^{\perp} \wedge V \neq 0_{\mathscr{P}},$$

and therefore that

$$\vee \{(yn^{-1})^{\perp} | \, n \in N\} = 1_{\mathscr{P}}.$$

Condition (a) now implies $y \in N$, hence $x \in N$.

PROPOSITION 5.11. *Suppose G is strongly projectable. Then an l-epimorphism on G is completely Boolean if and only if it preserves disjoint suprema.*

Proof. Suppose θ preserves disjoint suprema and has kernel N. Let $\mathscr{S}$ be any large subset of $\mathscr{P}_G$. By a Zorn's Lemma argument find a pairwise disjoint subset $\mathscr{R} \subseteq \mathscr{S}$ such that $\vee \mathscr{R} = 1_{\mathscr{P}}$. Suppose now that $1 < x \in G$ such that $x \wedge P \subseteq N$ for all $P \in \mathscr{R}$. In particular, N must contain the projection of x on each $P \in \mathscr{R}$. But these projections constitute a pairwise disjoint set with supremum x, hence $x \in N$.

Here is an example which shows the only way that a completely Boolean l-homomorphism θ: $G \to H$ can fail to be extendable to some $\hat{\theta}$: $\bar{G} \to \bar{H}$. Let

$$H = \left\{ \begin{bmatrix} m & m \\ n & p \end{bmatrix} \middle| \quad m, n, p \text{ integers} \right\},$$

where

$$\begin{bmatrix} m_1 & m_1 \\ n_1 & p_1 \end{bmatrix} + \begin{bmatrix} m_2 & m_2 \\ n_2 & p_2 \end{bmatrix}$$

is defined to be

$$\begin{bmatrix} m_1 + m_2 & m_1 + m_2 \\ n_1 + n_2 & p_1 + p_2 \end{bmatrix}$$

https://doi.org/10.4153/CJM-1982-042-5 Published online by Cambridge University Press

when m_2 is even and

$$\begin{bmatrix} m_1 + m_2 & m_1 + m_2 \\ p_1 + n_2 & n_1 + p_2 \end{bmatrix}$$

when m_2 is odd. H is a group with zero $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and negatives $-\begin{bmatrix} m & m \\ n & p \end{bmatrix} = \begin{bmatrix} -m & -m \\ -n & -p \end{bmatrix}$. Order H by declaring $\begin{bmatrix} m & m \\ n & p \end{bmatrix} \geqq 0$ whenever $m > 0$ or whenever $m = 0$ and both $n \geqq 0$ and $p \geqq 0$. H is an l-group under these conventions. Let

$$G = \left\{ \begin{bmatrix} m & m \\ n & p \end{bmatrix} \in H \middle| \, m \text{ even} \right\}$$

and let θ: $G \rightarrow H$ be the identity map. Then G is an order dense l-subgroup of H, so that θ is completely Boolean. Since G is abelian, $\bar{G}^\omega$ is the group of 2×2 integral matrices under addition with $\begin{bmatrix} m & n \\ p & q \end{bmatrix} \geqq 0$ whenever both of the following conditions are true: either $m > 0$ or $m = 0$ and $p \geqq 0$, and either $n > 0$ or $n = 0$ and $q \geqq 0$. It can also be verified that $\bar{H} = H$. Therefore no extension of θ to $\bar{G}$ or even $\bar{G}^\omega$ over G is possible.

An l-homomorphism θ: $G \rightarrow H$ is (*completely*) *tame* if it is (completely) Boolean and if for every $1 \leqq x \in G$, $s \in H$, and nonzero $P \in \mathscr{P}_H$ such that $1 \leqq s \leqq x\theta$ there is some $g \in G$ and $R \in \mathscr{P}_H$ such that $0_{\mathscr{P}} \neq R \subseteq P$, $1 \leqq g \leqq x$, and $R^s = R^{g\theta}$. The reader may wish to satisfy himself that this condition is violated in the previous example, that every (completely) Boolean l-epimorphism is (completely) tame, and that the identity map from G into ($\bar{G}$) $\bar{G}^\omega$ is (completely) tame.

THEOREM 5.12. *Every* (*completely*) *tame l-homomorphism θ: $G \rightarrow H$ has a unique extending l-homomorphism $\hat{\theta}$: $\bar{G}^\omega \rightarrow \bar{H}^\omega$ ($\hat{\theta}$: $\bar{G} \rightarrow \bar{H}$). $\hat{\theta}$ is* (*completely*) *Boolean.*

Proof. Suppose θ: $G \rightarrow H$ is completely tame and consider $x \in \bar{G}$. Then $x = [k]$ for some consistent map k. Let

$$\mathscr{R} = \{Q \in \mathscr{P}_H | \, Q \subseteq P\theta^{\perp\perp}, P \in \text{dom } (k)\},$$

a large set because θ is completely Boolean. Define m: $\mathscr{R} \rightarrow H$ by declaring $Qm = Pk\theta$, where $Q \subseteq P\theta^{\perp\perp}$ and $P \in \text{dom } (k)$. To see that m is well defined and consistent, consider $Q_1, Q_2 \in \mathscr{R}$, say $Q_i \subseteq P_i\theta^{\perp\perp}$ for $P_i \in \text{dom } (k)$. Then

$$\begin{aligned}(Q_1m)(Q_2m)^{-1} &= ((P_1k)(P_2k)^{-1})\theta \in (P_1 \wedge P_2)^{\perp}\theta \\ &\subseteq (P_1 \wedge P_2)^{\perp}\theta^{\perp\perp} = (P_1\theta^{\perp\perp} \wedge P_2\theta^{\perp\perp})^{\perp} \subseteq (Q_1 \wedge Q_2)^{\perp}.\end{aligned}$$

If read with $P_1 = P_2$, the foregoing argument shows m to be well defined; read with arbitrary P_1 and P_2 it shows m consistent. The task at hand is to show that m has properties (1) and (2) of Section 2.

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To show $\mathscr{R}^m$ large consider $0_{\mathscr{P}} \neq V \in \mathscr{P}_H$. Since dom $(k)^k$ is large and θ is completely Boolean, there is some $P \in$ dom (k) such that

$$(P^k)\theta^{\perp\perp} \wedge V \neq 0_{\mathscr{P}}.$$

But $(P^k)\theta^{\perp\perp} = (P\theta^{\perp\perp})^m \in \mathscr{R}^m$. To verify that m has property (2) consider disjoint nonzero polars $Q_1, Q_2 \in \mathscr{R}$ and element s such that

$$1 \leqq s \leqq (Q_1 m)(Q_2 m)^{-1} \vee 1 \quad \text{and} \quad Q_1{}^s = Q_2.$$

Let $P_1, P_2 \in$ dom (k) be such that $Q_i \subseteq P_i\theta^{\perp\perp}$. Now $P_1\theta^{\perp\perp} \wedge Q_2 = 0_{\mathscr{P}}$, for otherwise, by replacing Q_2 with $P_1\theta^{\perp\perp} \wedge Q_2$, we would have

$$Q_1 m = P_1 k\theta = Q_2 m,$$

forcing $s = 1$ and $Q_1 = Q_2$, a contradiction. By replacing P_2 with $P_2 \wedge P_1{}^\perp$ if necessary, we may assume $P_1 \wedge P_2 = 0_{\mathscr{P}}$. By the tameness condition applied to

$$1 \leqq s \leqq [(P_1 k)(P_2 k)^{-1} \vee 1]\theta,$$

there is some $g \in G$ and $R \in \mathscr{P}_H$ such that

$$0_{\mathscr{P}} \neq R \subseteq Q_1,\ 1 \leqq g \leqq (P_1 k)(P_2 k)^{-1} \vee 1, \quad \text{and}$$

$$R^{g\theta} = R^s \subseteq Q_2.$$

By condition (2) applied to k, $P_1{}^g \wedge P_2 = 0_{\mathscr{P}}$, whence

$$(P_1\theta^{\perp\perp})^{g\theta} \wedge (P_2\theta^{\perp\perp}) = 0_{\mathscr{P}} \quad \text{and}$$

$$R^{g\theta} \wedge Q_2 = R^s \wedge Q_2 = 0_{\mathscr{P}},$$

a contradiction.

The l-homomorphism $\hat{\theta}: \bar{G} \to \bar{H}$ is defined by declaring $x\hat{\theta} = [m]$. Observe that $\hat{\theta}$ takes $\bar{G}^\omega$ into $\bar{H}^\omega$ and that the argument in this case uses only the tameness of θ. We leave to the reader the verification that $\hat{\theta}$ is a well defined l-homomorphism.

Suppose $\psi: \bar{G} \to \bar{H}$ is an l-homomorphism which agrees with θ on G. Consider $x = [k] \in \bar{G}$ and let $x\psi = [m']$. Then for any $P \in$ dom (k), $x(Pk)^{-1} \in P^\perp$, which implies

$$x\psi(Pk\theta)^{-1} \in P^\perp\theta \subseteq (P\theta^{\perp\perp})^\perp.$$

This means $P\theta^{\perp\perp}$ is contained in the maximal domain of m', and that m' is equivalent to the m of the previous paragraphs. Succinctly put, $\hat{\theta} = \psi$.

COROLLARY 5.13. *Every (completely) Boolean l-homomorphism θ between representable l-groups G and H has a unique extending l-homomorphism*

$$\hat{\theta}: \bar{G}^\omega \to \bar{H}^\omega \ (\bar{G} \to \bar{H}).$$

$\hat{\theta}$ is (completely) Boolean.

Is the l-homomorphism $\hat{\theta}$ of Theorem 5.12 tame? The question is unresolved, though we might expect a negative answer on the basis of permutation sketches. By restricting the class of maps still further, however, it is possible to obtain the most symmetrical version of the universal mapping property. An l-homomorphism $\theta: G \rightarrow H$ is (completely) *docile* if it is (completely) Boolean and if for every $1 \leqq x \in G$, $s \in H$, and nonzero $P \in \mathscr{P}_H$ such that $1 \leqq s \leqq x\theta$ there is some $g \in G$ and $R \in \mathscr{P}_H$ such that $0_{\mathscr{P}} \neq R \subseteq P$, $1 \leqq g \leqq x$, and $s(g\theta)^{-1} \in R^{\perp}$. Observe that every (completely) Boolean l-epimorphism is (completely) docile, and that the identity map from G to $(\bar{G})$ $\bar{G}^{\omega}$ is (completely) docile.

THEOREM 5.14. *Every (completely) docile l-homomorphism $\theta: G \rightarrow H$ has a unique extending l-homomorphism $\hat{\theta}: \bar{G}^{\omega} \rightarrow \bar{H}^{\omega}$ $(\bar{G} \rightarrow \bar{H})$. $\hat{\theta}$ is (completely) docile.*

Proof. It remains to show $\hat{\theta}$ completely docile when θ is. Consider $1 < x \in \bar{G}$, $y \in \bar{H}$, and nonzero $Q \in \mathscr{P}_H$ such that $1 \leqq y \leqq x\hat{\theta}$. Suppose $x = [k]$ for some consistent map k. Since

$$\vee \{P\theta^{\perp\perp} | P \in \operatorname{dom}(k)\} = 1_{\mathscr{P}} \text{ in } \bar{H},$$

choose $P \in \operatorname{dom}(k)$ such that $P\theta^{\perp\perp} \cap Q = 0_{\mathscr{P}}$. Since $H \preccurlyeq \bar{H}$ one can also choose $R \in \mathscr{P}_H$ and $h \in H$ such that

$$0_{\mathscr{P}} \neq R \subseteq P\theta^{\perp\perp} \cap Q \quad \text{and} \quad yh^{-1} \in R^{\perp}.$$

By replacing h by $h \wedge Pk\theta$ if necessary, we may also assume $h \leqq Pk\theta$. The docility of θ now assures the existence of $g \in G$ and $S \in \mathscr{P}_H$ such that $0_{\mathscr{P}} \neq S \subseteq R$, $1 \leqq g \leqq Pk$, and $h(g\theta)^{-1} \in S^{\perp}$. Let $u = g \wedge x \in \bar{G}$. It remains only to show $y(u\hat{\theta})^{-1} \in S^{\perp}$. But

$$y(u\hat{\theta})^{-1} = (yh^{-1})(h(g\theta)^{-1})((g\hat{\theta})(u\hat{\theta})^{-1}).$$

Since $yh^{-1} \in R^{\perp} \subseteq S^{\perp}$, $h(g\theta)^{-1} \in S^{\perp}$, and $(P\theta^{\perp\perp})^{\perp} \subseteq S^{\perp}$, it remains only to show $gu^{-1} \in P^{\perp}$. But

$$1 \leqq gu^{-1} = 1 \vee gx^{-1} \leqq 1 \vee Pkx^{-1} \in P^{\perp},$$

implying $gu^{-1} \in P^{\perp}$.

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Boise State University,
Boise, Idaho