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SOME CHARACTERIZATIONS OF THE PROJECTION PROPERTY IN ARCHIMEDEAN RIESZ SPACES

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In this paper we give some new characterizations of the projection property in Archimedean Riesz spaces. Our approach primarily explores the inter-relationships between such things as the band structure or the prime ideal structure of an Archimedean vector lattice and corresponding structures of its Dedekind completion. Our results show that, in general, there is a "strong" relationship if and only if the original vector lattice has the projection property. The main result of this paper is Theorem 2.6 which both summarizes and extends all of the results we obtain prior to it.

1. Preliminaries. For the basic definitions and terminology, we refer the reader to [**1**; **4**; **5**; **7**]. We include in this section only certain definitions and results which the reader may not be familiar with. Near the end of this section some notational agreements are made which are used in the remainder of the paper.

A Riesz space E is said to have the *projection property* (P.P.) if every band (order closed ideal) in E is a projection band (i.e., a direct summand of E).

A band B in E is a projection band if and only if for every $v \in E^+$ we have that $\sup\{u \in B: 0 \leqq u \leqq v\}$ exists in E and hence in B (see [**4**; **5**]).

The complement in the Riesz space E of a set $A \subseteq E$, denoted $A^\perp$-in-E, is defined by

$$A^\perp\text{-in-}E = \{u \in E: |u| \wedge |v| = 0 \text{ for all } v \in A\}.$$

We will denote by $A^{\perp\perp}$-in-E the set $(A^\perp\text{-in-}E)^\perp$-in-$E$.

An ideal P in the Riesz space E is *prime* if $x \wedge y$ in P implies that x is in P or that y is in P. Equivalently, P is a prime ideal in E if $x \wedge y = 0$ implies that either x is in P or y is in P. (c.f.[**2**]).

In the sequel, E will always denote an Archimedean Riesz space, $\hat{E}$ its Dedekind completion, and E^+ its positive cone. We also agree that if I is any ideal in E, then by $\hat{I}$ we will mean the ideal in $\hat{E}$ defined by

$$\begin{aligned}\hat{I} &= \{\hat{x} \in \hat{E}: \text{there exists } y_1, y_2 \in I \text{ with } y_1 \leqq \hat{x} \leqq y_2\}\\ &= \{\hat{x} \in \hat{E}: \text{there exists } y \in I \text{ with } |\hat{x}| \leqq y\}.\end{aligned}$$

Remark 1.1. One of the properties we consider in the next section is the following property:

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If P is any prime ideal in E, then $\hat{P}$ is prime in $\hat{E}$.

This property was considered in [**6**] by J. J. Masterson and he showed that the projection property implies it (see [**6**, Theorem 2.1, p. 470]). We show that this property also implies the projection property, and hence that the converse to his theorem holds.

2. The characterizations. We first establish a result concerning prime ideals.

LEMMA 2.1. *Suppose that E has the property that if P is a prime ideal in E, then $\hat{P}$ is prime in $\hat{E}$. Then, for any $x \in E^+$ and $\bar{x} \in \hat{E}$ with $\bar{x} \wedge (x - \bar{x}) = 0$ (in $\hat{E}$) we have that $\bar{x} \in E$ (i.e., every component in $\hat{E}$ of a positive element in E is in E).*

Proof. Suppose there exist $0 < x \in E$ and an $\bar{x} \in \hat{E}$ with $\bar{x} \wedge (x - \bar{x}) = 0$ (in $\hat{E}$) such that $\bar{x} \notin E$. Let I be the ideal in E generated by $\{y \in E: |y| \leqq \bar{x}\}$ (that is, $I = \{y \in E$: for some real number $\alpha > 0$ we have $\alpha|y| \leqq \bar{x}\}$). Now, the ideal $\hat{I}$ in $\hat{E}$ has the property that $\bar{x} \notin \hat{I}$. To see this, suppose that $\bar{x} \in \hat{I}$. This means that there exist $y \in I$ and a real number $\alpha > 0$ such that $\alpha y \leqq \bar{x} \leqq y$. But then in $\hat{E}$ the principal ideals generated by y and by $\bar{x}$ are the same and it is an easy matter to check that this would imply that $\bar{x} \in E$, which is a contradiction. It follows from a straightforward Zorn's lemma argument that there exists a maximal ideal $M \subset \hat{E}$ containing $\hat{I}$ and not containing $\bar{x}$. In addition, it is known (c.f.[**2**, Lemma 6.2, p. 524; **3**, Theorem 1.5, p. 9]) that M is prime. It follows easily that $P = M \cap E$ is a prime ideal in E.

We show now that there does not exist a $y \in E$ with $y > x - \bar{x}$ and $y \in M$. It will follow from this that $\hat{P}$ is not prime in $\hat{E}$ (since $\bar{x} \notin \hat{P} \subseteq M$, $x - \bar{x} \notin \hat{P}$, and $\bar{x} \wedge (x - \bar{x}) = 0 \in \hat{P}$) and the assertion will follow.

So, assume $y \in P$ and $y > x - \bar{x}$. We have that $x \geqq y \wedge x > x - \bar{x}$ ($y \wedge x \neq x - \bar{x}$ since, otherwise, we would have that $x - \bar{x} \in E$ and $\bar{x} = x - (x - \bar{x}) \in E$, which is contrary to assumption). But, then

$$\bar{x} = x - (x - \bar{x}) > x - (y \wedge x).$$

By construction, $x - (y \wedge x) \in M$ and since $y \in M$, we have also that $y \wedge x \in M$ which implies that $x = (x - (y \wedge x)) + (y \wedge x) \in M$. But then $\bar{x} \in M$ (since $0 < \bar{x} < x$), which is a contradiction. So, $\hat{P}$ is not prime.

The assertion is proved.

It was shown in [**5**, Theorems 29.5 and 29.10] that if E is an Archimedean Riesz space and B is a band in E, then $B^{\perp\perp}$-in-$E = B$.

We will find this next lemma somewhat of a convenience. Its proof is easy and is omitted.

LEMMA 2.2. *Let B be a band in E. Then $f \in E$ is in $B^{\perp\perp}$-in-$\hat{E}$ if and only if $f \in B$.*

In this next lemma we establish one of the main portions of our characterization theorem.

LEMMA 2.3. *The following are equivalent:*

(i) *E has the projection property.*

(ii) *Given any $\hat{f} \in \hat{E}^+$ there exists an $f \in E$ such that $f \geqq \hat{f}$ and such that*

$$\{f\}^{\perp\perp}\text{-in-}\hat{E} = \{\hat{f}\}^{\perp\perp}\text{-in-}\hat{E}.$$

(iii) *For any band B in E, $B^{\perp\perp}$-in-$\hat{E} = \hat{B}$.*

Proof. (i) $\Rightarrow$ (ii). Let $\hat{f} \in \hat{E}^+$. Then, there exists an $f \in E^+$ such that $f \geqq \hat{f}$. Consider $B_1 = B_{\hat{f}} \cap E$ (where $B_{\hat{f}} = \{\hat{f}\}^{\perp\perp}$-in-$\hat{E}$); B_1 is a band in E; and, since E has the P. P., there exists a projection of f onto B_1, say f_0. Since $\hat{f} = \sup\{u \in B_1: 0 \leqq u \leqq \hat{f}\}$, $f_0 = \sup\{u \in B_1: 0 \leqq u \leqq f\}$, and $f \geqq \hat{f}$, we have that $f_0 \geqq \hat{f}$. Therefore,

$$\{f_0\}^{\perp\perp}\text{-in-}\hat{E} \supseteq \{\hat{f}\}^{\perp\perp}\text{-in-}\hat{E}.$$

But, $f_0 \in B_1 \subseteq B_{\hat{f}}$. Hence, $\{f_0\}^{\perp\perp}$-in-$\hat{E} \subseteq B_{\hat{f}}$. We have extablished that (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii). It is clear that $\hat{B} \subseteq B^{\perp\perp}$-in $\hat{E}$. Suppose now that $\hat{x} \in B^{\perp\perp}$-in-$\hat{E}$. Now, if $f \in E^+$ is such that $f \geqq \hat{x}$ and

$$\{f\}^{\perp\perp}\text{-in-}\hat{E} = \{\hat{x}\}^{\perp\perp}\text{-in-}\hat{E} \subseteq B^{\perp\perp}\text{-in-}\hat{E},$$

we have by Lemma 2.2 that $f \in B$ and hence that $\hat{B} \supseteq B^{\perp\perp}$-in-$\hat{E}$. We have established that (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (i). Let B be a band in E. We have that $B^{\perp\perp}$-in-$\hat{E} = \hat{B}$. Let $f \in E^+$. There exists $f_0 \in \hat{B}$ such that $f_0 \wedge (f - f_0) = 0$ and $(f - f_0) \in B^{\perp}$-in-$\hat{E}$. Let $g \in B$ be such that $g \geqq f_0$. We claim that $g \wedge f = f_0$. Clearly, $g \wedge f \geqq f_0 \wedge f = f_0$. Since $g \in B$, we have that $g \wedge (f - f_0) = 0$ and, hence, $(g \wedge f) \wedge (f - f_0) = 0$. But $g \wedge f \leqq f$ and $(f - f_0) \leqq f$; thus $(g \wedge f) + (f - f_0) = (g \wedge f) \vee (f - f_0) \leqq f = f_0 + (f - f_0)$. Therefore, $g \wedge f \leqq f_0$. Hence, $f_0 = g \wedge f \in B$ and B is seen to be a projection band.

The assertion is proved.

LEMMA 2.4. *Suppose that B is a band in E with the property that every band contained in B is a projection band. Then, $\hat{B} = B^{\perp\perp}$-in-$\hat{E}$.*

Proof. Again, that $\hat{B} \subseteq B^{\perp\perp}$-in-$\hat{E}$ is obvious. To show that $\hat{B} \supseteq B^{\perp\perp}$-in-$\hat{E}$, it is clear that we may restrict our attention to positive $\hat{x} \in B^{\perp\perp}$-in-$\hat{E}$. So, let $\hat{x} \in B^{\perp\perp}$-in-$\hat{E}$ be positive. We need only show that there exists $y \in B$ with $y \geqq \hat{x}$. Consider $B_1 = B_{\hat{x}} \cap E$ where

$$B_{\hat{x}} = \{\hat{x}\}^{\perp\perp}\text{-in-}\hat{E} \subseteq B^{\perp\perp}\text{-in-}\hat{E}.$$

Clearly, if $u \in B_1$, then $u \in (B^{\perp\perp}\text{-in-}\hat{E}) \cap E$ which is equal to B by virtue of

Lemma 2.2. Obviously B_1 is a band in E. Hence, for any $y \in E^+$ with $y \geqq \hat{x}$, since $B_1 \subseteq B$, the projection of y onto B_1 exists and is in $B_1 \subseteq E$. Let $y_0 = \sup\{u \in B_1: 0 \leqq u \leqq y\}$ be the projection of y onto B_1. Since $\hat{x} = \sup\{v \in B_1: 0 \leqq v \leqq \hat{x}\}$ and $\hat{x} \leqq y$, we have that $y_0 \geqq \hat{x}$ and $y_0 \in B_1 \subseteq B$. Hence, $\hat{B} = B^{\perp\perp}$-in-$\hat{E}$. The assertion is proved.

It is well-known that if x, y_1, y_2 are positive elements of a vector lattice such that $x \leqq y_1 + y_2$, there exists a decomposition of x as $x_1 + x_2$ where $0 \leqq x_1 \leqq y_1$ and $0 \leqq x_2 \leqq y_2$. This property is known as the Riesz dominated decomposition property. However, if $x \in E^+$ and $y_1, y_2 \in \hat{E}^+$, it is not in general true that x_1 and x_2 can be chosen to be in E(c.f. [**3**]). If we add the additional restriction that y_1 is perpendicular to y_2, the decomposibility of x as $x_1 + x_2$ where $x_1 \in E$ and $x_2 \in E$ for all $x \in E^+$ and for $y_1, y_2 \in \hat{E}^+$ with $x \leqq y_1 + y_2$, turns out to be equivalent to the projection property. Hence, we have the following lemma:

LEMMA 2.5. *E has the projection property if and only if for any $x \in E^+$ and $y_1, y_2 \in \hat{E}^+$ such that $y_1 \wedge y_2 = 0$ and $x \leqq y_1 + y_2$, x can be decomposed as $x_1 + x_2$ where $x_1, x_2 \in E^+$.*

Proof. Suppose that E does not have the projection property. Then, there exist a band B in E and an element $a \in E^+$ such that $a \notin B \oplus B^\perp$. Also we have that

$$\hat{E} = (B^{\perp\perp}\text{-in-}\hat{E}) \oplus (B^{\perp}\text{-in-}\hat{E}).$$

Therefore, $a = a_1 + a_2$, where $a_1 \in B^{\perp\perp}$-in-$\hat{E}$ and $a_2 \in B^\perp$-in-$\hat{E}$. Suppose that $a = a_1' + a_2'$ where $a_1' \leqq a_1, a_2' \leqq a_2, a_1', a_2' \in E$. But then $a_1' = a_1$ and $a_2' = a_2$ for, otherwise, we would have that $a = a_1' + a_2' < a_1 + a_2 = a$, which is a contradiction. By virtue of Lemma 2.2, it now follows that $a_1 \in B$ and that $a_2 \in B^\perp$-in-E, which implies that $a = a_1 + a_2 \in B \oplus B^\perp$, which is a contradiction. We have shown the sufficiency.

For the necessity, let $a \in E^+$, $b_1, b_2 \in \hat{E}^+$, $a \leqq b_1 + b_2$ and $b_1 \wedge b_2 = 0$. Let $K = \{b \in E^+: b \leqq b_1\}$. Let B be the band generated by K in E. Now, if E has the projection property, then $E = B \oplus (B^\perp$-in-$E)$ and $a = a_1 + a_2$ with $a_1 \in B$ and $a_2 \in B^\perp$-in-E. Also, $\hat{E} = \hat{B} \oplus (B^\perp$-in-$\hat{E})$. So, $b_1 \in \hat{B}$ and $b_2 \in B^\perp$-in-$\hat{E}$. Since $a \leqq b_1 + b_2$ and $a_1 \wedge b_2 = 0$, it follows that $a_1 \leqq b_1$. Similarly, it follows that $a_2 \leqq b_2$.

The assertion is proved.

We are now ready to establish our main result.

THEOREM 2.6. *The following are equivalent*:

(i) *E has the projection property.*

(ii) *If P is a prime ideal in E, then $\hat{P}$ is prime in $\hat{E}$.*

(iii) *Given any $x \in E^+$ and $\bar{x} \in \hat{E}$ with $\bar{x} \wedge (x - \bar{x}) = 0$, it follows that $\bar{x} \in E$.*

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(iv) *For any* $\hat{f} \in \hat{E}^+$ *there exists an* $f \in E$ *with* $f \geqq \hat{f}$ *such that* $\{f\}^{\perp\perp}$-in-$\hat{E} = \{\hat{f}\}^{\perp\perp}$-in-$\hat{E}$.

(v) *For any band* $B \subseteq E$ *it is true that* $B^{\perp\perp}$-in-$\hat{E} = \hat{B}$.

(vi) *For any* $x \in E^+$ *and* $y_1, y_2 \in \hat{E}^+$ *such that* $y_1 \wedge y_2 = 0$ *and* $x \leqq y_1 + y_2$, x *can be decomposed as* $x_1 + x_2$ *where* $x_1, x_2 \in E^+$.

Proof. We show first that (iii) $\Rightarrow$ (iv). To this end let $\hat{f} \in \hat{E}^+$. Since $\hat{E}$ is the Dedekind completion of E, there exists an $x \in E^+$ with $x \geqq \hat{f}$. Let $\bar{x}$ be the projection of the element x onto the principal band $\{\hat{f}\}^{\perp\perp}$-in-$\hat{E}$. Since $\bar{x} \wedge (x - \bar{x}) = 0$ (in $\hat{E}$) we have by (iii) that $\bar{x} \in E$. Taking $f = \bar{x}$, we have that $f \in E, f \geqq \hat{f}$, and

$$\{f\}^{\perp\perp}\text{-in-}\hat{E} = \{\hat{f}\}^{\perp\perp}\text{-in-}\hat{E},$$

which is the desired result.

By virtue of [**6**, Theorem 2.1, p.470], Lemma 2.1, Lemma 2.3, and what we have just shown, we have that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i). The equivalence of (i) and (vi) was established im Lemma 2.5. The theorem is proved.

Remark 2.7. As was mentioned in § 1, J. J. Masterson considered condition (ii) of Theorem 2.6 in [**6**] and showed that E has the P. P. implies (ii). He did not show the converse.

As far as we know, conditions (iii), (iv), (v), and (vi) have not appeared previously in the literature in this context.

The following corollary shows that every vector lattice "between" a vector lattice with the projection property and its Dedekind completion must have the projection property.

COROLLARY 2.8. *If* $E \subseteq E' \subseteq \hat{E}$ *and* E *has the projection property, then* E' *has the projection property.*

Proof. Condition (iv) of Theorem 2.6 is clearly inherited from E by E'.

A prime ideal P is called a *minimal prime ideal* if there is no prime ideal properly contained in P. It follows easily from a Zorn's lemma argument that every prime ideal contains a minimal prime ideal. Furthermore, it is easy to establish that if P and Q are ideals in E with $P \subseteq Q$ and P is prime then Q is prime. Combining these observations with Theorem 2.6, it is not difficult to establish the following corollary.

COROLLARY 2.9. *The following are equivalent:*

(i) *E has the projection property.*

(ii) *If P is a prime ideal in E, then $\hat{P}$ is a prime ideal in $\hat{E}$.*

(iii) *If P is a minimal prime ideal in E, then $\hat{P}$ is a prime ideal in $\hat{E}$.*

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