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A Free Product Formula for the Sofic Dimension

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Abstract. It is proved that if $G = G_1 *_{G_3} G_2$ is free product of probability measure preserving s-regular ergodic discrete groupoids amalgamated over an amenable subgroupoid G_3, then the sofic dimension $s(G)$ satisfies the equality

$$s(G) = \mathfrak{h}(G_1^0)s(G_1) + \mathfrak{h}(G_2^0)s(G_2) - \mathfrak{h}(G_3^0)s(G_3),$$

where $\mathfrak{h}$ is the normalized Haar measure on G.

1 Introduction

Let G be a group. The sofic dimension of G is an asymptotic invariant that accounts for the number of unital maps

$$\sigma\colon F_\pm^n \longrightarrow \mathrm{Sym}(d)$$

from the "Cayley ball" $F_\pm^n$ of radius n in G into the symmetric group $\mathrm{Sym}(d)$, where $F \subset G$ is a finite set, n is an integer, d is a "very large" integer and the maps σ are multiplicative and free up to an error $\delta > 0$ relative to the normalized Hamming distance on $\mathrm{Sym}(d)$ (see §2). If $\mathrm{SA}(F, n, \delta, d)$ is the (finite) set of all such maps, and $\mathrm{NSA} := |\{\sigma_{|F},\ \sigma \in \mathrm{SA}\}|$, then the sofic dimension of F is

$$s(F) = \inf_{n\in\mathbb{N}} \inf_{\delta>0} \limsup_{d\to\infty} \frac{\log \mathrm{NSA}(F, n, \delta, d)}{d \log d}$$

(so the limit is on d first, and then on δ and n). This definition was considered in [DKP1, DKP2]. It is a combinatorial version of Voiculescu's (microstate) free entropy dimension $\delta(F)$, which can be defined by a similar formula involving maps

$$\sigma\colon F_\pm^n \longrightarrow U(d)$$

into the unitary group $U(d)$ (see [Voi96, Jun1]). It can be shown that the value of $s(F)$ does not dependent on the finite generating set F of G and is therefore denoted $s(G)$. A limiting process allows to define of $s(G)$ for an arbitrary group G.

The definition of the sofic dimension can be extended to probability measure preserving (pmp) actions of countable groups, their orbit equivalence relations, and more generally to discrete pmp groupoids. We refer to [DKP2, Definition 2.3] for the general groupoid definition. An interesting feature of s is to provide combinatorial proofs of statements in orbit equivalence theory (for example, Corollary 7.5 in

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[DKP1] reproves Gaboriau's theorem that the free groups on p generators are pairwise non orbit equivalent using the counting method).

Let G be a pmp groupoid and assume that $G = G_1 *_{G_3} G_2$ is an amalgamated free product over a subgroupoid G_3. A free product formula of the form

$$s(G) = s(G_1) + s(G_2) - s(G_3)$$

is known to hold in the following cases (under some technical assumptions, for example "finitely generated" and/or "s-regularity"):

(a) G, G_1, G_2, and G_3 are pmp equivalence relations on (X, μ) and G_3 is amenable as an equivalence relation: see [DKP1, Theorem 1.2].
(b) G, G_1, G_2, and G_3 are countable groups and G_3 is an amenable group: see [DKP2, Theorem 4.10].
(c) G is the crossed product groupoid $G := G_1 * G_2 \ltimes X$ of a pmp action $(G_1 * G_2) \curvearrowright (X, \mu)$, where G_1 and G_2 are countable groups and X is a standard probability space. Here G_3 is assumed to be the trivial group but the action of $G_1 * G_2$ is not necessarily free (if free this is covered by (a)); see [DKP2, Theorem 6.4].

The general strategy to establish this sort of formula was devised by Voiculescu for the free entropy dimension: see in particular [Voi91, Voi96, Voi98].

The proofs of the above results in [DKP1, DKP2] apply distinct tools to handle the amenable amalgamated part, namely the Connes–Feldmann-Weiss theorem in (a), and the Ornstein–Weiss quasi-tiling theorem in (b). This was a reason why it was hardly conceivable to incorporate an amenable amalgamated subgroup G_3 in (c); in fact, the technical details would presumably (to quote [DKP2, §6]) be 'formidable' even if the action of $G_3 \curvearrowright X$ is essentially free.

We follow a different approach here, based on the use of Bernoulli shifts as a "correspondence principle":

$$\text{groupoids} \leftrightsquigarrow \text{equivalence relations}$$

by which we mean that proving a result for (pmp) equivalence relations automatically implies an a priori more general statement for (pmp) groupoids in a variety of situations, and in particular for the computation of s (see §10 for more details).

The exact assumptions that we need for the free product formula are described in the following statement, which is the main result of this paper.

Theorem 1.1 *Let G be a discrete pmp groupoid of the form $G = G_1 *_{G_3} G_2$, where G_1, G_2 are s-regular ergodic subgroupoids of G and G_3 is an amenable groupoid. Then*

$$s(G) = \mathfrak{h}(G_1^0)s(G_1) + \mathfrak{h}(G_2^0)s(G_2) - \mathfrak{h}(G_3^0)s(G_3),$$

where $\mathfrak{h}$ is the normalized Haar measure on G and G^0 is the object space of G.

The more technical assumptions in this result can be weakened slightly. For example, one way to remove the s-regularity assumption, following Voiculescu's idea (see *e.g.,* [Voi98, Remark 4.8]), is to replace the lim sup in the definition of $s(F)$ by a limit along a fixed ultrafilter ω. What is rather unclear is the extent to which the assumption that G_3 is amenable is essential. Cohomological tools can be used to prove

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a similar formula for the first L^2 Betti number β_1 under the much weaker assumption that $\beta_1(G_3) = 0$. (This is a result of Lück; see [BDJ, Theorem A.1] for the group case.) Furthermore, Mineyev and Shlyakhtenko [MS] have shown that Voiculescu's non-microstate free entropy dimension δ^* satisfies $\delta^*(G) = \beta_1(G) - \beta_0(G) + 1$ for any finitely generated group G, and therefore we have the formula

$$\delta^*(G_1 *_{G_3} G_2) = \delta^*(G_1) + \delta^*(G_2) - \delta^*(G_3),$$

where G_1 and G_2 are finitely generated groups and G_3 is a group such that $\beta_1(G_3) = 0$. A fundamental relation between the microstate and the non-microstate approach to free entropy is provided by the Biane–Capitaine–Guionnet inequality $\delta \leq \delta^*$ [BCG]. A free product formula for δ_0 has been established in [BDJ] for amalgamation of (δ_0-regular) groups over an amenable subgroup (where $\delta_0 \leq \delta$ is a technical modification of δ not depending on the generating set of the group; see [Voi96, Section 6] and [Voi98]). We also note that the above correspondence principle for δ_0 is probably less useful as the amenable part can always be handled uniformly using the hyperfiniteness of von Neumann algebra LG_3 (see in particular [Jun2]; for example, the proof in [BDJ] in the group case does not rely on quasi-tilings). Concerning pmp equivalence relations, a free product formula has been established by Gaboriau [Gab] for the cost, allowing for amalgamations over amenable subrelations, and by Shlyakhtenko [Shl] for δ_0, for free product with trivial amalgamation.

Question 1.2 Can the assumption that G_3 is amenable in Theorem 1.1 be weakened (for example to $\beta_1(G_3) = 0$)?

The paper is organized as follows. Sections 2 and 3 establish basic facts about pmp groupoids and their actions. In the case of $s(G)$, the correspondence "groupoids $\leftrightsquigarrow$ equivalence relations" is achieved by using the formula $s(G) = s(G \ltimes X_0^G)$, where $G \curvearrowright X_0^G$ is a Bernoulli shift; see Theorem 8.2 (other applications of the correspondence principle are given in §10). The proof of this formula uses the idea in a result of L. Bowen [Bow, Theorem 8.1] for the sofic entropy, as explained in Section5. Other difficulties inherent to the groupoid setting are dealt with in Sections 4, 6, and 7 (these difficulties were avoided in [DKP2] by working with groups and their actions rather than with general groupoids). The proof of Theorem 8.2 is given in Section 8. In Section 9 we prove a scaling formula for $s(G)$. In Section10 we prove Theorem 1.1 by putting together these ingredients.

2 Review of $s(G)$

Recall that a discrete standard Borel groupoid G with base (*i.e.*, set of objects) G^0, source map $\mathbf{s}\colon G \to G^0$ and range map $\mathbf{r}\colon G \to G^0$, is said to be *probability measure preserving (pmp)* with respect to a Borel probability measure μ on G^0 if the left and right Haar measures $\mathfrak{h}$ and $\mathfrak{h}^{-1}$ on G coincide: $\mathfrak{h} = \mathfrak{h}^{-1}$, where

$$\mathfrak{h}(A) := \int_{G^0} |A^e| \, \mathrm{d}\mu(e) \quad \text{and} \quad \mathfrak{h}^{-1}(A) := \int_{G^0} |A_e| \, \mathrm{d}\mu(e)$$

and $A \subset G$ is a Borel set with $A^e := \mathbf{r}^{-1}(e) \cap A$ and $A_e := \mathbf{s}^{-1}(e) \cap A$ for $e \in G^0$. Then $\mathfrak{h}_{|G^0} = \mathfrak{h}^{-1}_{|G^0} = \mu$, so we simply denote by $\mathfrak{h}$ the measure μ on G^0.

A *bisection* is a Borel subset $s \subset G$ such that the restrictions of $\mathbf{s}$ and $\mathbf{r}$ to s are Borel isomorphisms onto G^0. The set of bisections form a group called the *full group* of G and are denoted $[G]$. A *partial bisection* is a Borel subset $s \subset G$ such that $\mathbf{s}$ and $\mathbf{r}$ are injective in restriction to s. The set of partial bisections forms a Polish inverse monoid called the *full inverse semigroup* (or the full pseudogroup) of G, denoted $[\![G]\!]$. For $s \in [\![G]\!]$ let $\operatorname{dom}(s) := s^{-1}s \subset G^0$ and $\operatorname{ran}(s) := ss^{-1} \subset G^0$.

For example, if $G := \{1, \dots, d\}^2$ is the transitive equivalence relation on the set $\{1, \dots d\}$ with $d \in \mathbb{Z}_{\geq 1}$ elements, then $[G] = \operatorname{Sym}(d)$ is the symmetric group on d letters and $[\![G]\!]$ is the inverse semigroup of partial permutations. We denote the latter by $[\![d]\!]$.

The semigroup $[\![G]\!]$ (and $[G] \subset [\![G]\!]$) is Polish with respect to the *uniform distance*

$$|s - t| := \mathfrak{h}\{e \in G^0 \mid se \neq te\}.$$

If $G = \{1, \dots, d\}^2$, then the uniform distance is the normalized Hamming distance on $[\![d]\!]$.

The (von Neumann) *trace* on $[\![G]\!]$ is given by

$$\tau(s) := \mathfrak{h}(s \cap G^e) = \mathfrak{h}\{e \in G^0 \mid se = e\}.$$

It is the restriction to $[\![G]\!] \subset LG$ of the finite trace on the von Neumann algebra LG of G.

We have

$$|s - t| = \tau(s^{-1}s) + \tau(t^{-1}t) - \tau(s^{-1}st^{-1}t) - \tau(st^{-1}).$$

We will write tr for the trace on $[\![d]\!]$. So

$$\operatorname{tr}(\sigma) = \frac{1}{d} \times \text{ number of fixed points of } \sigma \in [\![d]\!].$$

If $F \subset [\![G]\!]$ is a finite subset and $n \in \mathbb{Z}_{\geq 1}$, then $F^n_{\pm}$ denotes the set of all products of at most n elements of $F_{\pm} := F \cup F^{-1} \cup \{\mathrm{Id}\}$. For $F \subset [\![G]\!]$ we let $\boldsymbol{\Sigma}\, F \subset [\![G]\!]$ denote the set of sums of elements of F with pairwise orthogonal domains and pairwise orthogonal ranges.

By definition, G is *sofic* if its full inverse semigroup $[\![G]\!]$ is sofic:

Definition 2.1 A pmp groupoid G is called sofic if for every finite set $F \subset [\![G]\!]$, $\delta > 0$ and $n \in \mathbb{Z}_{\geq 1}$ there exist $d \in \mathbb{Z}_{\geq 1}$ and a map

$$\sigma\colon\ \boldsymbol{\Sigma}\, F^n_{\pm} \longrightarrow [\![d]\!]$$

such that

(i) $$|\sigma(st) - \sigma(s)\sigma(t)| < \delta$$

for every $s, t \in \boldsymbol{\Sigma}\, F^n_{\pm}$ such that $st \in \boldsymbol{\Sigma}\, F^n_{\pm}$ (σ is δ-*multiplicative*) and

(ii) $$|\operatorname{tr} \circ \sigma(s) - \tau(s)| < \delta$$

for every $s \in F^n_{\pm}$ (σ is δ-*trace-preserving*).

Remark 2.2 If G is a group, one can replace $[\![d]\!]$ by $\operatorname{Sym}(d)$ as is easily seen.

Remark 2.3 The notion of sofic pmp equivalence relations was introduced by Elek and Lippner in [EL] in terms of graph approximation and in [Oz] by requiring that $[\![G]\!]$ be sofic as in the definition above (compare [DKP1, DKP2]). The sofic property was first considered for groups by Gromov and Weiss in terms of (Cayley) graph approximation and was studied by Elek and Szabo. It is a simultaneous generalization of amenability and the LEF property of Vershik and Gordon (see [Pe] for more details).

Let

$$\mathrm{SA}(F,n,\delta,d) := \{\sigma\colon \mathbf{\Sigma}\, F_{\pm}^{n} \to [\![d]\!] \text{ satisfying (i), (ii)}\}$$

and define

$$|\,\mathrm{SA}(F,n,\delta,d)|_E := |\{\sigma_{|E} \mid \sigma \in \mathrm{SA}(F,n,\delta,d)\}|$$

for $E \subset F$.

Definition 2.4 For $E \subset F \subset [\![G]\!]$ finite, $n \in \mathbb{Z}_{\geq 1}$ and $\delta > 0$ define successively

$$\begin{aligned} s_E(F,n,\delta) &:= \limsup_{d\to\infty} \frac{\log|\,\mathrm{SA}(F,n,\delta,d)|_E}{d\log d},\\ s_E(F,n) &:= \inf_{\delta>0} s_E(F,n,\delta),\\ s_E(F) &:= \inf_{n\in\mathbb{Z}_{\geq 1}} s_E(F,n). \end{aligned}$$

If $K \subset [\![G]\!]$ is an arbitrary subset, the sofic dimension of K is

$$s(K) := \sup_E \inf_F s_E(F)$$

where $E \subset F \subset K$ are finite subsets. The sofic dimension of G is $s(G) := s([\![G]\!])$. One defines similarly the lower sofic dimension $\underline{s}$ and the ω sofic dimension s^{ω} for a ultrafilter ω on $\mathbb{Z}_{\geq 1}$ by replacing $\limsup_{d\to\infty}$ by $\liminf_{d\to\infty}$ and $\lim_{d\to\omega}$ respectively.

Voiculescu's regularity condition reads as follows.

Definition 2.5 A pmp groupoid G is *s-regular* if $\underline{s}(G) = s(G)$.

Finally we recall the following definition.

Definition 2.6 A subset $K \subset [\![G]\!]$ is *transversally generating* if for any $t \in [\![G]\!]$ and $\varepsilon > 0$ there exist $n \in \mathbb{Z}_{\geq 1}$ and $s \in \mathbf{\Sigma}\, K_{\pm}^{n}$ such that $|t - s| \leq \varepsilon$.

This definition appears in [DKP1, Definition 2.4] where it is called "dynamically generating". The more classical notion of generating set for pmp equivalence relations (and groupoids) (as in [DKP1, Definition 2.2]) is that of Connes–Feldmann–Weiss. While being distinct notions, a groupoid is finitely generated in the Connes–Feldmann–Weiss sense if and only if it is transversally finitely generated (by an argument similar to that in [DKP1, Proposition 2.6]), so "finitely generated" is unambiguous for groupoids (and coincide with the usual notion in the group case).

The following result is proved in [DKP1, Theorem 4.1].

Theorem 2.7 (Invariant of s under orbit equivalence) *Let R be pmp equivalence relation and K, L be transversally generating sets. Then $s(K) = s(L)$, $\underline{s}(K) = \underline{s}(L)$, and $s^\omega(K) = s^\omega(L)$.*

The result in [DKP1] is stated for finitely generated equivalence relations, but the same proof works in the general case (as does the proof of [DKP1, Theorem 1.2]). The proof for groupoids is given in full generality in [DKP2, Theorem 2.11]. We will not use this more general result here but will rather deduce it from Theorem 2.7 as an illustration of the correspondence principle.

Remark 2.8 It is sometimes convenient to use the 2-norm on LG and its restriction to $[\![G]\!]$:

$$\|s-t\|_2^2 := \tau((s-t)(s-t)^{-1}) = \tau((s-t)^{-1}(s-t)).$$

Observe $\|s-t\|_2^2 \geq |s-t|$ (with an equality $\|s-t\|_2^2 = 2|s-t|$ on $[G] \subset [\![G]\!]$) as

$$\|s-t\|_2^2 = \tau(s^{-1}s) + \tau(t^{-1}t) - 2\tau(st^{-1}) \quad \text{and} \quad \tau(st^{-1}) \leq \tau(s^{-1}st^{-1}t).$$

3 Actions of Groupoids

Let X be a standard Borel space endowed with a Borel fibration $p\colon X \to G^0$, where G^0 is the base of G. If $(\mu^e)_{e\in G^0}$ is a Borel field of probability measures on X, we define a probability measure μ on X by $\mu := \int_{G^0} \mu^e d\mathfrak{h}(e)$, where $\mathfrak{h}$ is the invariant Haar measure on G^0. Recall that a pmp action of G on the fibered space (X, μ) is a measurable map

$$G *_{G^0} X \ni (g, x) \mapsto gx \in X$$

(where G fibers via the source map $\mathbf{s}\colon G \to X_G$) satisfying the usual axioms of an action, and such that $g_*\mu^{\mathbf{s}(g)} = \mu^{\mathbf{r}(g)}$ for ae $g \in G$. Groupoid actions are denoted $G \curvearrowright X$. The crossed product groupoid is the fiber bundle $G \times_{G^0} X$ endowed with groupoid law defined by $(s, x)(t, y) = (st, y)$ whenever $t(y) = x$.

Example 3.1 (Bernoulli shifts) Given a pmp groupoid G with invariant Haar measure $\mathfrak{h}$ and a probability space (X_0, μ_0), consider the probability space

$$(X_0^G, \mu) := \int_{G^0} (X_0^{G^e}, \mu_0^{\bigotimes G^e}) d\mathfrak{h}(e)$$

(where $X_0^{G^e} := \prod_{G^e} X_0$ is the infinite Cartesian product over $G^e := \mathbf{r}^{-1}(e)$) endowed with the fibration $X_0^G \to G^0$ and field of measures $(\mu_0^{\bigotimes G^e})_{e\in G^0}$. Every element $x \in X_0^G$ can be viewed as a sequence $x := (x_t)_{t\in G^e}$ of elements of X_0. The Bernoulli action $G \curvearrowright X_0^G$ is given by

$$s((x_t)_{t\in G^{\mathbf{r}(s)}}) := ((x_{s^{-1}t})_{t\in G^{\mathbf{r}(s)}}).$$

Remark 3.2 The notion of groupoid action has long been used in ergodic group theory (see for example [Ram]). They can equivalently be described as actions on bundles (cocycles) as above, or as groupoid extensions that are fiber bijective. In [Bow2] L. Bowen discusses Bernoulli shifts using the latter description.

We first prove a few lemmas that will be used in the proof of Theorem 1.1.

Lemma 3.3 *Let $G \curvearrowright X_0^G$ be a Bernoulli action and let $H \subset G$ be an ergodic subgroupoid with $G^0 = H^0$. Then the action $H \curvearrowright X_0^G$ is isomorphic to a Bernoulli shift over H.*

Proof By the von Neumann selection theorem, we can find a measurable section $s\colon RG \to G$ of $(\mathbf{s}, \mathbf{r})\colon G \rightrightarrows RG$ (*i.e.,* the pmp equivalence relation associated with G) such that that $s(RG^0) = G^0$ and $s(RH) \subset H$ and, since the set $G(e)/H(e)$ is countable for $e \in G^0$, measurable sections $(g_j\colon D_j \subset G^0 \to SG)_{j\in J}$ of $SG \to G^0$ such that for ae $e \in G^0$, $G(e) = \bigsqcup_{j\in J} H(e)g_j(e)$. By ergodicity we may assume $\mathfrak{h}(D_j) = 1$. Let $(\varphi_i)_{i\in I}$ be a sequence in $\mathrm{Aut}(G^0)$ such that $\{RH[\varphi_i e]\}_{i\in I}$ form a partition of $RG[e]$ for ae $e \in G^0$ (see [FSZ]). Then

$$G^e = \bigsqcup_{i\in I,\ j\in J} \ \bigsqcup_{(e,f)\in RH} H(e)g_j(e)s(e, \varphi_i^{-1} f),$$

since any $g \in G^e$ can be written uniquely in the form $h_0 g_j(e)s(e, \varphi_i^{-1} f)$ for i and f such that $(e, f) \in RH$ and $f = \varphi_i \mathbf{s}(g)$, so

$$gs(e, \varphi_i^{-1} f)^{-1} \in G(e) \quad \text{and} \quad h_0 = gs(e, \varphi_i^{-1} f)^{-1} g_j(e)^{-1} \in H(e).$$

Consider the measurable field of maps $\psi_e\colon X_0^{G^e} \to (X_0^{I\times J})^{H^e}$ defined by sending $x \in X_0^{G^e}$ to

$$\big((x_{h_0 g_j(e)s(e,\varphi_i^{-1} f)})_{(i,j)\in I\times J} \big)_{h_0\in H(e),(e,f)\in RH}.$$

These maps are measure preserving, and if we consider the Bernoulli action of H with base $X_0^{I\times J}$ then we see it is H-equivariant: for $h \in H^e$ say $h = h_1^{-1} s(d, e)^{-1}$, where $h_1 \in H(e)$

$$\begin{aligned}
\psi_d(h(x)) &= \psi(h(x_{h_0 g_j(e)s(e,\varphi_i^{-1} f)})_{(i,j)\in I\times J, h_0\in H(e),(e,f)\in RH})\\
&= \psi((x_{s(d,e)h_1 h_0 g_j(e)s(e,\varphi_i^{-1} f)})_{(i,j)\in I\times J, h_0\in H(e),(e,f)\in RH})\\
&= \big((x_{s(d,e)h_1 h_0 g_j(e)s(e,\varphi_i^{-1} f)})_{(i,j)\in I\times J} \big)_{h_0\in H(e),(e,f)\in RH}\\
&= h\big((x_{h_0 g_j(e)s(e,\varphi_i^{-1} f)})_{(i,j)\in I\times J} \big)_{h_0\in H(e),(e,f)\in RH}\\
&= h(\psi_e(x)).
\end{aligned}$$

∎

We say a groupoid action $G \curvearrowright X$ is *essentially free* if for ae $s \in G \setminus G^0$,

$$\mu^{\mathbf{s}(s)}(\mathrm{Fix}(s)) = 0,$$

where

$$\mathrm{Fix}(s) = \{x \in X^{\mathbf{s}(s)} \mid sx = x\}.$$

Lemma 3.4 *If the pmp groupoid pmp action $G \curvearrowright X$ is essentially free, then $G \ltimes X$ is isomorphic to a pmp equivalence relation.*

Proof Since $G \curvearrowright X$ is essentially free and G_e is countable, the set

$$X_0^e := \{x \in X^e \mid sx \neq x,\ \forall s \in G_e,\ s \neq e\}$$

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has measure 1 in X^e for every $e \in A \subset G^0$ a measurable subset with $\mathfrak{h}(A) = 1$. Let $(X_0^A, \mu) \to (A, \mathfrak{h})$ be the measure fibration corresponding to $(X_0^e)_{e\in A}$. Since $G \ltimes X$ is isomorphic to $G_{|A} \ltimes X_0^A$, we may assume that $sx \neq x$ for all $s \in G \setminus G^0$, $x \in X^{\mathbf{s}(s)}$. However, for $(s, x) \in G \ltimes X$,

$$\mathbf{r}(s, x) = \mathbf{s}(s, x) \Leftrightarrow x \in \mathrm{Fix}(s),$$

so $\mathbf{r}(s, x) \neq \mathbf{s}(s, x)$ for every $s \in G \setminus G^0$ and $x \in X$. This shows that $G \ltimes X$ is an equivalence relation. It is an easy exercise to check that it is pmp (more generally if G is pmp and $G \curvearrowright X$ is a pmp action, then $G \ltimes X$ is a pmp groupoid). ■

Lemma 3.5 *If G is transversally finitely generated, then so is $G \ltimes X$ for any ergodic pmp action $G \curvearrowright X$.*

Proof Let R be the orbit equivalence relation of $G \curvearrowright X$. Since R is ergodic, there exists an ergodic automorphism $\theta \in [\![R]\!]$, which is orbit equivalent to a Bernoulli shift $\mathbb{Z} \curvearrowright \{0, 1\}^{\mathbb{Z}}$ ([Dye]). For $i \in \{0, 1\}$, let B_i be the cylinder set $\{x \in \{0, 1\}^{\mathbb{Z}} \mid x_0 = i\}$. Let p_1, p_2 be the projection in $L^\infty(X)$ corresponding to B_0 and B_1 in $\mathbb{Z} \curvearrowright \{0, 1\}^{\mathbb{Z}}$. If $F \subset [\![G]\!]$ is a finite transversally generating set for G, then $F \cup \{\theta, p_1, p_2\}$ is a finite generating set for $G \ltimes X$. ■

Lemma 3.6 *The Bernoulli action $G \curvearrowright X_0^G$ is essentially free if G has infinite fibers (*i.e., *$|G^e| = \infty$ for ae $e \in G^0$) and the support of μ_0 contains at least two points. If in addition μ_0 is diffuse, then the action is essentially free.*

Proof Let $s \in G^e$, $s \neq e$, such that $|G^e| = \infty$. We show that $\mu^e(\mathrm{Fix}(s)) = 0$. If $\mathbf{r}(s) \neq \mathbf{s}(s)$, this is clear, so we assume $s \in G(e)$. Note that $(x_t)_{t\in G^e}$ is $\langle s\rangle$-invariant if and only if $x_{s^n t} = x_t$ for all $n \in \mathbb{Z}$, $t \in G^e$. Thus we can find an infinite family of pairwise disjoint pairs $\{s_i, t_i\}_{i\in I}$, $s_i \neq t_i \in G^e$, such that $x_{s_i} = x_{t_i}$ for every $x \in \mathrm{Fix}(s)$ and $i \in I$. Since $|I| = \infty$, the set of $x \in X_0^{G^e}$ such that $x_{s_i} = x_{t_i}$ is negligible, so $\mu^e(\mathrm{Fix}(s)) = 0$. If in addition μ_0 is diffuse, then the set of $x \in X_0^{G^e}$ such that $x_s = x_e$ is negligible so $\mu^e(\mathrm{Fix}(s)) = 0$ in this case. ■

Lemma 3.7 *If G is ergodic with infinite fibers then the Bernoulli action $G \curvearrowright X_0^G$ is ergodic for any base space (X_0, μ_0).*

Proof If $A \subset X_0^G$ is a nonzero G-invariant subset and $p\colon X_0^G \to G^0$, then $B := p(A)$ is G-invariant and nonzero therefore $\delta := \mu^e(A^e) \neq 0$ and is almost surely constant by the ergodicity of G. Let $\varepsilon > 0$ be arbitrary. Since $|G^e| = \infty$ almost surely, we can find by the von Neumann selection theorem a nonzero measure field $(F^e)_{e\in B}$ of finite subsets of $(G^e)_{e\in B}$ such that $\mu^e(A^e \cap A_{|F^e}) > \delta - \varepsilon$ and a section s of $\mathbf{s}$ such that $s_e F^e \cap F^{\mathbf{r}(s_e)} = \varnothing$ for ae $e \in B$. By invariance $\mu^{\mathbf{s}(e)}(A^{\mathbf{r}(s_e)} \cap A_{|s_e F^e}) > \delta - \varepsilon$, so

$$\mu^{\mathbf{s}(e)}(A_{|F^{\mathbf{r}(s_e)}} \cap A_{|s_e F^e}) > \delta - 2\varepsilon,$$
$$\mu^{\mathbf{s}(e)}(A_{|F^{\mathbf{r}(s_e)}} \cap A_{|s_e F^e}) = \mu^{\mathbf{s}(e)}(A_{|F^{\mathbf{r}(s_e)}})\mu^{\mathbf{s}(e)}(A_{|s_e F^e}),$$
$$\mu^{\mathbf{s}(e)}(A_{|F^{\mathbf{r}(s_e)}} \cap A_{|s_e F^e}) < \delta^2 + 6\varepsilon\delta + 2\varepsilon^2.$$

Letting $\varepsilon \to 0$, we get $\delta^2 \geq \delta$, so $\delta \geq 1$ and $\mu(A) = 1$. ■

Lemma 3.8 *Let $G \curvearrowright X$ be a groupoid action and suppose that $G = G_1 *_{G_3} G_2$. Then*

$$G \ltimes X \simeq (G_1 \ltimes X_{|1}) *_{(G_3 \ltimes X_{|3})} (G_2 \ltimes X_{|2}),$$

where $X_{|i} = p^{-1}(G_i^0)$.

Proof First note that G_i acts on X_i and that $G_i \ltimes X_{|i}$ is naturally a subgroupoid of $G \ltimes X$. Given an arbitrary groupoid H and groupoid morphisms $f_1\colon G_1 \ltimes X_{|1} \to H$, $f_2\colon G_2 \ltimes X_{|2} \to H$ with $f_1\big|_{G_3 \ltimes X_3} = f_2\big|_{G_3 \ltimes X_{|3}}$, we want to show there is a unique morphism $k\colon G \ltimes X \to H$ such that the following diagram commutes (where the unlabeled edges are the inclusion map).

$$\begin{tikzcd}
 & G_3 \ltimes X_{|3} \arrow[dl] \arrow[dr] & \\
G_1 \ltimes X_{|1} \arrow[r] \arrow[dr, "f_1"'] & G \ltimes X \arrow[d, "k"] & G_2 \ltimes X_{|2} \arrow[l] \arrow[dl, "f_2"] \\
 & H &
\end{tikzcd}$$

Since $k|_{G_1 \ltimes X_{|1}} = f_1|_{G_1 \ltimes X_{|1}}$, $k|_{G_2 \ltimes X_{|2}} = f_2|_{G_2 \ltimes X_{|2}}$, and the values of k are determined on $G_1 \ltimes X_{|1}$ and $G_2 \ltimes X_{|2}$, which generate $G \ltimes X$, this gives uniqueness. To show it is well defined note that if $g_i \in G_1 \ltimes X^{G_1} \cup G_2 \ltimes X^{G_2}$ and $g_1 \cdots g_n = \mathrm{Id}_{\mathbf{s}(g_1)}$ and if g_i' are the corresponding elements of G_1 and G_2 then $g_1' \cdots g_n' = e$, so using the fact that G_1 and G_2 are in free product over G_3 this shows that $k(g_1) \cdots k(g_n) = \mathrm{Id}_{\mathbf{s}(f_1(g_1))}$ or $\mathrm{Id}_{\mathbf{s}(f_2(g_1))}$ as appropriate. ■

4 Overlapping Generators

Let $F \subset [\![G]\!]$ be a finite subset, π be a partition of F, and $\sigma\colon F \to [\![d]\!]$ be a map. Denote $F_{|\pi}$ the set of $e \in G^0$ such that

$$e \leq \prod_{s \in F} ss^{-1}$$

and

$$\forall s, t \in F,\ es = et \iff \pi(s) = \pi(t).$$

Here and below we will view a partition π of F as a map from F to $\{1, \dots, \alpha\} =:$ $\operatorname{ran} \pi$ for some $\alpha \in \mathbb{Z}_{\geq 0}$.

Similarly, let $F^{\sigma}_{|\pi}$ be the set of $e \in \{1, \dots d\}$ (identified with the base space of the transitive relation on $\{1, \dots, d\}$) such that

$$e \leq \prod_{s \in F} \sigma(s)\sigma(s)^{-1}$$

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and

$$\forall s,t \in F,\ e\sigma(s) = e\sigma(t) \iff \pi(s) = \pi(t).$$

The following lemma will be useful in the proof of Theorem 8.2. We denote a projection onto a set A by p_A.

Lemma 4.1 *Given $F \subset [\![G]\!]$, $n \geq 1$, let*

$$F_n := F \cup \{p_{F_{0|\pi}} \mid F_0 \subset F_{\pm}^n, \pi \text{ partition of } F_0\}$$

If $\sigma \in \mathrm{SA}(F_n, 4n|F_{\pm}^n| + 1, \delta, d)$, $F_0 \subset F_{\pm}^n$, and π is a partition of F_0, then

$$\Big| \mathfrak{h}(F_{0|\pi}) - \frac{|F_{0|\pi}^{\sigma}|}{d} \Big| < c_1(F,n)\delta,$$

where $c_1(F,n) = 176(3^{2n})|F_{\pm}|^{2n}$.

Proof Fix some $F_0 \subset F_{\pm}^n$ and let $\pi_1, \pi_2, \dots, \pi_m$ be the family of partitions of F_0 (so $m = \mathrm{Be}(|F_0|)$ is the Bell number of $|F_0|$). For convenience let $p_j := p_{F_{0|\pi_j}}$ and $p_j^{\sigma} := p_{F^{\sigma}_{0|\pi_j}}$.

If $\pi_j(s) = \pi_j(t)$, $p_j = st^{-1}p_j$, so

$$\begin{aligned}
&\big|\sigma(s)\sigma(t)^{-1}\sigma(p_j) - \sigma(p_j)\big| \\
&\quad = \big|\sigma(s)\sigma(t)^{-1}\sigma(p_j) - \sigma(st^{-1}p_j)\big| \\
&\quad \leq \big|\sigma(s)\sigma(t)^{-1}\sigma(p_j) - \sigma(s)\sigma(t^{-1})\sigma(p_j)\big| + \big|\sigma(s)\sigma(t^{-1})\sigma(p_j) - \sigma(st^{-1}p_j)\big| \\
&\quad \leq 4\delta + 3\delta = 7\delta
\end{aligned}$$

by [DKP1, Lemma 3.1(8)].

Similarly, if $\pi_j(s) \neq \pi_j(t)$, then

$$\big|\sigma(s)\sigma(t)^{-1}\sigma(p_j) - \sigma(st^{-1}p_j)\big| \leq 7\delta.$$

As $\tau(st^{-1}p_j) = 0$, we have

$$\begin{aligned}
\big|\mathrm{tr}(\sigma(s)\sigma(t)^{-1}\sigma(p_j))\big| &= \big|\mathrm{tr}(\sigma(s)\sigma(t)^{-1}\sigma(p_j)) - \tau(st^{-1}p_j)\big| \\
&\leq \big|\mathrm{tr}(\sigma(s)\sigma(t)^{-1}\sigma(p_j)) - \mathrm{tr}\circ\sigma(st^{-1}p_j)\big| \\
&\quad + \big|\mathrm{tr}\circ\sigma(st^{-1}p_j) - \tau(st^{-1}p_j)\big| \\
&\leq 7\delta + \delta = 8\delta
\end{aligned}$$

by [DKP1, Lemma 3.1 (4)].

Moreover,

$$\big|\sigma(p_j) - \sigma(p_j)\sigma(p_j)\big| = \big|\sigma(p_j^2) - \sigma(p_j)\sigma(p_j)\big| < \delta,$$

and since $\sum_{j=1}^m p_j = \prod_{s\in F_0} ss^{-1}$ using [DKP1, Lemma 3.5]

$$\begin{aligned}
\Big|\sum_{j=1}^m \sigma(p_j) - \prod_{s\in F_0}\sigma(s)\sigma(s)^{-1}\Big| &< \Big|\sigma\Big(\prod_{s\in F_0} ss^{-1}\Big) - \prod_{s\in F_0}\sigma(s)\sigma(s)^{-1}\Big| + 150(2|F|+1)^{2n}\delta \\
&\leq |F_0|4\delta + |F_0|2\delta + \delta + 150(2|F|+1)^{2n}\delta \\
&\leq 157(3^{2n})|F_{\pm}|^{2n}\delta.
\end{aligned}$$

If V denotes the set of integers $1 \le k \le d$ such that for all $1 \le j \le m$ and $s, t \in F_0$:

$$\sigma(s)\sigma(t)^{-1}\sigma(p_j)k = \sigma(p_j)k$$
$$\sigma(s)\sigma(t)^{-1}\sigma(p_j)k \ne k$$
$$\sigma(p_j)k = \sigma(p_j)\sigma(p_j)k$$
$$\Big(\sum_{j=1}^{m} \sigma(p_j)\Big)\, k = \Big(\prod_{s\in F_0} \sigma(s)\sigma(s)^{-1}\Big)\, k,$$

then

$$\begin{aligned}|V| &\ge \big(1 - 7\delta|F_\pm^n| - 7\delta|F_\pm^n| - \delta|F_\pm^n| - \delta|F_\pm^n| - 157(3^{2n})|F_\pm|^{2n}\delta\big)\, d \\ &\ge \big(1 - 175(3^{2n})|F_\pm|^{2n}\delta\big)\, d.\end{aligned}$$

So, for all π_j we have that $\sigma(p_j)_{|V}$ is a projection and

$$\pi_j(s) = \pi_j(t) \Rightarrow \sigma(s)\sigma(t)^{-1}\sigma(p_j)_{|V} = \sigma(p_j)_{|V} \text{ and } \operatorname{tr}\big(\sigma(s)\sigma(t)^{-1}\sigma(p_j)_{|V}\big) = 0.$$

Hence $\operatorname{ran}(\sigma(p_j)p_V) \subset \operatorname{ran}(p_j^\sigma p_V)$, but since we also have

$$\Big(\sum_{j=1}^{m} \sigma(p_j)\Big)\, p_V = \Big(\prod_{s\in F_0} \sigma(s)\sigma(s)^{-1}\Big)\, p_V = \Big(\sum_{j=1}^{m} p_j^\sigma\Big)\, p_V,$$

we conclude that $\sigma(p_j)p_V = p_j^\sigma p_V$, which implies

$$\big|\operatorname{tr}\circ\sigma(p_j) - |F^\sigma_{0|\pi_j}|/d\big| < 175(3^{2n})|F_\pm|^{2n}\delta.$$

Therefore,

$$\begin{aligned}\big|\mathfrak{h}(F_0^{\pi_j}) - |F^\sigma_{0|\pi_j}|/d\big| &= \big|\operatorname{tr}(p_j) - |F^\sigma_{0|\pi_j}|/d\big| \\ &\le |\operatorname{tr}(p_j) - \operatorname{tr}\circ\sigma(p_j)| + \big|\operatorname{tr}\circ\sigma(p_j) - |F^\sigma_{0|\pi_j}|/d\big| \\ &\le \delta + 175(3^{2n})|F_\pm|^{2n}\delta \le 176(3^{2n})|F_\pm|^{2n}\delta.\end{aligned}$$ ■

5 Bernoulli Shifts and Random Partitions

For $1 \le i \le q$ let

$$B_i := \{x \in \{1, \dots, q\}^G \mid x_e = i\},$$

where $X = \{1, \dots, q\}^G$ is endowed with the Bernoulli action $G \curvearrowright X$, and for $d \in \mathbb{Z}_{\ge 1}$ let $A_1, \dots, A_q$ be a partition of $\{1, \dots, d\}$. Given a set $F \subset [\![G]\!]$ and function $\psi\colon F \to \{1, \dots, q\}$, we define

$$B_\psi := \bigcap_{s\in F} sB_{\psi(s)} \quad \text{and} \quad A_\psi := \bigcap_{s\in F_\psi} \sigma(s)A_{\psi(s)}$$

for any $\sigma\colon F \to [\![d]\!]$. (Here and below the dependency in σ is omitted from the notation A_ψ as the chosen map will always be clear from context.)

The next lemma adapts the proof of Theorem 8.1 from L. Bowen's paper [Bow].

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Lemma 5.1 *If d is large enough (depending on F, n, and δ) then there is a partition $\{A_1, A_2, \dots, A_q\}$ of $\{1, \dots, d\}$ such that if $\sigma \in \mathrm{SA}(F_n, 4n|F_{\pm}^n| + 1, \delta, d)$ then for every subset $F_\psi \subset F_{\pm}^n$ and every function $\psi\colon F_\psi \to \{1, \dots, q\}$*

$$\left| \mu(B_\psi) - \frac{|A_\psi|}{d} \right| < c_2(F, n)\delta$$

where $c_2(F, n) = 2c_1(F, n)\,\mathrm{Be}(|F_{\pm}|^n)$ and Be is the Bell number.

Proof Fix $F_\psi \subset F_{\pm}^n$ and $\psi\colon F_\psi \to \{1, \dots, q\}$. Create a random partition

$$\{A_1, A_2, \dots, A_q\}$$

of $\{1, \dots, d\}$ using the following scheme: for each $k \in \{1, \dots, d\}$ place k in A_i with probability $\mu_0(i)$.

We will find the probability that $|\mu(B_\psi) - |A_\psi|/d| < c_2(F, n)\delta$.

Let $\pi_1, \pi_2, \dots, \pi_m$ be the family of partitions of F_ψ. Let $s_j\colon \operatorname{ran}\pi_j \to F_\psi$ be an arbitrary section of π_j. If

$$\chi_j := \begin{cases} 0 & \text{if } \exists s, t \in F_\psi \text{ such that } \pi_j(s) = \pi_j(t) \text{ and } \psi(s) \neq \psi(t), \\ 1 & \text{otherwise,} \end{cases}$$

then

$$\begin{aligned} \mu(B_\psi) &= \mu\Big(\bigcap_{s \in F_\psi} sB_{\psi(s)} \Big) \\ &= \sum_{j=1}^{m} \chi_j \mathfrak{h}(F_{\psi|\pi_j}) \prod_{r \in \operatorname{ran}\pi_j} \mu_0\big(\psi(s_j(r))\big) \end{aligned}$$

Since for $\sigma \in \mathrm{SA}(F_n, 4n|F_{\pm}^n| + 1, \delta, d)$, we have

$$\frac{|A_\psi|}{d} = \sum_{j=1}^{m} \frac{|A_\psi \cap F^\sigma_{\psi|\pi_j}|}{d}$$

it will suffice to show by the triangle inequality that for all j

$$\left| \chi_j \mathfrak{h}(F_{\psi|\pi_j}) \prod_{k \in \operatorname{ran}\pi_j} \mu_0\big(\psi(s_j(k))\big) - \frac{|A_\psi \cap F^\sigma_{\psi|\pi_j}|}{d} \right| < 2c_1(F, n)\delta.$$

First suppose $\chi_j = 0$, so for some $s, t \in F_\psi$, $\pi_j(s) = \pi_j(t)$ and $\psi(s) \neq \psi(t)$. Then $A_\psi \cap F^\sigma_{\psi|\pi_j} = \varnothing$, for if $k \in A_\psi \cap F^\sigma_{\psi|\pi_j}$ then $k \in \sigma(s)A_{\psi(s)} \cap \sigma(t)A_{\psi(t)} \cap F^\sigma_{\psi|\pi_j}$ which implies $\sigma(s)^{-1}k = \sigma(t)^{-1}k \in A_{\psi(s)} \cap A_{\psi(t)}$, a contradiction.

Next suppose $\chi_j = 1$. For $1 \le k \le d$ let

$$Z_k = \begin{cases} 1 & \text{if } k \in F^\sigma_{\psi|\pi_j} \cap A_\psi \\ 0 & \text{otherwise} \end{cases}$$

We wish to compute $\mathbb{E}(Z_k)$.

For $k \notin F^\sigma_{\psi|\pi_j}$ we have $\mathbb{E}(Z_k) = 0$. Otherwise, $k \in F^\sigma_{\psi|\pi_j} \cap A_\psi$ if and only if $\sigma^{-1}_{s_j(r)}k \in A_{\psi(s_j(r))}$ for all $r \in \operatorname{ran}\pi_j$, and since

$$\sigma^{-1}_{s_j(r)}k \neq \sigma^{-1}_{s_j(r')}k,\ \forall r \neq r' \in \operatorname{ran}\pi_j,$$

we obtain

$$\mathbb{E}(Z_k) = \prod_{r\in \operatorname{ran} \pi_j} \mu_0\big(\psi(s_j(r))\big)$$

Thus if we let $Z = \sum_{k=1}^{d} Z_k = |F^{\sigma}_{\psi|\pi_j} \cap A_\psi|$, then

$$\mathbb{E}(Z) = |F^{\sigma}_{\psi|\pi_j}| \prod_{r\in \operatorname{ran} \pi_j} \mu_0\big(\psi(s_j(r))\big).$$

Now let us bound $\operatorname{var}(Z)$:

$$\operatorname{var}(Z) = \mathbb{E}(Z^2) - \mathbb{E}(Z)^2 = \sum_{k,l\in\{1\dots,d\}} \mathbb{E}(Z_kZ_l) - \mathbb{E}(Z)^2$$

For $k, l \notin F^{\sigma}_{\psi|\pi_j}$ we have $\mathbb{E}(Z_kZ_l) = 0 = \mathbb{E}(Z_k)\mathbb{E}(Z_l)$. On the other hand if $k, l \in F^{\sigma}_{\psi|\pi_j}$ then Z_k and Z_l are not independent if $\sigma^{-1}_{s_j(r)}k = \sigma^{-1}_{s_j(r')}l$ for some r, r'. Thus are at most $|F^n_{\pm}|^2|F^{\sigma}_{\psi|\pi_j}|$ non independent pairs (k, l). Now clearly for these pairs, $\mathbb{E}(Z_kZ_l) \leq \mathbb{E}(Z_k)\mathbb{E}(Z_l) + 1$. So returning to our equation above,

$$\begin{aligned}\sum_{1\leq k,l\leq d} \mathbb{E}(Z_kZ_l) - \mathbb{E}(Z)^2 \leq \sum_{1\leq k,l\leq d} &\mathbb{E}(Z_k)\mathbb{E}(Z_l) + |F^n_{\pm}|^2|F^{\sigma}_{\psi|\pi_j}| - \mathbb{E}(Z^2)\\ &= \mathbb{E}(Z^2) + |F^n_{\pm}|^2|F^{\sigma}_{\psi|\pi_j}| - \mathbb{E}(Z^2) = |F^n_{\pm}|^2|F^{\sigma}_{\psi|\pi_j}|\end{aligned}$$

Now we can apply Chebyshev's inequality to $\frac{Z}{d}$ for $a > 0$

$$\Pr\Big(\Big|\frac{Z}{d} - \frac{\mathbb{E}(Z)}{d}\Big| \geq a\Big) \leq \frac{\operatorname{var}(\frac{Z}{d})}{a^2} \leq \frac{|F^n_{\pm}|^2|F^{\sigma}_{\psi|\pi_j}|}{a^2d^2} \leq \frac{|F^n_{\pm}|^2}{a^2d}.$$

Since $Z = |A_\psi \cap F^{\sigma}_{\psi|\pi_j}|$ and $\mathbb{E}(Z) = |F^{\sigma}_{\psi|\pi_j}| \prod_{r\in \operatorname{ran}\pi_j} \mu_0(\psi(s_j(r)))$, and, by Lemma 4.1,

$$\Big|\mathfrak{h}(F_{\psi|\pi_j}) - \frac{|F^{\sigma}_{\psi|\pi_j}|}{d}\Big| < c_1(F, n)\delta,$$

we have

$$\Pr\Big(\Big|\mathfrak{h}(F_{\psi|\pi_j}) \prod_{k\in \operatorname{ran}\pi_j} \mu_0\big(\psi(s_j(k))\big) - |F^{\sigma}_{\psi|\pi_j} \cap A_\psi|/d\Big| \geq a + c_1(F, n)\delta\Big) \leq \frac{|F^n_{\pm}|^2}{a^2d}.$$

Let $a = c_1(F, n)\delta$, so then

$$\Pr\Big(\Big|\mathfrak{h}(F_{\psi|\pi_j}) \prod_{k\in \operatorname{ran}\pi_j} \mu_0\big(\psi(s_j(k))\big) - |F^{\sigma}_{\psi|\pi_j} \cap A_\psi|/d\Big| \geq 2c_1(F, n)\delta\Big) \leq \frac{|F^n_{\pm}|^2}{(c_1(F, n)\delta)^2d}.$$

So for d large enough there is some partition $(A_1, \dots, A_q)$ such that

$$\Big|\mathfrak{h}(F_{\psi|\pi_j}) \prod_{k\in \operatorname{ran}\pi_j} \mu_0\big(\psi(s_j(k))\big) - |F^{\sigma}_{\psi|\pi_j} \cap A_\psi|/d\Big| < 2c_1(F, n)\delta.$$

Thus, for d large enough there will be a partition such that this true for all j so that

$$\Big|\frac{|A_\psi|}{d} - \mu(B_\psi)\Big| < c_2(F, n)\delta.$$

Indeed for large enough d nearly all partitions will satisfy this and hence we will have nonzero probability that for all $F_\psi \subset F^n_{\pm}$ and $\psi\colon F_\psi \to \{1, \dots, q\}$ the inequality $|\mu(B_\psi) - |A_\psi|/d| < c_2(F, n)\delta$ holds. ∎

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6 A Lemma on Approximate Equivariance

Given a groupoid G acting on a set X for any finite set of projections $P \subset L^\infty(X,\mu)$ and finite set $F \subset [\![G]\!]$, let P_F denote the set of all projections of the form $\prod_{s\in F'} sp_s$, where $p_s \in P$ and $F' \subset F$.

Let $P = \{p_{B_i}\}$, where

$$B_i := \{x \in \{1,\dots,q\}^G \mid x_e = i\}$$

and $X = \{1,\dots,q\}^G$ is endowed with the Bernoulli action $G \curvearrowright X$. Fix $F \subset [\![G]\!]$ and a basis $\{p_{B_{\psi_1}}, p_{B_{\psi_2}}, \dots, p_{B_{\psi_\ell}}\}$ for $\operatorname{span}(P_{F^n_\pm})$ in $L^\infty(X,\mu)$ associated with $\psi_i\colon F_{\psi_i} \to \{1,\dots,q\}$, $1 \le i \le \ell$.

We set

$$\kappa = \max_{\substack{\psi,\psi_1,\dots,\psi_\ell,a_1,\dots a_\ell\\ p_{B_\psi}=\sum_{i=1}^{\ell} a_i p_{B_{\psi_i}}}} |a_i| \ge 1.$$

Both κ and ℓ depend only on F and n.

Lemma 6.1 *Let $\sigma \in \mathrm{SA}(F_n, 4n|F^n_\pm| + 1, \delta, d)$ and take a partition $\{A_1,\dots,A_q\}$ satisfying the conclusion of Lemma 5.1.*

If for some $\psi\colon F_\psi \to \{1,\dots,q\}$, $p_{B_\psi} = \sum_{i=1}^{\ell} a_i p_{B_{\psi_i}}$, then

$$\Big\| p_{A_\psi} - \sum_{i=1}^{\ell} a_i p_{A_{\psi_i}} \Big\|_2 < c_3(F,n)\sqrt{\delta},$$

where $c_3(F,n) = ((1 + (3+\kappa\ell)\operatorname{Be}(\ell)N_\ell))c_2(F,n)$ and $N_\ell = \ell 2^\ell$.

Proof For $I \subset \{1,\dots,\ell\}$ define

$$D_I := \{x \in X \mid x \in B_{\psi_i} \iff i \in I\}$$

and similarly

$$D_I^\sigma := \{1 \le k \le d \mid k \in A_{\psi_i} \iff i \in I\}.$$

We want to bound $|\mu(D_I) - |D_I^\sigma|/d|$.

Note that for any $I \subset \{1,\dots.\ell\}$,

$$\mu\{x \in X \mid \forall i \in I,\ x \in B_{\psi_i}\} = \mu\Big(\bigcap_{i\in I} B_{\psi_i}\Big).$$

Now either for some $s \in F_{\psi_{i_1}} \cap F_{\psi_{i_2}}$ we have $\psi_{i_1}(s) \ne \psi_{i_2}(s)$ and so $\bigcap_{i\in I} B_{\psi_i} = \varnothing$, or otherwise $\bigcap_{i\in I} B_{\psi_i} = B_{\psi'}$ for some $\psi'\colon F_{\psi'} \to \{1,\dots,q\}$. In the first case, $\bigcap_{i\in I} A_{\psi_i} = \varnothing$ as well, and in the second case, by Lemma 5.1,

$$\big|\mu(B_{\psi'}) - |A_{\psi'}|/d\big| < c_2(F,n)\delta$$

and

$$|A_{\psi'}|/d = \Big|\bigcap_{i\in I} A_{\psi_i}\Big|/d = \big|\{1 \le k \le d \mid \forall i \in I,\ k \in A_{\psi_i}\}\big|/d.$$

So in general we conclude that $\mu\{x \in X \mid \forall i \in I,\ x \in B_{\psi_i}\}$ is within $c_1(F,n)\delta$ of $|\{1 \le k \le d \mid \forall i \in I,\ k \in A_{\psi_i}\}|/d$. By inclusion-exclusion,

$$\mu(D_I) = \sum_{k=|I|}^{\ell} \sum_{\substack{I' \supset I \\ |I'|=k}} (-1)^{k-|I|} \mu(\{x \in X \mid \forall i \in I',\ x \in B_{\psi_i}\}),$$

and similarly

$$\frac{|D_I^\sigma|}{d} = \sum_{k=|I|}^{\ell} \sum_{\substack{I' \supset I \\ |I'|=k}} (-1)^{k-|I|} \frac{1}{d} \big| \{1 \le k \le d \mid \forall i \in I',\ k \in A_{\psi_i}\} \big|,$$

so by applying the triangle inequality at most $N_\ell = \ell 2^\ell$ times, we have

$$\big| \mu(D_I) - |D_I^\sigma|/d \big| \le c_2(F,n) N_\ell \delta.$$

Now,

$$\mu(B_\psi) = \sum_{\substack{I \subset \{1,\dots \ell\} \\ \sum_{i \in I} a_i = 1}} \mu(D_I).$$

So if $\sum_{i \in I} a_i = 1$, then $\mu(\bigcap_{i \in I} B_{\psi_i}) = \mu(\bigcap_{i \in I} B_{\psi_i} \cap B_\psi)$ so by a similar argument $\mu\{x \in B_\psi \mid \forall i \in I,\ x \in B_{\psi_i}\}$ is within $c_2(F,n)\delta$ of $|\{k \in A_\psi \mid \forall i \in I,\ k \in A_{\psi_i}\}|/d$. So if we let

$$\widetilde{D}_I^\sigma := \{k \in A_\psi \mid x \in A_{\psi_i} \Leftrightarrow i \in I\},$$

then $|\mu(D_I) - |\widetilde{D}_I^\sigma|/d| \le c_2(F,n) N_\ell \delta$, so

$$\Big| \frac{|D_I^\sigma|}{d} - \frac{|\widetilde{D}_I^\sigma|}{d} \Big| \le 2c_2(F,n) N_\ell \delta.$$

Let $a(I) := \sum_{i \in I} a_i$. Then

$$\begin{aligned}
&\Big\| p_{A_\psi} - \sum_{i=1}^{\ell} a_i p_{A_{\psi_i}} \Big\|_2 \\
&\quad = \Big\| p_{A_\psi} - \sum_{I \subset \{1,\dots \ell\}} \big(\sum_{i \in I} a_i\big) p_{D_I^\sigma} \Big\|_2 \\
&\quad \le \Big\| p_{A_\psi} - \sum_{a(I)=1} p_{D_I^\sigma} \Big\|_2 + \Big\| \sum_{a(I) \notin \{0,1\}} \big(\sum_{i \in I} a_i\big) p_{D_I^\sigma} \Big\|_2 \\
&\quad \le \Big\| p_{A_\psi} - \sum_{a(I)=1} p_{\widetilde{D}_I^\sigma} \Big\|_2 + \Big\| \sum_{a(I)=1} p_{\widetilde{D}_I^\sigma} - \sum_{a(I)=1} p_{D_I^\sigma} \Big\|_2 + \kappa\ell \sum_{a(I) \notin \{0,1\}} |D_I^\sigma|/d \\
&\quad \le \sqrt{\Big| \frac{|A_\psi|}{d} - \sum_{a(I)=1} \frac{|\widetilde{D}_I^\sigma|}{d} \Big|} + \sum_{a(I)=1} \sqrt{\Big| \frac{|\widetilde{D}_I^\sigma|}{d} - \frac{|D_I^\sigma|}{d} \Big|} + \kappa\ell\, \mathrm{Be}(\ell) c_2(F,n) \delta N_\ell \\
&\quad \le \sqrt{\Big| \frac{|A_\psi|}{d} - \mu(B_\psi) \Big| + \Big| \sum_{a(I)=1} \mu(D_I) - \sum_{a(I)=1} \frac{|\widetilde{D}_I^\sigma|}{d} \Big|} \\
&\qquad + \sqrt{\mathrm{Be}(\ell) 2c_2(F,n) \delta N_\ell} + \kappa\ell\, \mathrm{Be}(\ell) c_2(F,n) \delta N_\ell
\end{aligned}$$

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$$
\begin{aligned}
&\leq \sqrt{c_2(F,n)\delta + \sum_{a(I)=1} \Big| \mu(D_I) - \frac{|\widetilde{D}_I^{\sigma}|}{d} \Big|} + \sqrt{\mathrm{Be}(\ell) 2 c_2(F,n)\delta N_\ell} \\
&\quad + \kappa\ell\, \mathrm{Be}(\ell) c_2(F,n)\delta N_\ell \\
&\leq c_2(F,n)\sqrt{\delta} + (3+\kappa\ell) Be(\ell) N_\ell c_2(F,n)\sqrt{\delta} \\
&= c_3(F,n)\sqrt{\delta}.
\end{aligned}
$$

■

7 Sofic Dimension and Groupoid Actions

We now briefly recall the group action formulation of $s(G \ltimes X)$ given in [DKP2, Section 5], rephrasing it here in the framework of groupoid actions.

Let $G \curvearrowright X$ be a pmp action of a pmp groupoid. Let $1 \in F \subset [\![G]\!]$ and let P be a partition of X. We write $\mathrm{HA}(F,P,n,\delta,d)$ for the set of all pairs (σ,φ), where $\sigma \in \mathrm{SA}(F,n,\delta,d)$ and φ is a map $\varphi\colon \mathbf{\Sigma}\, P_{F^n_\pm} \to [\![d]\!]$ defined on $\mathbf{\Sigma}\, P_{F^n_\pm} \subset [\![G]\!]$ and satisfying

(a) $|\operatorname{tr} \circ \varphi(p) - \mu(p)| < \delta$ for all $p \in P_{F^n_\pm}$

(b) $|\varphi \circ s(p) - \sigma(s) \circ \varphi(p)| < \delta$ for all $p \in P$ and $s \in F^n_\pm$

(c) $|\varphi(p_1 p_2) - \varphi(p_1)\varphi(p_2)| < \delta$ for all $p_1, p_2, p_1 p_2 \in \mathbf{\Sigma}\, P_{F^n_\pm}$.

(d) $|\varphi(p_X) - p_d| < \delta$, where $p_d := p_{\{1,\dots,d\}}$.

(Note that since P partitions X and F contains the identity, we have $p_X \in \operatorname{span} P_{F^n_\pm}$.)

We observe that if φ satisfies these conditions, then it is automatically approximately linear.

Lemma 7.1 *If $(\sigma,\varphi) \in \mathrm{HA}(F,4n,\delta,d)$, then*

$$|\varphi(p_1+p_2) - (\varphi(p_1)+\varphi(p_2))| < 146\delta$$

for all $p_1, p_2, p_1+p_2 \in \mathbf{\Sigma}\, P_{F^n_\pm}$ with $p_1 p_2 = 0$ and with $\varphi(p_1)+\varphi(p_2)$ defined as in [DKP1, Def. 3.3].

Proof Using $(p_1+p_2)p_i = p_i$, we have

$$|\varphi(p_i) - \varphi(p_1+p_2)\varphi(p_i)| < \delta$$

for $i = 1,2$. Let $\pi_i(\varphi(p_1),\varphi(p_2))$ be defined as in [DKP1, Def. 3.3], so

$$\varphi(p_1)+\varphi(p_2) := \varphi(p_1)\pi_1(\varphi(p_1),\varphi(p_2)) + \varphi(p_2)\pi_2(\varphi(p_1),\varphi(p_2)).$$

By [DKP1, Lemma 3.4], using the approximate homomorphism property of φ, we obtain

$$|\pi_i(p_1,p_2) - \varphi(p_i)\varphi(p_i)^{-1}| < 40\delta$$

for $i = 1,2$. Since $|\varphi(p_i) - \varphi(p_i)\varphi(p_i)^{-1}| < 8\delta$ we obtain

$$|\varphi(p_1+p_2)\pi_i(p_1,p_2) - \varphi(p_i)| < 48\delta$$

and so

$$|\varphi(p_1+p_2)\pi(p_1,p_2) - (\varphi(p_1)+\varphi(p_2))| < 48\delta,$$

with $\pi(p_1,p_2) := \pi_1(\varphi(p_1),\varphi(p_2)) + \pi_2(\varphi(p_1),\varphi(p_2))$.

Then

$$\begin{aligned}\operatorname{tr}(\varphi(p_1+p_2)\pi(p_1,p_2)) &= \operatorname{tr}(\varphi(p_1+p_2)\pi_1(p_1,p_2))) + \operatorname{tr}(\varphi(p_1+p_2)\pi_2(p_1,p_2)) \\ &> \operatorname{tr}(\varphi(p_1)) + \operatorname{tr}(\varphi(p_2)) - 96\delta \\ &> \tau(p_1) + \tau(p_2) - 98\delta \\ &= \tau(p_1+p_2) - 98\delta.\end{aligned}$$

Therefore,

$$|\varphi(p_1+p_2) - (\varphi(p_1)+\varphi(p_2))| < 146\delta.$$

(Similarly one can also show that $\varphi(p)$ is approximately a projection for every $p \in \boldsymbol{\Sigma}\, P_{F^n_\pm}$.) ∎

Definition 7.2 Given E, Q, F, P, n, and δ, define successively

$$\begin{aligned} s_{E,Q}(F,P,n,\delta) &:= \limsup_{d\to\infty} \frac{1}{d\log(d)} \log(|\operatorname{HA}(F,P,n,\delta,d)|_{E,Q}), \\ s_{E,Q}(F,P,n) &:= \inf_{\delta>0} s_{E,Q}(F,P,n,\delta), \\ s_{E,Q}(F,P) &:= \inf_{n\in\mathbb{N}} s_{E,Q}(F,P,n). \end{aligned}$$

If $K \subset [\![G]\!]$ is a transversally generating set and R is a dynamically generating family of projections of $L^\infty(X,\mu)$ define

$$s(K,R) := \sup_E \sup_Q \inf_F \inf_P s_{E,Q}(F,P),$$

where E and F range over finite subsets of K and P, Q range over finite subpartitions of R. Set $s(G,X) = s([\![G]\!], L^\infty(X,\{0,1\}))$.

Since $[\![G]\!] \cup L^\infty(X,\{0,1\})$ is a transversally generating set of the crossed product groupoid, we obtain by Theorem 2.7 (if the action $G \curvearrowright X$ is essentially free) and [DKP2, Theorem 2.11] in general (compare [DKP2, Proposition 5.2]).

Proposition 7.3 $s(G \ltimes X) = s(G,X)$.

Moreover, if F is a finite transversally generating subset of $[\![G]\!]$ and P is a finite and dynamically generating partition of unity, then $s(G \ltimes X) = s_{F,P}(F,P)$.

Proposition 7.4 $s(G \ltimes X) \le s(G)$.

Proof For every $(\sigma,\varphi) \in \operatorname{HA}(F,P,n,\delta,d)$ there are at most $|Q|^d$ restrictions $\varphi|_Q$, and hence

$$|\operatorname{HA}(F,P,n,\delta,d)|_{E,Q} \le |Q|^d |\operatorname{SA}(F,n,\delta,d)|_E$$

so we have $s_{E,Q}(F,P) \le s_E(F)$, and the result follows directly by the proposition above. ∎

Remark 7.5 The set HA differs from the set HA introduced in [DKP2] in that we do not assume φ to be a strict homomorphism. Furthermore, the maps in [DKP2] are defined on $L^\infty(X)$ with values in M_d using the 2-norm. It is often convenient

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to adopt the latter point of view for computational purposes, and we will do so below. The definition of HA given above has the advantage of being purely finitary, in the spirit of the sofic property. We have maintained the notation HA in view of (c) and (d).

8 The Computation of $s(G \ltimes (X_0, \mu_0)^G)$

Lemma 8.1 *If $G \curvearrowright \{1,\dots,q\}^G$, then $s(G \ltimes (\{1,\dots,q\},\mu_0)^G) = s(G)$ for any probability measure μ_0 on $\{1,\dots,q\}$ and pmp groupoid G. The same holds true for $\underline{s}$ and s^ω.*

Proof Let $X = \{1,\dots,q\}^G$. By Proposition 7.4 we have to prove $s(G \ltimes X) \geq s(G)$. Let $P = \{p_{B_i}\}$ where $B_i = \{x \in \{1,\dots,q\}^G \mid x_e = i\}$. Let $\{p_{B_{\psi_1}}, p_{B_{\psi_2}} \dots p_{B_{\psi_\ell}}\}$ be a basis for $\operatorname{span}(P_{F^n_\pm}) \subset L^\infty(X,\mu)$, where $\psi_i \colon F_{\psi_j} \to \{1,\dots,q\}$.

Let $\sigma \in \mathrm{SA}(F_n, 4n|F^n_\pm| + 1, \delta, d)$. Let κ be as defined before Lemma 6.1. Let

$$
\begin{aligned}
\gamma_1 = {}& \min\Big\{ \Big|\sum_{i\in T} a_i\Big| : \psi, \psi_1, \dots . \psi_\ell, a_1 \dots, a_\ell, T \subset \{1,\dots,\ell\} \\
&\qquad\qquad \text{and } p_{B_\psi} = \sum_{i=1}^{\ell} a_i p_{B_{\psi_i}} \Big\} / \{0\}, \\
\gamma_2 = {}& \min\Big\{ \Big|\sum_{i\in T} a_i - 1\Big| : \psi, \psi_1, \dots . \psi_\ell, a_1 \dots, a_\ell, T \subset \{1,\dots,\ell\} \\
&\qquad\qquad \text{and } p_{B_\psi} = \sum_{i=1}^{\ell} a_i p_{B_{\psi_i}} \Big\} / \{0\}, \\
\gamma_3 = {}& \min\Big\{ \Big|\sum_{i\in T, j\in T'} a_i b_j\Big| : \psi, \psi' \psi_1, \dots . \psi_\ell, a_1 \dots, a_\ell, T, T' \subset \{1,\dots,\ell\} \\
&\qquad\qquad \text{and} p_{B_\psi} = \sum_{i=1}^{\ell} a_i p_{B_{\psi_i}}, p_{B_{\psi'}} = \sum_{i=1}^{\ell} b_i p_{B_{\psi_i}} \Big\} / \{0\} \\
\gamma = {}& \min\{\gamma_1, \gamma_2, \gamma_3\},
\end{aligned}
$$

so that γ depends only on F and n and $\gamma \leq 1$. We want to find φ such that

$$
(\sigma, \varphi) \in \mathrm{HA}\Big(F, P, n, 9|P_{F^n_\pm}|^2 \frac{1}{\gamma^2} \kappa^5 \ell^2 q c_3(F,n)^2 \sqrt{\delta}, d\Big)
$$

(for sufficiently large d). Using Proposition 7.3 will complete the proof.

Take a partition $\{A_1,\dots,A_q\}$ such that the conclusion of Lemma 5.1 holds (namely a random partition for d large). For each $p_{B_{\psi_i}}$, $i = 1,\dots,\ell$, let $\varphi_0(p_{B_{\psi_i}}) := p_{A_{\psi_i}}$ and extend φ_0 linearly to $\operatorname{span}(P_{F^n_\pm}) \subset L^\infty(X,\mu)$ with values in M_d. We will check that φ_0 satisfies the following properties, where $\delta_0 = 3\frac{1}{\gamma}\kappa^2 \ell^2 q c_3(F,n)^2 \sqrt{\delta}$ (compare Remark 7.5):

(1) $|\operatorname{tr} \circ \varphi_0(p) - \mu(p)| < \delta_0$ for all $p \in P_{F^n_\pm}$,
(2) $\|\varphi_0 \circ s(p) - \sigma(s) \circ \varphi_0(p)\|_2 < \delta_0$ for all $p \in P$ and $s \in F^n_\pm$,
(3) $\|\varphi_0(p_1 p_2) - \varphi_0(p_1)\varphi_0(p_2)\|_2 < \delta_0$ for all $p_1, p_2 \in \operatorname{span} P_{F^n_\pm}$,
(4) $\|\varphi_0(p_X) - p_d\|_2 < \delta_0$, where $p_d := p_{\{1,\dots,d\}}$.

One can see (as shown below) that properties (1)–(4) are closely related to the properties (a)–(d) defined above.

(1) Note that for any $x \in \{1, \dots, d\}$, ψ and $a_1, \dots, a_n$, if

$$\Big| \Big(\sum_{i=1}^{\ell} a_i p_{A_{\psi_i}} - p_{A_\psi} \Big) x \Big| \neq 0,$$

then

$$\Big| \Big(\sum_{i=1}^{\ell} a_i p_{A_{\psi_i}} - p_{A_\psi} \Big) x \Big| \geq \min\{\gamma_1, \gamma_2\} \geq \gamma.$$

Let $p_{B_\psi} \in P_{F^n_\pm}$, say $p_{B_\psi} = \sum_{i=1}^{\ell} a_i p_{B_{\psi_i}}$. Then

$$\varphi_0(p_{B_\psi}) = \sum_{i=1}^{\ell} a_i p_{A_{\psi_i}},$$

so

$$\begin{aligned}\big| \operatorname{tr} \circ \varphi_0(p_{B_\psi}) - \mu(p_{B_\psi}) \big| &\leq \Big| \operatorname{tr}\Big(\sum_{i=1}^{\ell} a_i p_{A_{\psi_i}} \Big) - \operatorname{tr} \circ p_{A_\psi} \Big| + \Big| \frac{|A_\psi|}{d} - \mu(B_\psi) \Big| \\ &\leq \frac{1}{\gamma} \Big\| \sum_{i=1}^{\ell} a_i p_{A_{\psi_i}} - p_{A_\psi} \Big\|_2^2 + c_2(F, n)\delta \\ &\leq \frac{1}{\gamma} c_3(F, n)^2 \delta^2 + c_2(F, n)\delta \leq \frac{1}{\gamma} c_3(F, n)^2 \sqrt{\delta}, \text{ for } \delta < 1,\end{aligned}$$

where we use Lemmas 5.1 and 6.1

(2) Here we use Lemma 6.1. Let $s \in F^n_\pm$ and suppose

$$s p_{B_i} = \sum_{i=1}^{\ell} a_i p_{B_{\psi_i}}.$$

Then

$$\varphi_0(s p_{B_i}) = \sum_{i=1}^{\ell} a_i p_{A_{\psi_i}} \quad \text{and} \quad \Big\| \sum_{i=1}^{\ell} a_i p_{A_{\psi_i}} - \sigma(s) p_{A_i} \Big\|_2 < c_3(F, n)\sqrt{\delta}.$$

On the other hand,

$$p_{B_i} = \sum_{i=1}^{\ell} b_i p_{B_{\psi_i}} \Rightarrow \varphi_0(p_{B_i}) = \sum_{i=1}^{\ell} b_i p_{A_{\psi_i}}$$

and

$$\Big\| \sum_{i=1}^{\ell} b_i p_{A_{\psi_i}} - \sigma(1) p_{A_i} \Big\|_2 < c_3(F, n)\sqrt{\delta}.$$

Thus,

$$\begin{aligned}&\|\varphi_0(s p_{B_i}) - \sigma(s)\varphi_0(p_{B_i})\|_2 \\ &\quad \leq \|\varphi_0(s p_{B_i}) - \sigma(s) p_{A_i}\|_2 + \|\sigma(s) p_{A_i} - \sigma(s)\sigma(1) p_{A_i}\|_2 \\ &\qquad + \|\sigma(s)\sigma(1) p_{A_i} - \sigma(s)\varphi_0(p_{B_i})\|_2 \\ &\quad < 2c_3(F, n)\sqrt{\delta} + \sqrt{\delta} = 3c_3(F, n)\sqrt{\delta}\end{aligned}$$

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(3) Let $p_1 = \sum_{i=1}^{\ell} a_i p_{B_{\psi_i}}$ and $p_2 = \sum_{i=1}^{\ell} b_i p_{B_{\psi_i}}$. Then

$$\begin{aligned}\big\| \varphi_0(p_1p_2) - \varphi_0(p_1)\varphi_0(p_2)\big\|_2 &= \Big\| \sum_{i,j=1}^{\ell} a_i b_i(\varphi_0(p_{B_{\psi_i}} p_{B_{\psi_j}}) - p_{A_{\psi_i}} p_{A_{\psi_j}})\Big\|_2 \\ &\leq \kappa^2 \sum_{i,j=1}^{\ell} \|\varphi_0(p_{B_{\psi_i}} p_{B_{\psi_j}}) - p_{A_{\psi_i}} p_{A_{\psi_j}}\|_2.\end{aligned}$$

So it is enough to show that for all i, j,

$$\|\varphi_0(p_{B_{\psi_i}} p_{B_{\psi_j}}) - p_{A_{\psi_i}} p_{A_{\psi_j}}\|_2 < c_3(F,n)\sqrt{\delta}.$$

But as we have seen in the proof of Lemma 6.1, either $p_{B_{\psi_i}} p_{B_{\psi_j}}$ and $p_{A_{\psi_i}} p_{A_{\psi_j}}$ are both 0, in which case we are done, or $p_{B_{\psi_i}} p_{B_{\psi_j}} = p_{B\psi'}$ and $p_{A_{\psi_i}} p_{A_{\psi_j}} = p_{A_{\psi'}}$ for some ψ', in which case if $p_{B_{\psi'}} = \sum_{i=1}^{\ell} c_i p_{B_{\psi_i}}$, then by Lemma 6.1,

$$\|\varphi_0(p_{B_{\psi_i}} p_{B_{\psi_j}}) - p_{A_{\psi_i}} p_{A_{\psi_j}}\|_2 = \Big\| \sum_{i=1}^{\ell} c_i p_{A_{\psi_i}} - p_{A_{\psi'}}\Big\|_2 < c_3(F,n)\sqrt{\delta}.$$

(4) $p_X = \sum_{i=1}^{q} p_{B_i}$ and $I_d = \sum_{i=1}^{q} p_{A_i}$ so by the triangle inequality it suffices to show

$$\|\varphi_0(p_{B_i}) - p_{A_i}\|_2 < 2c_3(F,n)\sqrt{\delta}$$

By Lemma 6.1

$$\|\varphi_0(p_{B_i}) - \sigma(1)p_{A_i}\|_2 < c_3(F,n)\sqrt{\delta}$$

so

$$\begin{aligned}\|\varphi_0(p_{B_i}) - p_{A_i}\|_2 &\leq \|\varphi_0(p_{B_i}) - \sigma(1)p_{A_i}\|_2 + \|\sigma(1)p_{A_i} - p_{A_i}\|_2 \\ &< 2c_3(F,n)\sqrt{\delta} + \sqrt{\delta} < 2c_3(F,n)\sqrt{\delta}.\end{aligned}$$

We now show how to define φ using φ_0. For each p_{B_ψ},

$$\big\| \varphi_0(p_{B_\psi}) - p_{A_\psi}\big\|_2 \leq c_3(F,n)\sqrt{\delta}$$

by Lemma 6.1. Let M be the number of $x \in \{1,\dots,d\}$ such that $\varphi_0(p_{B_\psi})x = p_{A_\psi}x$. Then

$$\big\| \varphi_0(p_{B_\psi}) - p_{A_\psi}\big\|_2 \geq \sqrt{\frac{d-M}{d}\gamma^2},$$

hence

$$M \geq d\Big(1 - c_3(F,n)^2\frac{1}{\gamma^2}\delta\Big).$$

Similarly for $p_{B_\psi}, p_{B_{\psi'}}$ we have

$$\big\| \varphi_0(p_{B_\psi})\varphi_0(p_{B_{\psi'}}) - \varphi(p_{B_\psi})\varphi(p_{B_{\psi'}})\big\|_2 \leq 2\kappa^2 c_3(F,n)\sqrt{\delta}$$

by (3), so

$$\big|\{x \in [d] : \varphi_0(p_{B_\psi})\varphi_0(p_{B_{\psi'}})x = 0\}\big| \geq d\Big(1 - c_3(F,n)^2\kappa^4\delta\frac{1}{\gamma^2}\Big).$$

Let V be the set of all $x \in \{1,\dots,d\}$ such that for all p_{B_ψ} (arbitrarily represented) we have $\varphi_0(p_{B_\psi})x = p_{A_\psi}x$, and for all $p_{B_\psi}, p_{B_{\psi'}}$ with $p_{B_\psi}p_{B_{\psi'}} = 0$ we have

$$\varphi_0(p_{B_\psi})\varphi_0(p_{B_{\psi'}})x = 0.$$

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Then

$$|V| \geq d\Big(1 - |P_{F^n_\pm}|c_3(F,n)^2\delta\frac{1}{\gamma^2} - |P_{F^n_\pm}|^2 c_3(F,n)^2\kappa^4\delta\frac{1}{\gamma^2}\Big)$$
$$\geq d\Big(1 - 2|P_{F^n_\pm}|^2 c_3(F,n)^2\kappa^4\delta\frac{1}{\gamma^2}\Big),$$

and for any $p \in \boldsymbol{\Sigma}\, P_{F^n_\pm}$, $\varphi_0(p)\big|_V \subset [\![d]\!]$. We finally define

$$\varphi\colon\ \boldsymbol{\Sigma}\, P_{F^n_\pm} \to [\![d]\!]$$
$$p \mapsto \varphi_0(p)\big|_V.$$

Let us check that φ satisfies (i)–(iv):

(i)

$$|\operatorname{tr}\circ\varphi(p_{B_\psi}) - \mu(B_\psi)| \leq |\operatorname{tr}\circ\varphi(p_{B_\psi}) - \operatorname{tr}\circ\varphi_0(p_{B_\psi})| + \delta_0$$
$$\leq \kappa\ell 2|P_{F^n_\pm}|^2 c_3(F,n)^2\kappa^4\delta\frac{1}{\gamma^2} + \delta_0,$$

(ii)

$$\begin{aligned}|\varphi(sp_{B_i}) - \sigma(1)\varphi(p_{B_i})| &\leq \|\varphi(sp_{B_i}) - \sigma(1)\varphi(p_{B_i})\|_2\\ &\leq \|\varphi(sp_{B_i}) - \varphi_0(sp_{B_i})\|_2\\ &\quad + \|\sigma(1)\varphi(p_{B_i}) - \sigma(1)\varphi_0(p_{B_i})\|_2 + \delta_0\\ &\leq \kappa\ell 4|P_{F^n_\pm}|c_3(F,n)\kappa^2\sqrt{\delta}\frac{1}{\gamma} + \delta_0,\end{aligned}$$

(iii)

$$\begin{aligned}|\varphi(p_1)\varphi(p_2) - \varphi(p_1p_2)| &\leq \|\varphi(p_1)\varphi(p_2) - \varphi(p_1p_2)\|_2\\ &\leq \|\varphi(p_1)\varphi(p_2) - \varphi_0(p_1)\varphi_0(p_2)\|_2\\ &\quad + \|\varphi(p_1p_2) - \varphi_0(p_1p_2)\|_2 + \delta_0\\ &\leq \kappa^2\ell^2 q4|P_{F^n_\pm}|c_3(F,n)\kappa^2\sqrt{\delta}\frac{1}{\gamma} + \delta_0,\end{aligned}$$

(iv)

$$|\varphi(p_X) - p_d| \leq \kappa\ell 2|P_{F^n_\pm}|c_3(F,n)\kappa^2\sqrt{\delta}\frac{1}{\gamma} + \delta_0.$$

Thus for every $\sigma \in \mathrm{SA}(F_n, 4n|F^n_\pm| + 1, \delta, d)$ with d sufficiently large, we found φ so that $(\sigma,\varphi) \in \mathrm{HA}(F, P, n, 9|P_{F^n_\pm}|^2\frac{1}{\gamma^2}\kappa^5\ell^2 q c_3(F,n)^2\sqrt{\delta}, d)$.

We now check that

$$s(G) = \sup_E \inf_F \inf_{n\in\mathbb{N}} \inf_{\delta>0} \limsup_{d\to\infty} \frac{1}{d\log d}\big|\,\mathrm{SA}(F_n, 4n|F^n_\pm| + 1, \delta, d)\big|_E.$$

Recall that $F_n := F \cup \{p_{F_{0|\pi}} \mid F_0 \subset F^n_\pm,\ \pi \text{ partition of } F_0\}$.

The right-hand side equals

$$\sup_E \inf_F \inf_{n\in\mathbb{N}} s_E(F_n, 4n|F^n_\pm| + 1).$$

Let $\epsilon > 0$ and choose n_0 so that

$$|s_E(F_{n_0}, 4n_0|F_{\pm}^{n_0}| + 1) - \inf_n s_E(F_n, 4n|F_{\pm}^{n}| + 1)| < \varepsilon$$

and

$$|S_E(F, 4n_0|F_{\pm}^{n_0}| + 1) - \inf_n s_E(F, 4n|F_{\pm}^{n}| + 1)| < \varepsilon.$$

Clearly $s_E(F_{n_0}, 4n_0|F_{\pm}^{n_0}| + 1) \leq s_E(F, 4n_0|F_{\pm}^{n_0}| + 1)$, but on the other hand if

$$\sigma \in \mathrm{SA}(F_{n_0}, 4n_0|(F_{n_0})_{\pm}^{n_0}| + 1, \delta, d),$$

then $\sigma \in \mathrm{SA}(F_{n_0}, 4n_0|F_{\pm}^{n_0}| + 1, \delta, d)$ since $4n_0|(F_{n_0})_{\pm}^{n_0}| + 1 \geq 4n_0|F_{\pm}^{n_0}| + 1$. So

$$\inf_F s_E(F, 4n_0|F_{\pm}^{n_0}| + 1) = \inf_F s_E(F_{n_0}, 4n_0|F_{\pm}^{n_0}| + 1).$$

Since ε was arbitrary

$$\inf_F \inf_n s_E(F_n, 4n|F_{\pm}^{n}| + 1) = \inf_F \inf_n s_E(F, 4n|F_{\pm}^{n}| + 1) = s_E(G),$$

since

$$\inf_n s_E(F, n) = \inf_n s_E(F, 4n|F_{\pm}^{n}| + 1).$$

However,

$$\big|\mathrm{SA}(F_n, 4n|F_{\pm}^{n}| + 1, \delta, d)\big|_E \leq |\,\mathrm{HA}\Big(F, P, n, 9|P_{F_{\pm}^{n}}|^2 \frac{1}{\gamma^2}\kappa^5\ell^2 q c_3(F, n)^2\sqrt{\delta}, d\Big)\,|_E,$$

so $s(G) \leq s(G \ltimes X)$ by Proposition 7.3. Replacing lim sup by lim inf or $\lim_{d\to\omega}$ above, we get a similar inequality for $\underline{s}$ and s^{ω}. ■

Theorem 8.2 *Let G be a pmp groupoid, let (X_0, μ_0) be a standard probability space, and let $G \curvearrowright X_0^G$ be the corresponding a Bernoulli action. Then $s(G \ltimes X_0^G) = s(G)$. The same holds true for $\underline{s}$ and s^{ω}.*

Proof Again we only need to prove $s(G) \leq s(G \ltimes X_0^G)$. Let $U = \{U_1, U_2, \dots, U_q\}$ be any finite partition of X_0 into measurable sets. Let

$$B_U = \{B_{U_1}, \dots, B_{U_q}\}$$

where

$$B_{U_i} = \{x \in X_0^G \mid \forall e \in G_0,\ x_e \in U_i\}.$$

Let $M := \{p_{U_i}\}$ for all possible choices of U and all i, so that M is a dynamically generating *-subalgebra of $L^\infty(X, \mu)$. Now if $P_U := \{p_{U_1}, \dots, p_{U_q}\}$, then

$$s(G \ltimes X) = \sup_E \sup_Q \inf_F \inf_{P_U} s_{E,Q}(F, P_U) = \sup_E \sup_Q \inf_{P_U} \inf_F s_{E,Q}(F, P_U).$$

Since the map $X_0 \to \{1, \dots, q\}$ defined by $U_i \ni x \mapsto i$ extends G-equivariantly to $X_0^G \to \{1, \dots, q\}^G$, we have

$$\inf_F s(F) \leq \inf_F s_{E,Q}(F, P_U).$$

Hence $s(G) \leq s(G \ltimes X)$. The same holds for $\underline{s}$ and s^{ω}. ■

Corollary 8.3 *A pmp groupoid G is s-regular if and only if the crossed product groupoid $G \ltimes X_0^G$ associated with the Bernoulli action $G \curvearrowright X_0^G$ is s-regular for any base space (X_0, μ_0).*

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Proof Since
$$\underline{s}(G) = \underline{s}(G \times X_0^G) \leq s(G \times X_0^G) = s(G),$$
$\underline{s}(G) = s(G)$ if and only if $\underline{s}(G \times X_0^G) = s(G \times X_0^G)$. ■

9 The Scaling Formula

Proposition 9.1 (Scaling formula) *Let G be an ergodic pmp groupoid, $0 \neq p \in L^\infty(G^0)$. Then*
$$s(G) - 1 = \mathfrak{h}(p)(s(pGp) - 1),$$
where pGp is endowed with the normalized Haar measure $\frac{\mathfrak{h}_{|pGp}}{\mathfrak{h}(p)}$. Furthermore, the same equality holds for $\underline{s}$ and s_ω; therefore,
$$G \text{ is } s\text{-regular} \Leftrightarrow pGp \text{ is } s\text{-regular.}$$

We will start by showing $\geq$ in the rational case, which is easier to handle:

Lemma 9.2 *If $\mathfrak{h}(p) \in \mathbb{Q}$, then*
$$s(pGp) \leq \frac{1}{\mathfrak{h}(p)} s(G) + 1 - \frac{1}{\mathfrak{h}(p)}.$$
Furthermore the same inequality holds for $\underline{s}$ and s_ω.

Proof Write $\mathfrak{h}(p) = \frac{N-k}{N}$ and choose $s_1, s_2, \dots, s_k \in [\![G]\!]$ with
$$s_i^{-1} s_i \leq p, \ \mathfrak{h}(s_i^{-1} s_i) = \frac{1}{N} \quad \text{and} \quad \sum_{i=1}^{k} s_i s_i^{-1} = 1 - p.$$
Let $S = \{s_1, \dots, s_k\}$, so $pGp \cup S$ is generating G and let $E, F \subset [\![pGp]\!]$ be finite subsets. We may assume that $p \in E \cap F$ and $s_i^{-1} s_i \in E \cap F$ for all i. Let $\sigma \in \mathrm{SA}_{pGp}(F, 4n+5, \delta, d)$. In light of [DKP2, Lemma 2.13] we may also assume that $N - k | d$. Let $d' = \frac{N}{N-k} d$ and partition $\{1, \dots, d'\}$ into sets $A_0, \dots, A_k$ with $A_0 = \{1, \dots, d\}$ and $|A_i| = \frac{d'}{N}$. Since
$$|\operatorname{Fix}\left(\sigma(s_i^{-1} s_i)\right)| \geq \frac{\mathfrak{h}(s_i^{-1} s_i)}{\mathfrak{h}(p)} d - \delta d,$$
we can choose a subset $B_i \subset A_0$ with exactly $\frac{d'}{N}$ elements such that
$$\left|\operatorname{Fix}\left(\sigma(s_i^{-1} s_i)\right) \triangle B_i\right| < \delta d'.$$
For each $i = 1 \dots k$ let $\gamma(s_i)$ be a bijection $B_i \to A_i$ (we have $(\frac{d'}{N}!)^k$ choices).

As
$$\begin{aligned} s_j^{-1} s_i^{-1} &= s_i s_j = 0 \ i, j \geq 1, \\ s_i^{-1} s_j &= 0 \ i \neq j \geq 1, \\ s s_i &= s_i^{-1} s = 0 \ i \geq 1, s \in pGp, \\ s_i^{-1} s_i &\in F, \end{aligned}$$

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each element of $\bigcup_n([\![pGp]\!] \cup S)^n$ can be written (not necessarily uniquely) as $s_i f s_j^{-1}$ for $i, j \geq 0$, where $s_0 = p$ and $f \in [\![pGp]\!]$. Let $\gamma(s_0) = p_{A_0}$ We define a map

$$\sigma_\gamma \colon (F \cup S)_{\pm}^{2n+5} \to [\![d']\!]$$

by

$$\sigma_\gamma(s_i f s_j^{-1}) := \gamma(s_i)\sigma'(s_i^{-1} s_i f s_j^{-1} s_j)\gamma(s_j)^{-1},$$

where we denote $\sigma'(f)$ the permutation $\left(\begin{smallmatrix} \sigma(f) & 0 \\ 0 & 0 \end{smallmatrix}\right)$ acting in $A_0 \subset \{1, \dots, d'\}$. Note that σ_γ is well defined: if $s_{i_1} f_1 s_{j_1}^{-1} = s_{i_2} f_2 s_{j_2}^{-1}$, then since the s_i have disjoint ranges, we have $i_1 = i_2$ $j_1 = j_2$ so $s_{i_1} f_1 s_{j_1}^{-1} = s_{i_1} f_2 s_{j_1}^{-1}$, hence $s_{i_1}^{-1} s_{i_1} f_1 s_{j_1}^{-1} s_{j_1} = s_{i_1}^{-1} s_{i_1} f_2 s_{j_1}^{-1} s_{j_1}$ so that

$$\gamma(s_{i_1})\sigma'(s_{i_1}^{-1} s_{i_1} f_1 s_{j_1}^{-1} s_{j_1})\gamma(s_{j_1})^{-1} = \gamma(s_{i_1})\sigma'(s_{i_1}^{-1} s_{i_1} f_2 s_{j_1}^{-1} s_{j_1})\gamma(s_{j_1})^{-1}.$$

Then extend σ_γ linearly to $\boldsymbol{\Sigma}(F \cup S)_{\pm}^{2n+5}$.

Let $\delta' = 5N^2\delta + 150N^2(2|F \cup S| + 1)^{2(2n+5)}\delta$. We claim that

$$\sigma_\gamma \in \mathrm{SA}_G(F \cup S, n, \delta', d').$$

Let us first see what this implies. Note if $\sigma^1(f) \neq \sigma^2(f)$, then

$$\sigma_\gamma^1(f) = p_{A_0}\sigma_\gamma^1(f)p_{A_0} = \sigma_\gamma^1(f) \neq \sigma_\gamma^2(f).$$

Next note if $\gamma_1(s_i) \neq \gamma_2(s_i) \colon B_i \to A_i$ are two bijections, then if we have

$$\sigma_{\gamma_1}(s_i) = \gamma_1(s_i)\sigma'(p) = \gamma_2(s_i)\sigma'(p) = \sigma_{\gamma_2}(s_i),$$

we must have

$$\gamma_1(s_i)\big|_{fix(\sigma(p))} = \gamma_2(s_i)\big|_{fix(\sigma(p))}.$$

But since $|\operatorname{Fix}(\sigma(p)) \triangle \{1, \dots, d\}| \leq \delta d$ we have at most $\big|[\![\lceil \delta d \rceil]\!]\big| \leq (\lceil \delta d \rceil + 1)^{\lceil \delta d \rceil}$ choices for γ_2 such that $\gamma_2(s_i) \neq \gamma_1(s_i)$ and $\sigma_{\gamma_2}(s_i) = \sigma_{\gamma_1}(s_i)$. Therefore, there are at most $(\lceil \delta d \rceil + 1)^{\lceil \delta d \rceil k}$ choices for γ_2 such that $\sigma_{\gamma_1}\big|_{E \cup S} \neq \sigma_{\gamma_2}\big|_{E \cup S}$ Thus each $\sigma_{|E} \in \mathrm{SA}_{pGp}(F, n, \delta, d)$ gives at least $(\frac{d'}{N}!)^k / (\lceil \delta d \rceil + 1)^{\lceil \delta d \rceil k}$ distinct elements $\sigma'_{|E \cup S} \in \mathrm{SA}_G(F \cup S, n, \delta', d')$, so we obtain

$$|\mathrm{SA}_{pGp}(F, n, \delta, d)|_E \leq |\mathrm{SA}_G(F \cup S, 4n + 5, \delta', d')|_{E \cup S} \Big/ \frac{(\frac{d'}{N}!)^k}{(\lceil \delta d \rceil + 1)^{\lceil \delta d \rceil k}}$$

So let us show that $\sigma_\gamma \in \mathrm{SA}_G(F \cup S, n, \delta', d')$. We first show that σ_γ is approximately linear. Suppose in the span we have

$$s_i f s_j = \sum_{\ell=1}^{M} s_{i_\ell} f_\ell s_{j_\ell},$$

then if we let $L(u, v)$ be the set of all ℓ with $i_\ell = u$ $j_\ell = v$ then $s_i f s_j = \sum_{\ell \in L(i,j)} s_i f_\ell s_j^{-1}$ and $s_u 0 s_v = 0 = \sum_{\ell \in L(u,v)} s_p f_\ell s_q^{-1}$ for $(u, v) \neq (i, j)$. So we may assume that $i_\ell =$

$i, j_\ell = j$ for all ℓ. Then

$$\begin{aligned}\sigma_\gamma(s_i f s_j) &= \gamma(s_i)\sigma'(s_i^{-1} s_i f s_j^{-1} s_j)\gamma(s_j)^{-1}\\ &= \gamma(s_i)\sigma'\Big(s_i^{-1}\Big(\sum_{\ell=1}^{M} s_i f_\ell s_j^{-1}\Big) s_j\Big)\gamma(s_j)^{-1}\\ &= \gamma(s_i)\sigma'\Big(\sum_{\ell=1}^{M} s_i^{-1} s_i f_\ell s_j^{-1} s_j\Big)\gamma(s_j)^{-1}\end{aligned}$$

so by [DKP1, Lemma 3.5],

$$\begin{aligned}\Big|\sigma_\gamma(s_i f s_j) - \sum_{\ell=1}^{M}\sigma_\gamma(s_i f_\ell s_j)\Big| &= \Big|\sigma_\gamma(s_i f s_j) - \sum_{\ell=1}^{M}\gamma(s_i)\sigma'(s_i^{-1} s_i f_\ell s_j^{-1} s_j)\gamma(s_j)^{-1}\Big|\\ &< 150(2|F\cup S|+1)^{2(n+5)}\delta.\end{aligned}$$

Thus if $f = \sum_{\ell=1}^{M} f_\ell$, then

$$\begin{aligned}\Big|\sigma_\gamma(f) - \sum_{\ell=1}^{M}\sigma_\gamma(f_\ell)\Big| &= \Big|\sigma_\gamma(\sum_{i,j=1}^{N} s_i f s_j) - \sum_{\ell=1}^{M}\sigma_\gamma(\sum_{i,j=1}^{N} s_i f_\ell s_j)\Big|\\ &\le \sum_{i,j=1}^{N}\Big|\sigma_\gamma(s_i f s_j) - \sum_{\ell=1}^{M}\sigma_\gamma(s_i f_\ell s_j)\Big|\\ &< 150N^2(2|F\cup S|+1)^{2(n+5)}\delta.\end{aligned}$$

Thus to show that σ_γ is δ'-multiplicative it will suffice to show that for $a, b \in (F\cup S)^n_\pm$ such that $ab \in (F\cup S)^n_\pm$ we have

$$|\sigma_\gamma(ab) - \sigma_\gamma(a)\sigma_\gamma(b)| < 5\delta.$$

Indeed then for $\sum_i a_i, \sum_i b_i \in \boldsymbol{\Sigma}(F\cup S)^n_\pm$ with $(\sum_i a_i)(\sum_j b_j) \in \boldsymbol{\Sigma}(F\cup S)^n_\pm$

$$\begin{aligned}&\Big|\sigma_\gamma\Big(\Big(\sum_i a_i\Big)\Big(\sum_j b_j\Big)\Big) - \sigma_\gamma\Big(\sum_i a_i\Big)\sigma_\gamma\Big(\sum_j b_j\Big)\Big|\\ &\quad\le \Big|\sigma_\gamma\Big(\sum_{i,j} a_i b_j\Big) - \Big(\sum_i \sigma_\gamma(a_i)\Big)\Big(\sum_j \sigma_\gamma(b_j)\Big)\Big|\\ &\qquad + \big(150N^2(2|F\cup S|+1)^{2(2n+5)}\delta\big)^2\\ &\quad\le \Big|\sum_{i,j}\sigma_\gamma(a_i b_j) - \sum_{i,j}\sigma_\gamma(a_i)\sigma_\gamma(b_j)\Big| + 2\big(150N^2(2|F\cup S|+1)^{2(2n+5)}\big)^2\delta\\ &\qquad\qquad (\text{assuming } \delta < 1)\\ &\quad\le 5N^2\delta + 2\big(150N^2(2|F\cup S|+1)^{2(2n+5)}\big)^2\delta = \delta'.\end{aligned}$$

So say we have $a = s_{i_1} f_1 s_{j_1}^{-1}$ and $b = s_{i_2} f_2 s_{j_2}^{-1}$, where $f_1, f_2 \in F^n_\pm$. If $j_1 \ne i_2$, then $s_{j_1}^{-1} s_{i_2} = \gamma(s_{j_1})^{-1}\gamma(s_{i_2}) = 0$, so we are done. Suppose otherwise; then

$$\begin{aligned}|\sigma_\gamma(ab) - \sigma_\gamma(a)\sigma_\gamma(b)| &= |\gamma(s_{i_1})\sigma'(s_{i_1}^{-1} s_{i_1} f_1 (s_{j_1}^{-1} s_{j_1})^3 f_2 s_{j_2}^{-1} s_{j_2})\gamma(s_{j_2})^{-1}\\ &\quad - \gamma(s_{i_1})\sigma'(s_{i_1}^{-1} s_{i_1} f_1 s_{j_1}^{-1} s_{j_1})\gamma(s_{j_1})^{-1}\gamma(s_{j_1})\sigma'(s_{j_1}^{-1} s_{j_1} f_2 s_{j_2}^{-1} s_{j_2})\gamma(s_{j_2})^{-1}|.\end{aligned}$$

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Note first that for any $i > 0$,

$$|\gamma(s_i)^{-1}\gamma(s_i) - \sigma'(s_i^{-1}s_i)| \leq \frac{1}{d'}|\operatorname{Fix}(\sigma(s_i^{-1}s_i)) \triangle B_i| + |\sigma(s_i^{-1}s_i) - p_{\operatorname{Fix}(\sigma(s_i^{-1}s_i))}| \\ \leq 2\delta,$$

and for $g_1, g_2 \in F_{\pm}^{2n+5}$ such that $g_1g_2 \in F_{\pm}^{2n+5}$,

$$|\sigma'(g_1g_2) - \sigma'(g_1)\sigma'(g_2)| = \frac{1}{d'}d|\sigma(g_1g_2) - \sigma(g_1)\sigma(g_2)| \leq \delta,$$

so by repeated applications of the triangle inequality

$$|\sigma_\gamma(ab) - \sigma_\gamma(a)\sigma_\gamma(b)| \leq 2\delta + 3\delta = 5\delta.$$

Let us now show that σ_γ is δ'-trace-preserving. Let $s_i f s_j^{-1} \in (F \cup S)_{\pm}^{n}$. If $i \neq j$, then

$$\operatorname{tr}(s_i f s_j^{-1}) = \operatorname{tr}(\gamma(s_i)\sigma'(s_i^{-1}s_i f s_j^{-1}s_j)\gamma(s_j)^{-1}) = 0.$$

Suppose otherwise; then

$$\begin{aligned}
&\left|\operatorname{tr}(\sigma_\gamma(s_i f s_i^{-1})) - \tau(s_i f s_i^{-1})\right| \\
&\quad = \left|\operatorname{tr}\big(\gamma(s_i)\sigma'(s_i^{-1}s_i f s_i^{-1}s_i)\gamma(s_i)^{-1}\big) - \tau(s_i f s_i^{-1})\right| \\
&\quad = \left|\operatorname{tr}\big(\sigma'(s_i^{-1}s_i f s_i^{-1}s_i)\gamma(s_i)^{-1}\gamma(s_i)\big) - \tau(f s_i^{-1}s_i)\right| \\
&\quad \leq \left|\operatorname{tr}\big(\sigma'(s_i^{-1}s_i f s_i^{-1}s_i)\gamma(s_i)^{-1}\gamma(s_i)\big)\big) - \operatorname{tr}\big(\sigma'(s_i^{-1}s_i f s_i^{-1}s_i)\big)\right| \\
&\qquad + \left|\operatorname{tr}\big(\sigma'(s_i^{-1}s_i f s_i^{-1}s_i)\big) - \frac{(N-k)\tau(f s_i^{-1}s_i)}{N\mathfrak{h}(p)}\right| \\
&\quad \leq (2\delta + \delta) + \delta < \delta',
\end{aligned}$$

where we used the computation above.

We have proved that

$$|\operatorname{SA}_{pGp}(F, 4n+5, \delta, d)|_E \leq |\operatorname{SA}_G(F \cup S, n, \delta', d')|_{E\cup S} \Big/ \frac{\left(\frac{d'}{N}!\right)^k}{(\lceil \delta d\rceil + 1)^{\lceil \delta d\rceil k}},$$

so

$$\begin{aligned}
\frac{\log|\operatorname{SA}_{pGp}(F, 4n+5, \delta, d)|_E}{d\log d} &\leq \frac{\log\Big(|\operatorname{SA}_G(F \cup S, n, \delta', d')|_{E\cup S} \Big/ \frac{\left(\frac{d'}{N}!\right)^k}{(\lceil \delta d\rceil + 1)^{\lceil \delta d\rceil k}}\Big)}{d\log d} \\
&= \frac{N}{N-k}\frac{\log|\operatorname{SA}_G(F \cup S, n, \delta', d')|_{E\cup S}}{d'\log d} \\
&\qquad - k\frac{\log(\frac{d'}{N})!}{d\log d} + \lceil \delta d\rceil k \frac{\log(\lceil \delta d + 1\rceil)}{d\log d}.
\end{aligned}$$

Now for $\varepsilon > 0$ arbitrary and d sufficiently large we have $\frac{1}{\log d} \leq \frac{1}{\log d'} + \frac{\varepsilon}{\log d'}$, so the left-hand side is at most

$$\frac{N}{N-k}(1+\epsilon)\frac{\log|\operatorname{SA}_G(F \cup S, n, \delta', d')|_{E\cup S})}{d'\log d'} - k\frac{\log(\frac{d'}{N})!}{d\log d} + \lceil \delta d\rceil k\frac{\log(\lceil \delta d + 1\rceil)}{d\log d}.$$

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By Stirling's approximation,

$$\log\Big(\frac{d'}{N}\Big)! = \frac{d'}{N}\log\frac{d'}{N} + \frac{d'}{N} + O\Big(\log\frac{d'}{N}\Big),$$

therefore

$$s_{pGp,E}(F, 2n+5, \delta) \le \frac{N}{N-k}(1+\varepsilon)\limsup_{d\to\infty} s_{G,E\cup S}(F\cup S, n, \delta', d') + \frac{k}{N} + k\delta$$

which gives

$$\begin{aligned} s_{pGp,E}(F, 4n+5) &\le \frac{1}{\mathfrak{h}(p)}(1+\varepsilon)s_{G,E\cup S}(F\cup S, n) + \frac{k}{N} \\ &= \frac{1}{\mathfrak{h}(p)}(1+\varepsilon)s_{G,E\cup S}(F\cup S, n) + 1 - \frac{1}{\mathfrak{h}(p)}. \end{aligned}$$

From here it is clear that

$$s(pGp) \le \frac{1}{\mathfrak{h}(p)}s(G) + 1 - \frac{1}{\mathfrak{h}(p)}.$$

The same proof works with $\liminf_{d\to\infty}$ and $\lim_{d\to\omega}$ instead of $\limsup_{d\to\infty}$. ■

Next we prove the other direction in the general case.

Lemma 9.3 *$s(G) \le \mathfrak{h}(p)s(pGp) + 1 - \mathfrak{h}(p)$ and similarly for $\underline{s}$ and s_ω.*

Proof Let $S = \{s_1, s_2, \dots, s_k\} \subset [\![G]\!]$ be such that

$$s_i^{-1}s_i \le p \quad \text{and} \quad \sum_{i=1}^{k} s_i s_i^{-1} = 1 - p,$$

and let $E, F \subset [\![pGp]\!]$ be finite subsets. We may assume that $p \in E\cap F$ and $s_i s_i^{-1}, s_i^{-1}s_i \in E\cap F$ for all i.

Let $\sigma \in \mathrm{SA}_G(F\cup S, n+4, \delta, d)$ and assume that d is large enough so that $d' := \lfloor \mathfrak{h}(p)d\rfloor > \delta^{-1}$. Let $B_0 = \operatorname{Fix}\sigma(p)$ so that

$$\Big|\frac{|B_0|}{d} - \mathfrak{h}(p)\Big| < \delta.$$

Thus we can arbitrarily extend or shrink B_0 to a subset B such that either $B \subset B_0$ or $B \supset B_0$ and $|B| = d'$, where

$$\begin{aligned} \big|\sigma(p) - p_B\big| &\le \big|\sigma(p) - p_{\operatorname{Fix}\sigma(p)}\big| + \big|p_{\operatorname{Fix}\sigma(p)} - p_B\big| \\ &= \frac{1}{d}|\{x\in\{1,\dots,d\} \mid \sigma(p)^2x \ne \sigma(p)\}| \\ &\quad + \Big|\frac{1}{d}\Big|B_0| - \mathfrak{h}(p)| + \Big|\frac{1}{d}\Big|B| - \mathfrak{h}(p)| \\ &< 3\delta. \end{aligned}$$

Let $\delta' = \frac{20\delta}{\mathfrak{h}(p)}$. We will show that

$$\sigma' := \sigma_{|_B} \in \mathrm{SA}_{pGp}(F, n, \delta', d').$$

Clearly σ' is well defined. Let us show it is δ'-multiplicative; that is, for $a, b \in \boldsymbol{\Sigma}\, F^n_{\pm}$ with $ab \in \boldsymbol{\Sigma}\, F^n_{\pm}$,

$$\big|\,\sigma'(ab) - \sigma'(a)\sigma'(b)\big| < \delta',$$

so

$$\begin{aligned}\big|\,\sigma'(ab) - \sigma'(a)\sigma'(b)\big| &= \frac{1}{d'} \sum_{x \in B} \big|\,\{x \in B \mid \sigma'(ab)x \neq \sigma'(a)\sigma'(b)x\}\big| \\ &= \frac{d}{d'} \big|\, p_B\sigma(ab)p_B - p_B\sigma(a)p_B\sigma(b)p_B\big| \\ &\leq \frac{\delta}{\mathfrak{h}(p)}(15\delta + 5\delta) = \frac{20\delta^2}{\mathfrak{h}(p)} < \delta' \quad \text{for } \delta < 1.\end{aligned}$$

We now show σ' is δ'-trace-preserving. Let $a \in F^n_{\pm}$. Then

$$\begin{aligned}&\Big|\frac{|\operatorname{Fix}\sigma'(a)|}{d'} - \frac{\tau_G(a)}{\mathfrak{h}(p)}\Big| \\ &\quad= \frac{1}{\mathfrak{h}(p)}\Big|\frac{|\operatorname{Fix}\sigma'(a)|}{d'/\mathfrak{h}(p)} - \tau_G(a)\Big| \\ &\quad\leq \frac{1}{\mathfrak{h}(p)}\Big|\frac{\mathfrak{h}(p)}{d'} - \frac{1}{d}\Big| + \frac{1}{\mathfrak{h}(p)}\Big|\frac{|\operatorname{Fix}\sigma'(a)|}{d} - \tau_G(a)\Big| \\ &\quad\leq \frac{1}{\mathfrak{h}(p)}\Big(\Big|\frac{|\operatorname{Fix} p_B\sigma(a)p_B|}{d} - \frac{|\operatorname{Fix}\sigma(a)|}{d}\Big| + \big|\operatorname{tr}\circ\sigma(a) - \tau_G(a)\big| + \delta^2\Big) \\ &\quad\leq \frac{1}{\mathfrak{h}(p)}\big(|p_B\sigma(a)p_B - \sigma(a)| + \delta + \delta^2\big) \\ &\quad\leq \frac{1}{\mathfrak{h}(p)}\big((6\delta + 2\delta) + \delta + \delta^2\big) = \frac{9\delta + \delta^2}{\mathfrak{h}(p)} < \delta' \quad \text{for } \delta < 1,\end{aligned}$$

so $\sigma' \in \mathrm{SA}_{pGp}(F, n, \delta', d')$, as claimed.

Next we study the map

$$\begin{gathered}\mathrm{SA}_G(F \cup S, n + 4, \delta, d) \to \mathrm{SA}_{pGp}(F, n, \delta', d') \\ \sigma \mapsto \sigma'.\end{gathered}$$

Note that

$$\begin{aligned}&\Big|\frac{|\operatorname{Fix}\sigma(s_i)\sigma(s_i)^{-1}|}{d} - \mathfrak{h}(s_i s_i^{-1})\Big| \\ &\quad\leq \Big|\frac{|\operatorname{Fix}\sigma(s_i)\sigma(s_i)^{-1}|}{d} - \frac{|\operatorname{Fix}\sigma(s_i s_i^{-1})|}{d}\Big| + \big|\operatorname{tr}\circ\sigma(s_i s_i^{-1}) - \mathfrak{h}(s_i s_i^{-1})\big| \\ &\quad\leq \big|\sigma(s_i)\sigma(s_i)^{-1} - \sigma(s_i s_i^{-1})\big| + \delta \\ &\quad\leq 5\delta + \delta < 6\delta\end{aligned}$$

and for $i \neq j$

$$\begin{aligned}|\operatorname{ran}(\sigma(s_i)) \cap \operatorname{ran}(\sigma(s_j))| &= \big|\{x \in \{1, \dots, d\} \mid \sigma(s_j)\sigma(s_j)^{-1}\sigma(s_i)\sigma(s_i)^{-1}x \neq 0\}\big| \\ &\leq d\big|\sigma(s_j)\sigma(s_j)^{-1}\sigma(s_i)\sigma(s_i)^{-1}\big| < 9\delta d,\end{aligned}$$

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so

$$\begin{aligned}|\operatorname{ran}(\sigma(s_i))\cap B| &= \big|\{x\in\{1,\dots,d\}\mid p_B\sigma(s_i)\sigma(s_i)^{-1}x\neq 0\}\big|\\ &\leq d\big|\,p_B\sigma(s_i)\sigma(s_i)^{-1}\big|\leq(3\delta+5\delta)d\leq 8\delta d.\end{aligned}$$

So for all d sufficiently large,

$$\operatorname{ran}(\sigma(s_i))\setminus\big(\bigcup_{j\neq i}\operatorname{ran}(\sigma(s_j))\cup B\big)$$

contains at least $d_i := \lfloor \mathfrak{h}(s_i s_i^{-1})d-(14\delta+9\delta k)d\rfloor$ elements so we can find a subset

$$A_i\subset\operatorname{ran}(\sigma(s_i))\setminus\big(\bigcup_{j\neq i}\operatorname{ran}(\sigma(s_j))\cup B\big)$$

with size $|A_i| = d_i$ and $A_i\cap A_j=\varnothing$ for $i\neq j$ and $A_i\cap B=\varnothing$. Similarly we have

$$\Big|\frac{|\operatorname{Fix}\sigma(s_i)^{-1}\sigma(s_i)}{d}-\mathfrak{h}(s_i^{-1}s_i)\Big|<6\delta$$

and

$$\begin{aligned}&|\operatorname{dom}\sigma(s_i)\bigtriangleup\operatorname{Fix}\sigma(s_i^{-1}s_i)\cap B|\\ &\qquad\leq\big|\{x\in\{1,\dots,d\}\mid P_BP_{\operatorname{Fix}\sigma(s_i^{-1}s_i)}x\neq\sigma(s_i)^{-1}\sigma(s_i)x\}\big|\\ &\qquad\leq d\big|\,P_BP_{\operatorname{Fix}\sigma(s_i^{-1}s_i)}-\sigma(s_i)^{-1}\sigma(s_i)\big|<d6\delta,\end{aligned}$$

so we can find subsets $B_i\subset\operatorname{dom}(\sigma(s_i))\cap\operatorname{Fix}\sigma(s_i^{-1}s_i)\cap B$ with $|B_i|=d_i$. Let $d_{k+1}=d-d'-\sum_{i=1}^k d_i$. Also recall that $|\operatorname{Fix}\sigma(s_is_i^{-1})|\leq d(\mathfrak{h}(s_i^{-1}s_i)+\delta)$.

Let $|\operatorname{SA}_G(F\cup S,n{+}4,\delta,d)\big|'_{E\cup S}$ be the number of elements of $\operatorname{SA}_G(F\cup S,n+4,\delta,d)$ where we distinguish elements by their values on $\{p_B\sigma(f)p_B\mid f\in E\}$ and $\{p_{A_i}\sigma(s_i)p_{B_i}\}$. We have shown with the above computations that

$$\begin{aligned}&|\operatorname{SA}_G(F\cup S,n+4,\delta,d)\big|'_{E,S}\leq\\ &\qquad\binom{d}{d',d_1,\dots,d_{k+1}}\big|\operatorname{SA}_{pGp}(F,n,\delta',d')\big|_E\prod_{i=1}^k\binom{d(\mathfrak{h}(s_i^{-1}s_i)+\delta)}{d_i}\prod_{i=1}^k d_i!,\end{aligned}$$

where the former term accounts for the choice of the A_i and B and the latter terms for the choice of the B_i and $\sigma(s_i)$, respectively.

Now note that for $f\in E$,

$$\big|\,p_B\sigma(f)p_B-\sigma(f)\big|\leq(6\delta+2\delta)=8\delta,$$

and, similarly, for $i\neq j$,

$$\begin{aligned}&\big|\,p_{A_i}\sigma(s_i)p_{B_i}-\sigma(s_i)\big|\\ &\quad\leq\big|\,p_{A_i}-\sigma(s_i)^{-1}\sigma(s_i)\big|+\big|\,p_{B_i}-\sigma(s_i)\sigma(s_i)^{-1}\big|+3\delta\\ &\quad\leq(14\delta+9k\delta+6\delta)+(14\delta+9k\delta+6\delta)+\delta+3\delta,\text{ for d sufficiently large}\\ &\quad=(44+18k)\delta\end{aligned}$$

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so we can find $\kappa > 0$ with $\kappa \to 0$ as $\delta \to 0$ so that for a given choice of $p_B\sigma(f)p_B$ there are $d^{\kappa d}$ choices for $\sigma(f)$. Similarly, for $\sigma(s_i)$ (see [DKP2, Lemma 2.5]) so

$$\big|\,\mathrm{SA}_G(F\cup S, n+4, \delta, d)\big|_{E,S} \le \binom{d}{d', d_1, \dots, d_{k+1}} \big|\,\mathrm{SA}_{pGp}(F, n, \delta', d')\big|_E \prod_{i=1}^{k} \binom{d(\mathfrak{h}(s_i^{-1}s_i)+\delta)}{d_i} \prod_{i=1}^{k} (d_i!) d^{\kappa d|E\cup S|}.$$

Therefore,

$$\begin{aligned}\log\big|\,\mathrm{SA}_G(F\cup S, n+4, \delta, d)\big|_{E,S} \le \log \frac{d!}{d'!d_{k+1}!} &+ \log\big|\,\mathrm{SA}_{pGp}(F, n, \delta', d')\big|_E \\ &+ \sum_{i=1}^{k} \log \frac{d(\mathfrak{h}(s_i^{-1}s_i)+\delta)!}{d_i!(d(\mathfrak{h}(s_i^{-1}s_i)+\delta)-d_i)!} \\ &+ \kappa d|E\cup S|\log d,\end{aligned}$$

hence

$$\log\big|\,\mathrm{SA}_G(F\cup S, n+4, \delta, d)\big|_{E,S} + \log d'! \le \log d! + \log\big|\,\mathrm{SA}_{pGp}(F, n, \delta', d')\big|_E + \varepsilon(\delta, d)$$

with

$$\varepsilon(\delta, d) := \sum_{i=1}^{k} \log \frac{d(\mathfrak{h}(s_i^{-1}s_i)+\delta)!}{d_i!(d(\mathfrak{h}(s_i^{-1}s_i)+\delta)-d_i)!} + \kappa d|E\cup S|\log d.$$

So since $\lim_{d\to\infty} \frac{\log d'!}{d\log d} = \mathfrak{h}(p)$ and $\inf_\delta \lim_{d\to\infty} \varepsilon(\delta, d) = 0$, we finally obtain

$$s_{G,E\cup S}(F\cup S, n+4) + \mathfrak{h}(p) \le 1 + \inf_{\delta>0} \limsup_{d\to\infty} \frac{|\,\mathrm{SA}_{pGp}(F, n, \delta', d')|_E}{d\log d}.$$

Using [DKP2, Lemma 2.13]

$$s_{G,E\cup S}(F\cup S, n+4) + \mathfrak{h}(p) \le 1 + \mathfrak{h}(p)s_{pGp,E}(F, n),$$

and we deduce

$$s(G) \le \mathfrak{h}(p)s(pGp) + 1 - \mathfrak{h}(p).$$

The same proof works with $\liminf_{d\to\infty}$ and $\lim_{d\to\omega}$ instead of $\limsup_{d\to\infty}$. ■

Finally we deduce the scaling formula.

Proof of Proposition 9.1 Let $p_n \le p \le q_n$ be projections with $\mathfrak{h}(p_n), \mathfrak{h}(q_n) \in \mathbb{Q}$ and $|p_n - q_n| \to 0$. We have

$$\mathfrak{h}(p)s(pGp) \le \mathfrak{h}(p_n)s(p_nGp_n) + \mathfrak{h}(p) - \mathfrak{h}(p_n)$$

by the previous lemma, so $\liminf_{n\to\infty} s(p_nGp_n) \ge s(pGp)$. Similarly,

$$\mathfrak{h}(q_n)s(q_nGq_n) \le \mathfrak{h}(p)s(pGp) + \mathfrak{h}(q_n) - \mathfrak{h}(p)$$

by the previous lemma, so $\limsup_{n\to\infty} s(q_nGq_n) \le s(pGp)$. However, the two lemmas combined imply that

$$s(G) = \mathfrak{h}(p_n)s(p_nGp_n) + 1 - \mathfrak{h}(p_n),$$

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so

$$\limsup_{n\to\infty} s(p_nGp_n) = \frac{1}{\mathfrak{h}(p)}(s(g)-1)+1$$

and

$$s(G) = \mathfrak{h}(q_n)s(q_nGq_n) + 1 - \mathfrak{h}(q_n),$$

so

$$\liminf_{n\to\infty} s(q_nGq_n) = \frac{1}{\mathfrak{h}(p)}(s(G)-1)+1.$$

Therefore,

$$s(pGp) = \frac{1}{\mathfrak{h}(p)}(s(G)-1)+1,$$

and the same argument works for $\underline{s}$ and s_ω.

So

$$\underline{s}(G) = s(G) \Leftrightarrow \underline{s}(pGp) = s(pGp),$$

and G is s-regular if and only if pGp is s-regular. ∎

10 Applications of the Correspondence Principle and the Proof of Theorem 1.1

Since the work of [Dye], orbit equivalence relations have been used to prove results in group theory on several occasions, with the Bernoulli action $\Gamma \curvearrowright X_0^\Gamma$ serving as a link. A recent example is the use of the Gaboriau–Lyons theorem [GL] (that the orbit relation of $\Gamma \curvearrowright [0,1]^\Gamma$ of a nonamenable group Γ contains a nonamenable subtreeing) as a way to replace the assumption "contains a nonamenable free group" on Γ by "Γ is nonamenable". The 'correspondence principle' discussed here is a straightforward but useful extension of this well-known idea from groups to groupoids.

For example, let us prove the following result using the principle.

Proposition 10.1 *Let $G \curvearrowright X$ be a pmp action of an amenable pmp groupoid G. Then $s(G) = s(G \ltimes X)$. In particular $s(G \ltimes X)$ does not depend on the action.*

Let us first observe that the measure of the set of finite orbits of an action $G \curvearrowright X$ does not depend on the action.

Lemma 10.2 *If $G \curvearrowright X$ is an essentially free pmp action of a pmp groupoid G and $D \subset X$ is a fundamental domain for the set of finite orbits of G in X, then*

$$\mu(D) := \int_{G^0} \frac{1}{|G_e|}\,\mathrm{d}\mathfrak{h}(e),$$

so $\mu(D)$ depends only on G.

Proof Let

$$X_{\mathrm{f}} := \{x \in X \mid |Gx| < \infty\}.$$

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Since $G \curvearrowright X$ is free, $|Gx| = |G_{\mathbf{r}(x)}|$ almost surely, so

$$\mu^e(X_{\mathrm{f}}^e) = \begin{cases} 0 & \text{if } |G_e| = \infty, \\ 1 & \text{otherwise}, \end{cases}$$

for ae $e \in G^0$ and if $D \subset X_{\mathrm{f}}$ is a fundamental domain then if $e_1, e_2 \ldots, e_n$ are the units isomorphic to e, we have

$$\sum_{i=1}^{n} \mu^{e_i}(D^{e_i}) = \frac{n}{|G_e|}.$$

Therefore,

$$\mu(D) = \int_{G^0} \frac{1}{|G_e|} \mathrm{d}\mathfrak{h}(e)$$

using invariance of μ. ■

Remark 10.3 An analogue of the so-called "fixed price problem" [Gab] for s is the question of whether $s(G \ltimes X)$ depends on G only and not on the essentially free pmp action $G \curvearrowright X$. This "fixed sofic dimension problem" holds for example for groupoids with fixed price 1 (*i.e.,* all their free pmp actions have cost 1) provided all their free pmp actions are sofic, but it is open in general. Also open is whether a groupoid is sofic if and only if all its free pmp actions are sofic.

Proof of Proposition 10.1 Since the Bernoulli action $G \curvearrowright [0,1]^G$ is essentially free $G \ltimes [0,1]^G$ is a pmp equivalence relation, so by Theorem 8.2 and [DKP1, Corollary 5.2]

$$s(G) = s(G \ltimes [0,1]^G) = 1 - \mu(D) = 1 - \int_{G^0} \frac{1}{|G_e|} \mathrm{d}\mathfrak{h}(e),$$

where D is the fundamental domain of the set of finite classes of the amenable pmp equivalence relation $G \ltimes [0,1]^G$.

If $G \curvearrowright X$ is a pmp action, then the action $G \ltimes X \curvearrowright [0,1]^{G \ltimes X}$ is pmp and essentially free, so by Theorem 8.2

$$s(G \ltimes X) = s\big((G \ltimes X) \ltimes [0,1]^{G \ltimes X}\big).$$

Now the measure isomorphism

$$(G \ltimes X) \ltimes [0,1]^{G \ltimes X} \to G \ltimes \big(X \times [0,1]^{G \ltimes X}\big)$$
$$((s,x),y) \mapsto (s,(x,y))$$

is an isomorphism of pmp equivalence relations, where the action $G \curvearrowright X \times [0,1]^{G \ltimes X}$ is diagonal and $G \curvearrowright [0,1]^{G \ltimes X}$ on the first coordinate.

Then

$$\begin{aligned} s(G \ltimes X) &= s\big(G \ltimes \big(X \times [0,1]^{G \ltimes X}\big)\big) \\ &= 1 - \mu(D') \\ &= 1 - \int_{G^0} \frac{1}{|G_e|} \mathrm{d}\mathfrak{h}(e), \end{aligned}$$

where D' is the is the fundamental domain of the set of finite classes of the amenable pmp equivalence relation $G \curvearrowright X \times [0,1]^{G \ltimes X}$. So $s(G) = s(G \ltimes X)$. ■

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Another example is the proof that the invariance of s under orbit equivalence (Theorem 2.7) established in [DKP1, Theorem 4.1] is equivalent to the statement for groupoids in [DKP2, Theorem 2.1].

Theorem 10.4 *Let G be a pmp groupoid and let E, F be transversally generating sets. Then $s(E) = s(F)$, $\underline{s}(E) = \underline{s}(F)$ and $s^{\omega}(E) = s^{\omega}(F)$.*

Proof If F is a transversally generating set of G (which we assume to have infinite fibers) then $F \cup \{p_{B_0}, p_{B_1}\}$ (as defined in Lemma 3.5) is a transversally generating set of the pmp equivalence relation $G \ltimes \{0,1\}^G$. By Theorem 8.2

$$s(F) = s(F \cup \{p_{B_0}, p_{B_1}\})$$

so by Theorem 2.7 we have

$$s(E) = s(E \cup \{p_{B_0}, p_{B_1}\}) = s(F \cup \{p_{B_0}, p_{B_1}\}) = s(F).$$

The same applies to $\underline{s}$ and s_{ω}. ∎

We conclude with the proof Theorem 1.1, which is our main illustration of the correspondence principle.

Proof of Theorem 1.1 Let $\widetilde{G}_i := G_{i|G_3^0}$ and $\widetilde{G} := G_{|G_3^0} = \widetilde{G}_1 *_{G_3} \widetilde{G}_2$ then for every essentially free pmp action $\widetilde{G} \curvearrowright X$ of $\widetilde{G}$ we have

$$\begin{aligned}
s(G) - 1 &= \mathfrak{h}(G_3^0)\big(s(\widetilde{G}_1 *_{G_3} \widetilde{G}_2) - 1\big) && \text{by Proposition 9.1}\\
&\geq \mathfrak{h}(G_3^0)\big(s(\widetilde{G}_1 *_{G_3} \widetilde{G}_2 \ltimes X) - 1\big) && \text{by Proposition 7.4}\\
&= \mathfrak{h}(G_3^0)\Big(s\big((\widetilde{G}_1 \ltimes X_{|1}) *_{(G_3 \ltimes X_{|3})} (\widetilde{G}_2 \ltimes X_{|2})\big) - 1\Big) && \text{by Lemma 3.8}\\
&= \mathfrak{h}(G_3^0)\big(s(R_1 *_{R_3} R_1) - 1\big)\\
&\qquad \text{where } R_i := \widetilde{G}_i \ltimes X_{|i} \text{ are pmp equivalence relations by Lemma 3.4}\\
&= \mathfrak{h}(G_3^0)\big(s(R_1) + s(R_2) - s(R_3) - 1\big) && \text{by [DKP1, Theorem 1.2]}\\
&= \mathfrak{h}(G_3^0)\big(s(R_1) + s(R_2) - s(G_3) - 1\big) && \text{by Proposition 10.1}
\end{aligned}$$

If $\widetilde{G} \curvearrowright X$ is Bernoulli, then

$$s(\widetilde{G}_1 *_{G_3} \widetilde{G}_2) = s(\widetilde{G}_1 *_{G_3} \widetilde{G}_2 \ltimes X)$$

by Theorem 8.2 and $\widetilde{G}_i \curvearrowright X_{|i}$ for $i = 1, 2$ are (isomorphic to) Bernoulli actions by Lemma 3.3. Therefore, $s(R_i) = s(\widetilde{G}_i)$, $i = 1, 2$ by Theorem 8.2. Then

$$\begin{aligned}
s(G) - 1 &= \mathfrak{h}(G_3^0)\big(s(R_1) + s(R_2) - s(G_3) - 1\big) \\
&= \mathfrak{h}(G_3^0)\Big(s(\widetilde{G}_1) + s(\widetilde{G}_2) - s(G_3) - 1\Big) \qquad \text{by Lemma 3.3 and Theorem 8.2} \\
&= \mathfrak{h}(G_3^0)\big(s(\widetilde{G}_1) - 1\big) + \mathfrak{h}(G_3^0)\big(s(\widetilde{G}_2) - 1\big) - \mathfrak{h}(G_3^0)\big(s(G_3) - 1\big) \\
&= \mathfrak{h}(G_1^0)\big(s(G_1) - 1\big) + \mathfrak{h}(G_2^0)\big(s(G_2) - 1\big) - \mathfrak{h}(G_3^0)\big(s(G_3) - 1\big) \\
&\qquad \text{by Proposition 9.1} \\
&= \mathfrak{h}(G_1^0)s(G_1) + \mathfrak{h}(G_2^0)s(G_2) - \mathfrak{h}(G_3^0)s(G_3) - \mathfrak{h}(G_1^0) - \mathfrak{h}(G_2^0) + \mathfrak{h}(G_3^0) \\
&= \mathfrak{h}(G_1^0)s(G_1) + \mathfrak{h}(G_2^0)s(G_2) - \mathfrak{h}(G_3^0)s(G_3) - 1.
\end{aligned}$$

(We note that the inequality $s(G) \le \mathfrak{h}(G_1^0)s(G_1) + \mathfrak{h}(G_2^0)s(G_2) - \mathfrak{h}(G_3^0)s(G_3)$ can also be proved by a direct argument.) ∎

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